and

$$z\psi(x) = \int_1^x L\,dn^z * dn^{-z}$$

$$= \int_1^x (z\,dt * dn^z + b_1 dn + b_2 dn^z) * dn^{-z} + O\Big\{\int_1^x \frac{x}{t}(\log ex/t)^{z-2} dn^z(t)\Big\}.$$

Integration of the O-term by parts yields $O\{N_z(x)\}$. Also, $\int_1^x dn * dn^{-z} = N_{1-z}(x)$, and each of the last two expressions is of magnitude $o(x)$. Thus $\psi(x) = x + o(x)$, and the proof is complete.

References

[1] H. G. Diamond, *Asymptotic distribution of Beurling's generalized integers* (to appear).

[2] R. D. Dixon, *On a generalized divisor problem*, J. Indian Math. Soc. N. S. 28 (1964), pp. 187-195.

[3] A. Kienast, *Über die asymptotische Darstellung der summatorischen Funktion von Dirichletreihen mit positiven Koeffizienten*, Math. Z. 45 (1939), pp. 554-558.

[4] A. Selberg, *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. 18 (1954), pp. 83-87.

[5] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, London 1951.

[6] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc. 10 (1935), pp. 286-293.

[7] — *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. (2) 46 (1940), pp. 389-408.

Reçu par la Rédaction le 31. 1. 1967

ACTA ARITHMETICA
XIII (1967)

Approximation to real numbers by quadratic irrationals

by

H. Davenport (Cambridge) and Wolfgang M. Schmidt (Boulder, Colo.)

1. Introduction. It is well known that if ξ is any real number, not itself rational, there are infinitely many rational approximations p/q to ξ which satisfy

$$|\xi - p/q| < q^{-2}. \tag{1}$$

Many different proofs have been given (see [1], chapters 1-3).

In this paper we investigate the analogous problem of approximation to a real number ξ, not itself rational or a quadratic irrational, by rationals or quadratic irrationals. If α is rational or quadratic irrational, then α satisfies a unique equation

$$x\alpha^2 + y\alpha + z = 0 \tag{2}$$

with relatively prime integral coefficients x, y, z, not all zero, and with the polynomial

$$f(\theta) = x\theta^2 + y\theta + z$$

irreducible over the rationals. We define the *height* $H(\alpha)$ of α by

$$H(\alpha) = \max(|x|, |y|, |z|). \tag{3}$$

Our main result is as follows.

THEOREM. *For any real ξ which is not rational or quadratic irrational, there are infinitely many rational or real quadratic irrational α which satisfy*

$$|\xi - \alpha| < CH(\alpha)^{-3}, \tag{4}$$

where

$$C = \begin{cases} C_0 & \text{if} \quad |\xi| < 1, \\ C_0\xi^2 & \text{if} \quad |\xi| > 1, \end{cases} \tag{5}$$

and C_0 is any fixed number greater than $\frac{160}{9} = 17\cdot 77\ldots$

The relation between the cases $|\xi| < 1$ and $|\xi| > 1$ is very simple. If $|\xi| < 1$ and $\xi_1 = 1/\xi$, and if $\alpha_1 = 1/\alpha$, then $H(\alpha_1) = H(\alpha)$ and

$$|\xi_1 - \alpha_1| = |(\xi - \alpha)\xi_1\alpha_1|.$$

Thus if (4) holds for ξ and α then

$$|\xi_1-\alpha_1| < CH(\alpha)^{-3}|\xi_1\alpha_1| < C(1+\varepsilon)\xi_1^2 H(\alpha_1)^{-3}$$

for any fixed $\varepsilon > 0$, provided α is sufficiently near to ξ. Hence if C is a permissible constant for $|\xi| < 1$, we infer that $C(1+\varepsilon)\xi^2$ is a permissible constant for $|\xi| > 1$.

No particular importance attaches to the number $160/9$, which could be reduced at the cost of further complications.

Apart from the value of the constant, the result of the theorem is best possible. For if ξ is a cubic irrational, the fact that the norm of $x\xi^2+y\xi+z$ has a positive lower bound for all integers x, y, z, not all 0, implies that

$$|\xi^2 x+\xi y+z| > D\{\max(|x|, |y|, |z|)\}^{-2},$$

where $D = D(\xi) > 0$; and from this it is easily deduced that if α is a root of (2) then

$$|\xi-\alpha| > D_1 H(\alpha)^{-3}, \quad D_1 = D_1(\xi) > 0.$$

The problem of the present paper is a particular case of one investigated by Wirsing [3], arising from the relationship between Mahler's and Koksma's classifications of transcendental numbers (see [2], chapter 3). In this particular case, Wirsing's inequality (9) would give an exponent $-2-\frac{1}{2}\sqrt{2}$ in place of -3; but he is primarily concerned with the more general problem of approximating to real ξ by algebraic numbers of degree at most n. We have no contribution to make to this more general problem.

The proof of the theorem will be indirect. We can confine ourselves to the range $0 < \xi < 1$. So we assume that for a particular ξ in this range and some $C_1 > 160/9$ we have

(6) $$|\xi-\alpha| > C_1 H(\alpha)^{-3}$$

for all rational or real quadratic irrational α with $H(\alpha)$ sufficiently large. This will ultimately lead to a contradiction, and the theorem will follow.

2. A lemma. Let $\boldsymbol{x} = (x, y, z)$ be any set of 3 integers, and write

(7) $$|\boldsymbol{x}| = \max(|x|, |y|, |z|).$$

If $\boldsymbol{x} \neq 0$, and α satisfies (2), then $H(\alpha) \leqslant |\boldsymbol{x}|$.

We shall be concerned with two linear forms:

(8) $$P(\boldsymbol{x}) = 2\xi x+y,$$

(9) $$L(\boldsymbol{x}) = \xi^2 x+\xi y+z.$$

Lemma 1. *There is a number $C_2 < 9/160$ with the following property: if $\boldsymbol{x} = (x, y, z)$, where x, y, z are relatively prime integers, and if $|\boldsymbol{x}|$ is sufficiently large, then*

(10) $$|P(\boldsymbol{x})| < C_2|\boldsymbol{x}|^3|L(\boldsymbol{x})|.$$

Proof. Take C_2 so that $C_1^{-1} < C_2 < 9/160$, where C_1 is the constant of (6). If the lemma is false, there will be infinitely many $\boldsymbol{x} \neq 0$ satisfying

(11) $$|L(\boldsymbol{x})| \leqslant C_2^{-1}|P(\boldsymbol{x})|\,|\boldsymbol{x}|^{-3}.$$

We now distinguish two cases.

(a) For infinitely many of these $\boldsymbol{x}$ we have $|P(\boldsymbol{x})| > |\boldsymbol{x}|^{-1}$. For the polynomial $f(\theta) = \theta^2 x+\theta y+z$ we have

$$f(\xi) = L(\boldsymbol{x}), \quad f'(\xi) = P(\boldsymbol{x}),$$

and therefore

$$|f(\xi)| \leqslant C_2^{-1}|f'(\xi)|\,|\boldsymbol{x}|^{-3},$$

$$|f''(\xi)| = 2|x| \leqslant 2|\boldsymbol{x}| < 2|f'(\xi)|\,|\boldsymbol{x}|^2.$$

We may suppose without loss of generality that $f(\xi) > 0$. Let η be the real number determined by $|\eta-\xi| = C_1|\boldsymbol{x}|^{-3}$ and $(\eta-\xi)f'(\xi) < 0$. Then

$$f(\eta) = f(\xi)+(\eta-\xi)f'(\xi)+\tfrac{1}{2}(\eta-\xi)^2 f''(\xi)$$
$$< |f'(\xi)|\,|\boldsymbol{x}|^{-3}\{C_2^{-1}-C_1+C_1^2|\boldsymbol{x}|^{-1}\} < 0,$$

provided $|\boldsymbol{x}|$ is sufficiently large. Hence $f(\theta) = 0$ has a real root α with $|\xi-\alpha| < C_1|\boldsymbol{x}|^{-3}$, and for this α we have $H(\alpha) \leqslant |\boldsymbol{x}|$. Thus we have a contradiction to (6).

(b) In the alternative case, we have $|P(\boldsymbol{x})| \leqslant |\boldsymbol{x}|^{-1}$ for all $\boldsymbol{x}$ satisfying (11) with $|\boldsymbol{x}|$ sufficiently large. From the identity

$$y^2-4xz = P^2(\boldsymbol{x})-4xL(\boldsymbol{x})$$

we obtain

$$|y^2-4xz| \leqslant |\boldsymbol{x}|^{-2}+4|x|C_2^{-1}|\boldsymbol{x}|^{-4} < 1,$$

whence $y^2 = 4xz$. Since x, y, z are relatively prime, this implies that $x = \pm u^2$, $y = \pm 2uv$, $z = \pm v^2$, where u, v are integers, and now

$$L(\boldsymbol{x}) = \pm(\xi u+v)^2.$$

Since $|L(\boldsymbol{x})|$ is small, and $0 < \xi < 1$, we have $0 < |v| < |u|$ and $|\boldsymbol{x}| \leqslant u^2$. By (11) and the hypothesis of the present case,

$$|\xi u+v|^2 \leqslant C_2^{-1}|\boldsymbol{x}|^{-4}.$$

Taking $\alpha = -v/u$ we get

$$|\xi-\alpha| \leqslant C_2^{-1/2}|u|^{-5} = C_2^{-1/2}H(\alpha)^{-5},$$

which contradicts (6).

3. The sequence of minimal points. For each real $X > 1$ we consider the finite set of integer points $\boldsymbol{x} \neq 0$ satisfying

$$|\boldsymbol{x}| \leqslant X.$$

The values of $L(\boldsymbol{x})$ at these points are distinct, since $L(\boldsymbol{x})$ does not vanish at any integer point other than the origin. We choose the unique point for which $L(\boldsymbol{x})$ has its least positive value, and call this the *minimal point corresponding to* X.

It is obvious that if $\boldsymbol{x}$ is the minimal point corresponding both to X' and to X'', it is also the minimal point corresponding to any X between X' and X''. Hence there is a sequence of integers

(12) $$X_1 < X_2 < \ldots$$

such that the same minimal point corresponds to all X in the range $X_i \leqslant X < X_{i+1}$ but to no X outside this range. Denoting this point by $\boldsymbol{x}_i$, we obviously have

(13) $$|\boldsymbol{x}_i| = X_i.$$

We write for brevity

(14) $$L_i = L(\boldsymbol{x}_i), \quad P_i = |P(\boldsymbol{x}_i)|.$$

Plainly

(15) $$L_1 > L_2 > \ldots$$

Lemma 1 applies to the point $\boldsymbol{x}_i$ if i is sufficiently large, and gives

(16) $$P_i < C_2 X_i^3 L_i.$$

Another inequality can be deduced from Minkowski's theorem in the Geometry of Numbers. Consider the convex polyhedron, with centre at the origin, defined by

(17) $$|x| < X, \quad |y| < X, \quad |z| < X, \quad |\xi^2 x + \xi y + z| < \tfrac{4}{3} X^{-2}.$$

This contains the polyhedron defined by

$$|x| < X, \quad |y| < X, \quad |\xi^2 x + \xi y| < X - \tfrac{4}{3} X^{-2} \quad |\xi^2 x + \xi y + z| < \tfrac{4}{3} X^{-2};$$

hence its volume is at least $\frac{8}{3} X^{-2}$ times the area of the hexagon in the x, y plane defined by the first three of the last inequalities. Since $0 < \xi < 1$, this hexagon contains

$$|x| < X, \quad |y| < X, \quad |x + y| < X$$

if X is sufficiently large, and the area of the latter is $3X^2$. Hence the volume of the polyhedron (17) is greater than 8, and so it contains an integer point other than the origin. Taking $X = X_{i+1}$, and recalling that the least value of $|L(\boldsymbol{x})|$ subject to $|\boldsymbol{x}| < X_{i+1}$ is L_i, we deduce that

(18) $$L_i < \tfrac{4}{3} X_{i+1}^{-2}.$$

It follows in particular from (18) that

(19) $$x_i \neq 0$$

if i is sufficiently large. For if $x_i = 0$, then (18) gives

$$|\xi y_i + z_i| < \tfrac{4}{3} X_{i+1}^{-2}$$

and since

$$\max(|y_i|, |z_i|) = X_i < X_{i+1},$$

this contradicts (6) with $\alpha = -z_i/y_i$.

Finally we observe, for future reference, that by the definition of the points $\boldsymbol{x}_i$, there is no integer point $\boldsymbol{x}$ other than the origin satisfying

(20) $$|\boldsymbol{x}| < X_{i+1}, \quad |L(\boldsymbol{x})| < L_i.$$

4. Linear dependence and independence. The further development of the argument requires us to consider the linear dependence or independence of each set of three successive minimal points $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$. On this subject we prove two lemmas.

LEMMA 2. *If the points* $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ *are linearly dependent then, for one of the two signs,*

(21) $$\boldsymbol{x}_{i-1} \pm \boldsymbol{x}_{i+1} = u\boldsymbol{x}_i,$$

where u *is an integer.*

Proof. We show first that, for any i, the points $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ constitute an integral basis for all integer points in the plane through the origin and these two points. If this were not so, there would exist an integer point, other than the origin, of the form

$$\boldsymbol{x} = r\boldsymbol{x}_i + s\boldsymbol{x}_{i+1},$$

where r, s are rational numbers satisfying $|r| \leqslant \frac{1}{2}$, $|s| \leqslant \frac{1}{2}$. For such a point we should have

$$|\boldsymbol{x}| \leqslant \tfrac{1}{2} X_i + \tfrac{1}{2} X_{i+1} < X_{i+1},$$
$$|L(\boldsymbol{x})| \leqslant \tfrac{1}{2} L_i + \tfrac{1}{2} L_{i+1} < L_i.$$

This point would therefore satisfy (20), which is impossible.

The hypothesis of linear dependence now implies that

$$\boldsymbol{x}_{i-1} = u\boldsymbol{x}_i + v\boldsymbol{x}_{i+1},$$

where u, v are integers. Similarly

$$\boldsymbol{x}_{i+1} = u'\boldsymbol{x}_{i-1} + v'\boldsymbol{x}_i,$$

where u', v' are integers. Comparison of these relations gives $u'v = 1$, whence $v = \pm 1$, whence (21).

LEMMA 3. *There are infinitely many values of n for which the points $\boldsymbol{x}_{n-1}, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}$ are linearly independent.*

Proof. If the points $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ are linearly dependent, it follows from (21) that

$$x_{i-1}L(\boldsymbol{x}_i) - x_i L(\boldsymbol{x}_{i-1}) = \pm\{x_i L(\boldsymbol{x}_{i+1}) - x_{i+1}L(\boldsymbol{x}_i)\}.$$

If this holds for all i in the range $m \leqslant i < n$, then

$$|x_{m-1}L(\boldsymbol{x}_m) - x_m L(\boldsymbol{x}_{m-1})| = |x_{n-1}L(\boldsymbol{x}_n) - x_n L(\boldsymbol{x}_{n-1})| < \tfrac{4}{3}X_{n-1}X_{n+1}^{-2} + \tfrac{4}{3}X_n X_n^{-2}$$

by (18). If this holds for all $n > m$, then since the limit of the last expression as $n \to \infty$ is 0, we get

$$x_{m-1}L(\boldsymbol{x}_m) - x_m L(\boldsymbol{x}_{m-1}) = 0.$$

But the only integer point at which $L(\boldsymbol{x})$ vanishes is the origin. Hence

$$x_{m-1}\cdot\boldsymbol{x}_m - x_m\cdot\boldsymbol{x}_{m-1} = 0,$$

and this is impossible by (19) if m is sufficiently large.

It follows that there exist arbitrarily large values of n for which the points $\boldsymbol{x}_{n-1}, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}$ are linearly independent.

5. Three independent points. The inequality (16) gives an upper bound for P_i, but we have as yet no useful lower bound. The object of the present section is to establish a good lower bound for P_{n+1} in the case when $\boldsymbol{x}_{n-1}, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}$ are linearly independent.

LEMMA 4. *If n is large and $\boldsymbol{x}_{n-1}, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}$ are linearly independent, then*

$$P_{n+1}X_n L_{n-1} > \tfrac{1}{2} - \tfrac{32}{9}C_2. \tag{22}$$

Proof. The determinant

$$\begin{vmatrix} x_{n-1} & P(\boldsymbol{x}_{n-1}) & L(\boldsymbol{x}_{n-1}) \\ x_n & P(\boldsymbol{x}_n) & L(\boldsymbol{x}_n) \\ x_{n+1} & P(\boldsymbol{x}_{n+1}) & L(\boldsymbol{x}_{n+1}) \end{vmatrix}$$

is equal to the similar determinant formed with $x_{n-1}, y_{n-1}, z_{n-1}$ etc., and so is an integer. By the hypothesis of linear independence, this integer is not zero and so has absolute value at least 1.

The cofactor of $P(\boldsymbol{x}_{n-1})$ in the determinant is

$$x_{n+1}L(\boldsymbol{x}_n) - x_n L(\boldsymbol{x}_{n+1}),$$

and this has absolute value less than $\frac{8}{3}X_{n+1}^{-1}$ by (18). Hence the contribution to the determinant made by the terms containing $P(\boldsymbol{x}_{n-1})$ is

$$\begin{aligned} &< \tfrac{8}{3}X_{n+1}^{-1}P_{n-1} \\ &< \tfrac{8}{3}C_2 X_{n+1}^{-1}X_{n-1}^{3}L_{n-1} \\ &< \tfrac{32}{9}C_2 X_{n-1}^{3}X_n^{-3} < \tfrac{32}{9}C_2 \end{aligned}$$

by (16) and (18).

Similarly the contribution of the terms containing $P(\boldsymbol{x}_n)$ is

$$\begin{aligned} &< 2X_{n+1}L_{n-1}P_n \\ &< 2C_2 X_{n+1}L_{n-1}X_n^{3}L_n \\ &< \tfrac{32}{9}C_2. \end{aligned}$$

Hence the contribution of the terms containing $P(\boldsymbol{x}_{n+1})$ is greater in absolute value than

$$1 - \tfrac{64}{9}C_2.$$

But this contribution is less in absolute value than

$$2P_{n+1}X_n L_{n-1},$$

and this proves (22).

6. Completion of the proof. Let m be a large integer for which the points $\boldsymbol{x}_{m-1}, \boldsymbol{x}_m, \boldsymbol{x}_{m+1}$ are linearly independent, and let n be the least integer greater than m for which $\boldsymbol{x}_{n-1}, \boldsymbol{x}_n, \boldsymbol{x}_{n+1}$ are linearly independent. If $m < i < n$, the points $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ are linearly dependent, and therefore

$$\boldsymbol{x}_{i-1} \pm \boldsymbol{x}_{i+1} = u_i \boldsymbol{x}_i$$

by Lemma 2. The same relation holds for the corresponding values of the linear forms P and L, and therefore

$$|P(\boldsymbol{x}_{i-1})L(\boldsymbol{x}_i) - P(\boldsymbol{x}_i)L(\boldsymbol{x}_{i-1})| = |P(\boldsymbol{x}_i)L(\boldsymbol{x}_{i+1}) - P(\boldsymbol{x}_{i+1})L(\boldsymbol{x}_i)|.$$

By repeated use of this relation, we obtain

$$|P(\boldsymbol{x}_m)L(\boldsymbol{x}_{m+1}) - P(\boldsymbol{x}_{m+1})L(\boldsymbol{x}_m)| = |P(\boldsymbol{x}_{n-1})L(\boldsymbol{x}_n) - P(\boldsymbol{x}_n)L(\boldsymbol{x}_{n-1})|.$$

Naturally this reduces to an identity if $n = m+1$.

We have

$$\begin{aligned} |P(\boldsymbol{x}_{n-1})L(\boldsymbol{x}_n) - P(\boldsymbol{x}_n)L(\boldsymbol{x}_{n-1})| &< C_2 X_{n-1}^{3}L_{n-1}L_n + C_2 X_n^{3}L_n L_{n-1} \\ &< 2C_2\tfrac{4}{3}X_n^{3}X_n^{-2}L_n = \tfrac{8}{3}C_2 X_n L_n, \end{aligned}$$

by (16) and (18).

On the other hand, by (22),

$$|P(\boldsymbol{x}_{m+1})L(\boldsymbol{x}_m)-P(\boldsymbol{x}_m)L(\boldsymbol{x}_{m+1})|$$
$$> (\tfrac{1}{2}-\tfrac{32}{9}C_2)X_m^{-1}L_{m-1}^{-1}L_m-\tfrac{4}{3}C_2X_mL_m$$
$$> (\tfrac{3}{8}-\tfrac{8}{3}C_2)X_mL_m-\tfrac{4}{3}C_2X_mL_m$$
$$= (\tfrac{3}{8}-4C_2)X_mL_m.$$

We have $\frac{8}{3}C_2 < \frac{3}{8}-4C_2$ since $C_2 < 9/160$. Thus we deduce that

$$X_mL_m < X_nL_n.$$

But this is impossible, since it leads to an infinite sequence of values of n for which X_nL_n increases, whereas we know that $X_nL_n \to 0$ as $n \to \infty$ by (18).

In view of the remarks at the end of § 1, this contradiction proves the theorem.

Note addet in proof. We have since extended the basic result of this paper to a general theorem on $n-1$ linear forms in n variables, the result of the present paper being the case $n=3$ with the linear forms $P(\boldsymbol{x})$, $L(\boldsymbol{x})$. See a forthcoming paper *A theorem on linear forms* in this journal. The more general result does not, however, solve the problem investigated by Wirsing and mentioned in § 1.

References

[1] J. F. Koksma, *Diophantische Approximationen*, Ergebn. Math. IV, 4.

[2] Th. Schneider, *Einführung in die transzendenten Zahlen*, Berlin 1957.

[3] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades*, Journ. Math. 206 (1960), pp. 67-77.

TRINITY COLLEGE, CAMBRIDGE, ENGLAND,
UNIVERSITY OF COLORADO, BOULDER, COLORADO

Reçu par la Rédaction le 31. 1. 1967

ACTA ARITHMETICA
XIII (1967)

On two theorems of Gelfond and some of their applications

by

A. Schinzel (Warszawa)

§ 1. Introduction. The theorems mentioned in the title are concerned with the ordinary and $\mathfrak{p}$-adic measure of irrationality of the ratio of two logarithms of algebraic numbers. A. O. Gelfond, having estimated this measure [9], [10] was able in 1940 to deduce [10] for two elements α, β of an algebraic number field R and a prime ideal $\mathfrak{p}$ of R the inequalities

$$G_0 = \log|\alpha^n-\beta^m|-\max\{n\log|\alpha|, m\log|\beta|\} > -\log^{3+\varepsilon}\max\{|n|, |m|\},$$
$$G_{\mathfrak{p}} = \operatorname{ord}_{\mathfrak{p}}(\alpha^n-\beta^m) < \log^{3+\varepsilon}\max\{|n|, |m|\},$$

provided $\log|\alpha|/\log|\beta|$ is irrational and $n > n_0(\varepsilon, \alpha, \beta)$ or $\alpha^u\beta^v \neq 1$ for all integer pairs $(u, v) \neq (0, 0)$, α, β are $\mathfrak{p}$-adic units and $n > n_{\mathfrak{p}}(\varepsilon, \alpha, \beta)$, respectively.

In his book [11] published first in 1952 Gelfond has improved the estimates for the measure of irrationality of $\log\alpha_2/\log\alpha_1$ in a manner which permits to replace exponent $3+\varepsilon$ by $2+\varepsilon$ in the inequality for G_0. The same new method works *mutatis mutandis* in the $\mathfrak{p}$-adic case. It has also the advantage of being applicable if $\log\alpha_2/\log\alpha_1$ is irrational but $\alpha_1^u\alpha_2^v = 1$ for some integer $(u, v) \neq (0, 0)$, while the earlier method failed in this case as pointed out by V. Jarnik [13]. Therefore, the estimation for G_0 is true not only if $\log|\alpha|/\log|\beta|$ is irrational but as originally asserted by Gelfond in [10] if $\alpha^n-\beta^m \neq 0$ and the case $|\alpha| = |\beta| = 1$, $\alpha^u\beta^v \neq 1$ for all integer pairs $(u, v) \neq (0, 0)$ is excepted.

The applications I have in view require estimates for G_0 and $G_{\mathfrak{p}}$ that are explicit, i.e. do not involve the unspecified functions n_0 and $n_{\mathfrak{p}}$. For the purpose of finding such estimates earlier Gelfond's method is much more suitable than the very involved method of 1952. Therefore in § 2 I reproduce the arguments of [9] and [10] with such modifications as to replace $\log^{3+\varepsilon}\max\{|n|, |m|\}$ by $C(\alpha, \beta)\big(\log\max\{|n|, |m|\}+C'(\alpha, \beta)\big)^3$ or $C(\alpha, \beta, \mathfrak{p})\big(\log\max\{|n|, |m|\}+C'(\alpha, \beta, \mathfrak{p})\big)^3$ in the inequality for G_0 or $G_{\mathfrak{p}}$, respectively. $C(\alpha, \beta)$, $C'(\alpha, \beta)$, $C(\alpha, \beta, \mathfrak{p})$, $C'(\alpha, \beta, \mathfrak{p})$ are constants written