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Interpolation of the Dirichlet divisor problem*

by

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1. The classical divisor problem of Dirichlet is that of finding an asymptotic estimate as $x \to \infty$ for the number of ordered pairs of positive integers (m, n) whose product does not exceed x. In the language of multiplicative convolution, the problem is one of estimating $\int_1^x dn * dn$, where dn is the counting measure of positive integers. The estimation of $\int_1^x dn^k$, the k-fold convolution of dn for k some positive integer is also well known ([5], Ch. 12).

In this paper we give a reasonable interpretation to the expression dA^z for z an arbitrary complex number and dA any real valued Borel measure on $[1, \infty)$ with positive point mass at 1. Moreover, we shall show that there exist numbers $c_j = c_{jz}$, a positive number a', and a function $J = J(x) = (\log x)^{a'}$ such that as $x \to \infty$

$$(1.1)\qquad \int_1^x dn^z = \sum_{j=1}^{J} c_j x(\log x)^{z-j} + O\big(x \exp\{-(\log x)^{a'}\}\big).$$

Let $\zeta(s)$ be the Riemann zeta function and z a complex number. The branch of $\zeta(s)^z$ which is real and positive for s and z real and $s > 1$, $z > 0$ has a Dirichlet representation $\sum_{n=1}^{\infty} a_{nz} n^{-s}$, and the problem of estimating $\int_1^x dn^z$ may be rephrased in zeta function terms as the estimation of $\sum_{n \leqslant x} a_{nz}$.

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This type of problem has been considered in [2], [3], and [4], and asymptotic expansions for $\int_1^x dn^z$ have been produced. In the present paper, the error term of (1.1) is obtained.

Our method depends on the representation of a measure by an exponential, and has been exposed in [1]. This method requires the assumption of the prime number theorem with remainder term. As a converse to our main result, we show how knowledge of the asymptotic behavior of $\int_1^x dn^z$ leads to the prime number theorem.

An idea important to the present problem, that of an exponential of a measure, has been set out in detail in a previous paper [1]. Consequently, we shall only give a sketch of this material here.

2. Let $\mathfrak{M}$ denote the collection of complex valued set functions with support contained in $[1, \infty)$ which, when restricted to bounded Borel sets, are finite measures. In the sequel, "measure" is always to be understood as "element of $\mathfrak{M}$" and "set" as "bounded Borel set". Measures will be represented by such symbols as dA. The associated distribution function $A(x)$ is defined to be $\int_{1-}^{x+} dA$, which we write more simply as $\int_1^x dA$.

Define $dA*dB$, the multiplicative convolution of the measures dA and dB, by

$$\int_E dA*dB = \int_{st\in E} dA(s)dB(t),$$

for E any set. $dA*dB$ also belongs to $\mathfrak{M}$ and satisfies the following equation:

$$\int_1^x dA*dB = \int_1^x A(x/t)dB(t) = \int_1^x B(x/t)dA(t).$$

Let dp be the measure which assigns the value 1 (resp. 0) to any set which includes (resp. excludes) the point 1. $(\mathfrak{M}, +, *)$ is a commutative algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ with dp as unit. Define $dA^0 = dp$ for all $dA\in\mathfrak{M}$ and dA^n by $dA^n = dA^{n-1}*dA$, $n = 1, 2, \ldots$ A mapping L: $\mathfrak{M}\to\mathfrak{M}$ is defined by

$$\int_E L\,dA = \int_{t\in E} \log t\, dA(t).$$

L satisfies $L(dA*dB) = (L\,dA)*dB + dA*(L\,dB)$.

A set of seminorms $\|\cdot\|_x$, $1\leqslant x<\infty$, is defined on $\mathfrak{M}$ by the total variation of an element of $\mathfrak{M}$ on $[1, x]$. These seminorms induce a topology on $\mathfrak{M}$ satisfying the first axiom of countability. For each $x\in[1,\infty)$, $(\mathfrak{M}, +, *, \|\cdot\|_x)$, the normed algebra of restrictions of elements of $\mathfrak{M}$ to $[1, x]$, is a Banach algebra.

We say that a measure $dA\in\mathfrak{M}$ has an inverse if there exists a $dB\in\mathfrak{M}$ such that $dA*dB = dp$. If an inverse exists, it is unique. The invertible elements of $\mathfrak{M}$ are precisely those with non zero point mass at 1, and we designate this subset of $\mathfrak{M}$ by $\mathfrak{M}_1$. An inversion formula that will be of use later is the following: $(dp+dt)*(dp-dt/t) = dp$, where dt is Borel-Lebesgue measure on $[1, \infty)$.

If dA is an arbitrary element of $\mathfrak{M}_1$, then there exists $da\in\mathfrak{M}$ such that $dA = e^{da}$, where the exponential is defined by $\sum_{n=0}^{\infty} da^n/n!$ and is convergent in the topology of $\mathfrak{M}$. The measure da, which we call the logarithm of dA, is uniquely determined, modulo $2\pi i dp$, by the following equations:

$$\begin{cases} L\,da = (L\,dA)*dA^{-1} \text{ and} \\ da\{1\} = \log(dA\{1\}). \end{cases} \tag{2.1}$$

Conversely, a pair of measures da and dA which satisfy (2.1) also satisfy the equation $dA = e^{da}$. If $dA = e^{da}$ and $dB = e^{db}$, then $dA*dB = e^{da+db}$.

In particular we have the following exponential formulas which we will later need: $dp+dt = e^{d\tau}$, where $\tau(x) = \int_1^x (1-t^{-1})\,dt/\log t$; and $dn = e^{d\Pi}$, where $\Pi(x) = \sum 1/k$, the sum extending over all numbers of the form p^k, p a prime and k a positive integer, satisfying $p^k\leqslant x$. In consequence of (2.1), these formulas are proved by establishing the truth of the following equations:

$$L\,d\tau = \{L(dp+dt)\}*(dp+dt)^{-1} \quad\text{and}\quad L\,d\Pi = L\,dn*dn^{-1}.$$

Alternatively, the exponential formulas may be obtained by applying the Mellin transform:

$$\int x^{-s}\big(dp(x)+dx\big) = s/(s-1), \qquad \int x^{-s}d\tau(x) = \log s/(s-1),$$

$$\int x^{-s}dn(x) = \zeta(s) \quad\text{and}\quad \int x^{-s}d\Pi(x) = \log\zeta(s).$$

From the exponential representation we see that any measure $dA\in\mathfrak{M}_1$ possesses precisely n distinct nth roots in $\mathfrak{M}$, where an nth root of dA is a measure dB satisfying $dB^n = dA$.

For da a logarithm of dA, the nth roots are given by

$$dB_k = \omega_k e^{da/n},$$

ω_k an nth root of unity, $k = 0, 1, \ldots, n-1$. The dB_k are nth roots and are distinct. Any nth root of dA is itself a member of $\mathfrak{M}_1$ and as such has an exponential representation $dB = e^{db}$. Since

$$n\,db = da \ (\text{modulo } 2\pi i\,dp), \qquad db = da/n \ (\text{modulo } 2\pi i\,dp/n),$$

and

$$e^{db} = e^{da/n + 2\pi i k dp/n} = \omega_k e^{da/n}.$$

Thus our list of nth roots is exhaustive.

Let $\mathfrak{M}_{1+}$ be the subset of $\mathfrak{M}_1$ consisting of real valued measures with positive point mass at 1. If $dA \in \mathfrak{M}_{1+}$, by (2.1), there exists a real valued da whose exponential is dA. If z is an arbitrary complex number and $dA \in \mathfrak{M}_{1+}$, define dA^z by $dA^z = e^{zda}$, where da is the unique real valued logarithm of dA.

The following three remarks show our definition of dA^z to be a reasonable one. We assume here that $dA \in \mathfrak{M}_{1+}$ and we have selected the determination of the logarithm da which is real at 1. First, in the case that z is a rational number p/q, dA^z already exists as $(dA^p)^{1/q}$ or $(dA^{1/q})^p$. If we take the determination of these expressions that is positive at 1, they each equal e^{zda}. Thus the new definition of dA^z agrees with the previous one in the case that both are applicable.

Second, suppose z is a real irrational number and $\{z_n\}$ is any sequence of rational numbers whose limit is z. Assume further that for each n we have selected the determination of dA^{z_n} which is positive at 1. Then $dA^{z_n} \to dA^z$ in the topology of $\mathfrak{M}$. The proof of this statement follows from the continuity of the exponential map ([1], Ch. 2).

Third, under suitable convergence hypotheses, the Mellin transform of a multiplicative convolution equals the product of the transforms. Our definition of zth power interpolates this property to non integer values of z. Specifically, assume that there exists an n such that as $x \to \infty$,

$$\int_1^x |da| = O(x^n).$$

Then the three Mellin integrals

$$\int x^{-s} da(x), \quad \int x^{-s} dA(x), \quad \text{and} \quad \int x^{-s} dA^z(x)$$

exist, at least for $\operatorname{Re} s > n$, and on this set

$$\int x^{-s} dA^z(x) = \Big\{\int x^{-s} dA(x)\Big\}^z.$$

In the sequel we will want to express zth powers using the binomial theorem. We now prove this representation equivalent to that given by the exponential. With an obvious normalization, assume $dA\{1\} = 1$.

Lemma 2. *Let $dA = e^{da} = dp + dB$, with $dB\{1\} = 0$. For any $z \in \mathbf{C}$,*

$$e^{zda} = \sum_{j=0}^{\infty} \binom{z}{j} dB^j.$$

Proof. If z is a positive integer, e^{zda} is the z-fold convolution of dA, and

$$\sum_{j=0}^{z} \binom{z}{j} dB^j = (dp + dB)^z.$$

Each is equal to dA^z and there is nothing more to prove.

If z is not a positive integer, set $dq = \sum \binom{z}{j} dB^j * e^{-zda}$. It suffices to prove that $dq = dp$. The crux of the argument is to show that $Ldq = 0$, which in turn implies that $dq = 0$ on $(1, \infty)$. Verification that $dq\{1\} = 1$ completes the demonstration that $dq = dp$.

In the remainder of the argument we shall make rearrangements of and do termwise operations on power series in measures. The justification, which is analogous to that for functions of a complex variable, is explicitly carried out in [1], Ch. 2.

Recall that for the operator L the following equation is valid for arbitrary dA and $dB \in \mathfrak{M}$:

$$L(dA * dB) = (L\,dA) * dB + dA * (L\,dB).$$

Thus

$$Ldq = \sum_{j=1}^{\infty} \binom{z}{j} j\, dB^{j-1} * LdB * e^{-zda} + \sum_{j=0}^{\infty} \binom{z}{j} dB^j * e^{-zda} * (-zLda).$$

By (2.1),

$$Lda = L(dp + dB) * (dp + dB)^{-1} = (LdB) * (dp + dB)^{-1}.$$

Thus

$$Ldq = e^{-zda} * LdB * (dp + dB)^{-1} * \Big\{\sum \binom{z}{j} j\, dB^{j-1} * (dp + dB) - z \sum \binom{z}{j} dB^j\Big\}.$$

The last "factor" of the convolution may be rewritten as

$$\sum_{j=0}^{\infty} \Big\{\binom{z}{j+1}(j+1) + \binom{z}{j} j - \binom{z}{j} z\Big\} dB^j.$$

The three terms in the curly bracket sum to zero, for all $j \geq 0$, and thus $Ldq = 0$. This completes the proof of Lemma 2.

We conclude this section with the remark that while membership in $\mathfrak{M}_1$ is a sufficient condition for a measure to have nth roots, it is by no means necessary. An example of a measure not in $\mathfrak{M}_1$ which has nth roots is dt which, for each positive integer n equals $\{(\log t)^{\nu-1} dt / \Gamma(\nu)\}^n$, with $\nu = 1/n$. This is so since the Mellin transform of $(\log t)^{\nu-1} dt$ is $\Gamma(\nu)(s-1)^{-\nu}$, and the claimed representation holds by the Fourier uniqueness theorem.

3. In the previous section we established the existence of zth roots of arbitrary measures in $\mathfrak{M}_{1+}$. We now consider the problem of estimating the asymptotic behavior of $\int_1^x dn^z$. Our method depends on approximating dn^z by $(dp+dt)^z$, and we begin by deriving an asymptotic formula for $\int_1^x (dp+dt)^z$.

THEOREM 3. *Let $z \in K$, K a compact set, define*

$$T_z(x) = \int_1^x (dp+dt)^z,$$

and for $n = 1, 2, \ldots$ let

$$a_n = a_{nz} = \binom{z-1}{n-1}^2 \frac{\Gamma(n)}{\Gamma(z)}. \tag{3.1}$$

There exists a constant k depending only on K such that as $x \to \infty$

$$T_z(x) = \sum_{n=1}^{\log x} a_n x(\log x)^{z-n} + O(\log^k x). \tag{3.2}$$

The O- is uniform for $z \in K$. Further, if $x \geqslant x_0 = x_0(K)$ and N is any number less than $\frac{1}{2}\log x$,

$$T_z(x) = \sum_{n=1}^{N} a_n x(\log x)^{z-n} + 2a_N\,\theta x(\log x)^{z-N} + O(\log^k x), \tag{3.3}$$

where θ is a number of modulus smaller than one. If z is any positive integer, the series terminates after z terms and is exact.

Remarks. 1) It is important to note in (3.3) that x_0 is independent of N; otherwise we would have only an asymptotic expansion, which is inadequate for our purposes.

2) We soon will see that

$$T_z(x) = \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} \frac{\Gamma(n+z)}{\Gamma(n+1)^2} (\log x)^n.$$

The series is a generalized hypergeometric function of $\log x$ and is known ([6], [7]) to have an asymptotic expansion. However, we know of no treatment of this function which yields an error term of the type given here.

3) One might suspect that an asymptotic series whose terms initially decrease and then increase would best be approximated by terminating near the term of lowest value. For our function, that proves indeed to be the case.

Proof. It is always assumed in what follows that z belongs to a compact set K. Under this restriction, all estimates will be uniform in z. The constants arising in estimates will be denoted by k, and may have different values in different places.

If z is a positive integer we note that

$$T_0(x) = 1, \quad x \geqslant 1 \quad \text{and} \quad T_{z+1}(x) = \int_1^x T_z(x/t)(dp+dt).$$

The proof in this case is completed in the obvious way by induction.

Henceforth we assume that z is not a positive integer. We represent $(dp+dt)^z$ by $(dp - t^{-1}dt)^{-z}$ and expand the latter measure by the binomial theorem:

$$T_z(x) = \int_1^x (dp - t^{-1}dt)^{-z} = \sum_{j=0}^{\infty} \binom{-z}{j} \int_1^x (-t^{-1}dt)^j = \sum_{j=0}^{\infty} \binom{-z}{j} (-\log x)^j / j!.$$

For z not a negative integer we write $\binom{-z}{j}$ as $(-1)^j \dfrac{\Gamma(z+j)}{\Gamma(z)\Gamma(j+1)}$, giving the claimed generalized hypergeometric function.

For $y > 1$, set

$$F(y) = \sum_{j=0}^{\infty} \binom{-z}{j} (-y)^j / j!.$$

We will represent F by a finite series plus an error term. The method of proof is to represent F as a contour integral, appropriately deform the contour, and expand the integrand as a series in y^{-1}.

The contour integral. By evaluating the residue at the origin,

$$F(y) = \frac{1}{2\pi i} \int_{|t|=r} (1-t)^{-z} e^{y/t} t^{-1} dt,$$

where $r \in (0,1)$. Setting $u = 1-t$, we have

$$F(y) = \frac{-1}{2\pi i} \int_{\mathfrak{C}} u^{-z} \exp\left\{\frac{y}{1-u}\right\} \frac{du}{1-u},$$

where $\mathfrak{C}$ is a curve of winding number 1 with respect to $u = 1$ and which does not cross the negative real axis. Let $\mathfrak{C} = \mathfrak{C}_1 - \mathfrak{C}_2$, $\mathfrak{C}_1$ the circle $u = 1 + ye^{i\theta}$ with θ varying from $-\pi+0$ to $\pi-0$ and $\mathfrak{C}_2$ a curve from $1-y-i0$ to $1-y+i0$ which does not cross the negative axis and is at positive distance from the origin. We have

$$\int_{\mathfrak{C}_1} = O(y^k),$$

and thus

$$F(y) = \frac{1}{2\pi i}\int_{\mathfrak{C}_2} + O(y^k).$$

We want to let $\mathfrak{C}_2$ come near the origin. If $\operatorname{Re} z < 1$, this may be done with no further changes. If $\operatorname{Re} z \geqslant 1$, we first integrate by parts $q = [\operatorname{Re} z]$ times and obtain

$$F(y) = \frac{1}{2\pi i}\cdot\frac{\Gamma(1-z)}{\Gamma(1-z')}\int_{\mathfrak{C}_2} u^{-z'}\exp\left\{\frac{y}{1-u}\right\}\sum_{n=q+1}^{2q+1} P_n(y)(1-u)^{-n}\,du + O(y^k),$$

where $z' = z-q$ and the P_n's are some polynomials. The exact form of the P_n's does not concern us, but we note that P_{2q+1} is of highest degree, q. Note also that since z is assumed not to be a positive integer, $\Gamma(1-z)/\Gamma(1-z')$ is finite.

Now we take $\mathfrak{C}_2$ as a curve along the (slit) negative axis from $1-y-i0$ to zero and back to $1-y+i0$ and write

$$F(y) = \frac{\Gamma(1-z)}{\Gamma(1-z')}\cdot\frac{\sin \pi z'}{\pi} f(y) + O(y^k),$$

re

$$f(y) = \int_0^{y-1} t^{-z'}\exp\left\{\frac{y}{1+t}\right\}\sum_n P_n(y)(1+t)^{-n}\,dt.$$

nsion of the integrand. Now, in place of $\exp\left\{\frac{y}{1+t}\right\}$ we write

e^y ex t} and set $x = ty/(1+t)$:

$$) = e^y \sum_n P_n(y) y^{z'-1}\int_{x=0}^{y-1} x^{-z'}(1-x/y)^{z'+n-2} e^{-x}\,dx.$$

Expand $)^{z'+n-2}$ in a power series in x/y about the origin. For fixed series is absolutely and uniformly convergent for $0 \leqslant x \leqslant y-$ have now

$$f(y) = e^y \sum_n P_n(y) y^{z'-1}\sum_{j=0}^{\infty}\binom{z'+n-2}{j}(-y)^{-j}\int_0^{y-1} x^{j-z'} e^{-x}\,dx.$$

Estimate the integral differently, depending on whether or not j is less than y: If $j < y$,

$$\int_0^{y-1} x^{j-z'} e^{-x}\,dx = \Gamma(j+1-z') - \int_{y-1}^{\infty},$$

and

$$\left|\int_{y-1}^{\infty}\right| \leqslant \max_{x\geqslant y-1}|x^{j+2-z'}e^{-x}|\int_{y-1}^{\infty} x^{-2}\,dx \leqslant k_1(y-1)^{j+k}e^{-y}.$$

If $j \geqslant y$, we estimate $\int_0^{y-1} x^{j-z'}e^{-x}\,dx$ by $(y-1)$ times the maximum of the integrand on $[0, y-1]$: $\left|\int_0^{y-1}\right| \leqslant k_1(y-1)^{j+k}e^{-y}$. The constants k and k_1 are independent of y and j.

We have

$$f(y) = e^y\sum_n P_n(y)\sum_{j<y}\binom{z'+n-2}{j}(-1)^j y^{z'-j-1}\Gamma(j+1-z') + E(y),$$

where

$$E(y) = O\left\{\sum_n |P_n(y)| y^k \sum_{j=0}^{\infty}\left|\binom{z'+n-2}{j}\right|(1-y^{-1})^j\right\}$$

$$= O\left\{y^{k'}\sum_{j=0}^{\infty} j^{k'}(1-y^{-1})^j\right\} = O\{y^{k''}\}.$$

Rearranging the series for f in decreasing powers of y, we find that there exist coefficients $b_j = b_{jz}$ and a number $k = k(K)$ such that

$$f(y) = e^y\sum_{0\leqslant j<y} b_j y^{z'+q-j-1} + O(y^k).$$

Recalling that $z'+q = z$ and setting $a_{j+1} = (-1)^q\dfrac{\Gamma(1-z)}{\Gamma(1-z+q)}\cdot\dfrac{\sin\pi z}{\pi}b_j$, we have an expansion of the desired form.

Evaluation of the coefficients. The simplest method we know for finding the a_j is first to do so under the assumption that $\operatorname{Re} z < 1$, and then show that the formula is valid to the right of that line.

If $\operatorname{Re} z < 1$, we need perform no integration by parts, $z' = z$, and $\sum P_n(y)(1-u)^{-n} = (1-u)^{-1}$. We find that

$$F(y) = \frac{\sin\pi z}{\pi}e^y\sum_{0\leqslant j<y}\binom{z-1}{j}\Gamma(j+1-z)(-1)^j y^{z-j-1} + O(y^k),$$

and, by the fact that $\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z$, the numbers a_j are as in (3.1).

If $\operatorname{Re} z \geqslant 1$, we proceed inductively, using the following relation between T_{z+1} and T_z:

$$T_z(x) = \int_1^x T_{z+1} * (dp+dt)^{-1} = \int_1^x T_{z+1}(x/t)(dp - t^{-1}dt)$$
$$= T_{z+1}(x) - \int_1^x T_{z+1}(t) t^{-1} dt.$$

Now replace T_{z+1} by its asymptotic expansion and repeatedly integrate by parts the expression

$$\sum_n a_{n,z+1} \int_{x_0}^x (\log t)^{z+1-n} dt.$$

We find the following equations connecting $\{a_{n,z}\}_{n=1}^{\infty}$ and $\{a_{n,z+1}\}_{n=1}^{\infty}$:

$$(3.4) \qquad a_{n,z} = -\sum_{j=1}^{n} a_{j,z+1} \frac{\Gamma(n-z)}{\Gamma(j-1-z)}, \qquad n = 1, 2, \ldots$$

We know that formula (3.1) is valid for $\operatorname{Re} z < 1$. Assume its truth for $\operatorname{Re} z < q$, q an arbitrary positive integer. Solving equation (3.4) recursively for $a_{j,z+1}$, $j = 1, 2, \ldots$, we find that (3.1) is valid for $\operatorname{Re} z < q+1$, completing the induction.

Formula (3.3) is established by using (3.2) and estimating $\sum_{N+1}^{\log x} a_n x(\log x)^{z-n}$ by the geometric series $a_N x(\log x)^{z-N} \sum_1^{\infty} (2/3)^n$ on $N+1 \leqslant n \leqslant \frac{3}{5}\log x$ and by $(\frac{2}{5}\log x)\big(x a_W (\log x)^{z-W}\big)$ on $[\frac{3}{5}\log x] = W < n \leqslant \log x$.

This completes the proof of Theorem 3.

4. In this section we apply the results of the earlier sections to the problem of estimating $\int_1^x dn^z$.

THEOREM 4. *Let dn be the counting measure of positive integers, $z \in K$, K a compact set in $\mathbf{C}$, and $N_z(x) = \int_1^x dn^z$. There exist numbers $c_j = c_{jz}$, a positive number a', and a function $J = J(x) = (\log x)^{a'}$ such that as $x \to \infty$*

$$N_z(x) = \sum_{j=1}^{J} c_j x(\log x)^{z-j} + O\big(x \exp\{-(\log x)^{a'}\}\big).$$

The estimate is uniform in z for $z \in K$. The $\{c_j\}$ are defined by the formula

$$c_j = \sum_{n=1}^{j} \binom{z-n}{j-n} \binom{z-1}{n-1}^2 \frac{\Gamma(n)}{\Gamma(z)} F^{(j-n)}(1),$$

where

$$F(s) = F_z(s) = \{\zeta(s)(s-1)/s\}^z;$$

and $a' = a/(1+a)$, where a is a number for which the inequality

$$\Big| x - \int_1^x \log t \, d\Pi(t) \Big| \leqslant Ax \exp\{-(\log x)^a\}$$

is true for all sufficiently large x.

Remark. If z is a positive integer, $c_k = 0$ for $k > z$, and our result reduces to a form of the classical theorem on the divisor problem. Our method is of little interest in this case, since the usual proof ([5], Ch. 12) is simpler, makes no appeal to the prime number theorem, and gives a sharper error estimate. However, the classical proof applies only when z is an integer.

Proof. In order to compare $N_z(x)$ with $T_z(x)$ we need some integral estimates which we give here.

LEMMA 4.1. *Let $\tau(x)$ and $\Pi(x)$ be as defined in Section 2. Let $\nu(x) = \int_1^x t^{-1}(d\Pi - d\tau)(t)$ and assume the prime number theorem in the form*

$$\int_1^x \log t \, d\Pi(t) = x + O(x \exp\{-\log^a x\}).$$

Then $\nu(x) = O(\exp\{-\log^a x\})$.

Proof. $\nu(x) = \int_1^{\infty} - \int_x^{\infty}$. The first integral is convergent and may be evaluated by noting that

$$\int_1^{\infty} t^{-s}(d\Pi - d\tau) = \log\{\zeta(s)(s-1)/s\}.$$

Since the last expression tends to zero as $s \to 1+$, we have, by Abel's continuity theorem, $\int_1^{\infty} = 0$.

The second integral may be evaluated by partial integration, with the aid of the prime number theorem, giving

$$O\left(\frac{\exp\{-(\log^a x)\}}{\log x}\right) + O\left\{\int_x^{\infty} \exp(-\log^a t) \frac{dt}{t \log t}\right\}.$$

If x is sufficiently large, the last integral is less than

$$\exp(-\log^a x) \log x \int_x^{\infty} t^{-1} \log^{-2} t \, dt = \exp\{-\log^a x\},$$

and the proof is complete.

LEMMA 4.2. *Suppose that* $|\nu(x)| \leqslant A_0 \exp\{-\log^a x\}$, $1 \leqslant x < \infty$. *Let* $\nu_n(x) = \int_1^x (d\nu)^n$. *Then there exists a constant* A_1 *such that for each positive integer* n *and all* $x \geqslant 1$,

$$|\nu_n(x)| \leqslant nA_0(2\log\log ex + A_1)^{n-1}\exp\{-(n^{-1}\log x)^a\}.$$

The proof proceeds by using induction on n and the representation $\nu_{n+1}(x) = \int_1^x d\nu_n * d\nu$. The convolution integral is evaluated by iterated integration — in one order over one region, in reverse order over another. The details are carried out in [1], section 3.3.

LEMMA 4.3. *Let*

$$\varphi(x) = \varphi_z(x) = \int_1^x e^{zd\nu} - 1 \quad \textit{and} \quad a' = a/(1+a).$$

For $b = 2(1/a')^{1/a'}$,

$$\varphi(x) = O\big(\exp\{-(b\log x)^{a'}\}\big).$$

Proof. By the preceding lemma,

$$|\varphi(x)| \leqslant A_0 \sum_1^\infty |z|^n (2\log\log ex + A_1)^{n-1}\exp\{-(n^{-1}\log x)^a\}/\Gamma(n).$$

Take $N = [(\log x)^{a'}(\log\log x)^{-a'/a}]$ and write the last sum as $\sum_1^N + \sum_{N+1}^\infty$.

$$\sum_1^N \leqslant \exp\{-(N^{-1}\log x)^a\} A_0 |z| \exp\{2|z|\log\log ex + A_1|z|\}$$

$$\leqslant k_1 \exp\{-(\log x \log\log x)^{a'}\}(\log x)^{k_2},$$

$$\sum_{N+1}^\infty \leqslant A_0 \sum_{N+1}^\infty |z|^n (2\log\log ex + A_1)^{n-1}/\Gamma(n)$$

$$\leqslant 2A_0|z|^{N+1}(2\log\log ex + A_1)^N/\Gamma(N+1)$$

$$\leqslant k\exp\{-\tfrac{1}{2}(\log x\log\log x)^{a'}\}.$$

Adding the two sums we find that

$$\varphi(x) = O\big(\exp\{-\tfrac{1}{2}(\log\log x)^{a'}(\log x)^{a'}\}\big),$$

which is somewhat more than was claimed in the statement of the lemma. The $(\log\log x)^{a'}$ factor has been deleted and the b entered in the interest of simplicity.

LEMMA 4.4.

$$\int_1^x e^{z(d\Pi - d\tau)} = O\big(x\exp\{-(b'\log x)^{a'}\}\big),$$

any fixed b' *satisfying* $1 < b' < b$.

Proof. $\int_1^x e^{z(d\Pi - d\tau)} = \int_1^x te^{zd\nu}$, since multiplication by t is a homomorphism with respect to multiplicative convolution. Now the last integral equals $1 + \int_1^x t\,d\varphi(t)$, and the proof is completed by integration by parts and use of the preceding lemma.

Proof of the theorem. We have

$$N_z(x) = \int_1^x e^{zd\Pi} = \int_1^x dT_z * e^{z(d\Pi - d\tau)}$$

$$= \int_1^{\sqrt{x}} T_z(x/t)\, e^{z(d\Pi - d\tau)} + \int_1^{\sqrt{x}} \Big\{\int_1^{x/t} e^{z(d\Pi - d\tau)}\Big\} dT_z(t) - T_z(\sqrt{x})\int_1^{\sqrt{x}} e^{z(d\Pi - d\tau)}$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

II and III may be estimated by using the leading term of the asymptotic approximation of T, the estimate of $\int_1^y e^{z(d\Pi - d\tau)}$, and the fact that

$$|dT_z| = |e^{zd\tau}| \leqslant e^{|z|d\tau} \leqslant e^{kd\tau} = (dp + dt)^k \leqslant dp + k_1 dt + k_2(\log t)^{k-1}dt.$$

We find that

$$N_z(x) = \int_{t=1}^{\sqrt{x}} T_z(x/t)\, e^{(zd\Pi - d\tau)(t)} + O(xe^{-J}).$$

In the last integral, t lies in $[1, \sqrt{x}]$. By (3.3) we may estimate $T_z(x/t)$ by

$$\frac{x}{t}\sum_{n=1}^J a_n\Big(\log\frac{x}{t}\Big)^{z-n} + 2\theta\frac{x}{t}a_J\Big(\log\frac{x}{t}\Big)^{z-J},$$

with $J = (\log x)^{a'} < \frac{1}{4}\log x \leqslant \frac{1}{2}\log(x/t)$. Using the fact that $a_J = O\{\Gamma(J+k)\}$, we find that

$$\int_{t=1}^{\sqrt{x}} 2\theta\frac{x}{t}\Big|a_J\Big(\log\frac{x}{t}\Big)^{z-J}\Big|\,|e^{z(d\Pi - d\tau)}|$$

$$\leqslant 2x|(\log\sqrt{x})^{z-J}|\,\Gamma(J+k)\int_1^{\sqrt{x}} t^{-1}e^{k(d\Pi + d\tau)} = O(xe^{-J}).$$

To evaluate

$$\int_1^{\sqrt{x}} \sum_{n=1}^J a_n\frac{x}{t}\Big(\log\frac{x}{t}\Big)^{z-n} e^{z(d\Pi - d\tau)(t)},$$

we write

$$\left(\log\frac{x}{t}\right)^{z-n} \quad \text{as} \quad (\log x)^{z-n}\left(1-\frac{\log t}{\log x}\right)^{z-n}$$

and expand the last factor by the binomial theorem. We must now estimate

$$\int_1^{\sqrt{x}} (-\log t)^j e^{z d\nu(t)} = \int_1^{\sqrt{x}} (-\log t)^j (dp + d\varphi)(t).$$

For $j \leqslant J = (\log x)^{a'}$, write the last expression as $\int_1^\infty - \int_{\sqrt{x}}^\infty$. As in earlier arguments,

$$\int_1^\infty = F^{(j)}(1) = \{(\zeta(s)(s-1)/s)^z\}^{(j)}|_{s=1}.$$

$\int_{\sqrt{x}}^\infty$ is estimated by integrating by parts and using Lemma 4.3 and the fact that for $b = 2(1/a')^{1/a'}$, $\log^j t \exp\{-(b\log t)^{a'}\}$ is decreasing for $t \geqslant \sqrt{x}$. We find that

$$\left|\int_{\sqrt{x}}^\infty\right| < A(\log\sqrt{x})^j \exp\{-(b\log\sqrt{x})^{a'}\}.$$

For $j > J$, $\log^j t\exp\{-(b\log t)^{a'}\}$ is increasing for $1 \leqslant t \leqslant \sqrt{x}$, and also in this case

$$\left|\int_1^{\sqrt{x}} (-\log t)^j d\varphi(t)\right| \leqslant A(\log\sqrt{x})^{j+1}\exp\{-(b\log\sqrt{x})^{a'}\}.$$

The same constant A is valid for all j, $0 \leqslant j < \infty$.

We now have

$$\int_{t=1}^{\sqrt{x}} \sum_{n=1}^{J} a_n \frac{x}{t}\left(\log\frac{x}{t}\right)^{z-n} e^{z(d\Pi - d\tau)(t)}$$
$$= \sum_{n=1}^{J} a_n x(\log x)^{z-n} \sum_{j=0}^{J} \binom{z-n}{j} F^{(j)}(1)(\log x)^{-j} + R(x),$$

where $R(x)$ is of the order

$$\sum_{n=1}^{J} |a_n x(\log x)^{z-n}| \sum_{j=0}^{\infty} \left|\binom{z-n}{j}\right| \frac{(\log\sqrt{x})^{j+1}}{(\log x)^j} \exp\{-(b\log\sqrt{x})^{a'}\}.$$

To estimate R, majorize $\binom{z-n}{j}$ by $\binom{-n-|z|}{j}(-1)^j$ and sum the series in j to yield $k\,2^n\log x$. The series in n may be estimated by its first term, for its terms decrease geometrically. We find that $R(x) = O(xe^{-J})$.

We change the order of summation of

$$\sum_{n=1}^{J} a_n x(\log x)^{z-n} \sum_{j=0}^{J} \binom{z-n}{j} F^{(j)}(1)(\log x)^{-j}$$

to

$$\sum_{i=1}^{J} \sum_{n+j=i} + \sum_{i=J+1}^{2J} \sum_{\substack{n+j=i\\ n,j\leqslant J}}.$$

We estimate the second double sum:

$$|a_n| \leqslant \Gamma(n+k); \quad \left|\binom{z-n}{j}\right| \leqslant \frac{\Gamma(n+j+k)}{\Gamma(n)\Gamma(j+1)};$$

and, by Cauchy's inequalities, for ϱ a fixed number between 0 and 1,

$$|F^{(j)}(1)| \leqslant k\varrho^{-j}\Gamma(j+1).$$

With these approximations, we see that the second double sum is $O(xe^{-J})$.

Collecting the various O-estimates and setting $c_i = \sum_{j+n=i} a_n\binom{z-n}{j}F^{(j)}(1)$, we have

$$N_z(x) = \sum_{i=1}^{J} c_i x(\log x)^{z-i} + O(xe^{-J}),$$

$J = (\log x)^{a'}$, $a' = a/(1+a)$, where a is the index from the prime number theorem. This completes the proof of Theorem 4.

5. In the present section we prove a converse of Theorem 4: an assumption upon $N_z(x)$ leads to the prime number theorem. We give three results, for z in different ranges. The first of these, with $z = -1$ is a form of the classical theorem that $\sum_{n\leqslant x}\mu(n) = o(x)$ (μ the Möbius function) implies the prime number theorem. The second, with z near zero, uses an idea that appeared in [4]; namely, to interpret suitably the equation

$$\Pi(x) = \frac{\partial}{\partial z} N_z(x)|_{z=0}.$$

The third, which uses any value of z in $(0, 1)$, exploits the fact that $N_z(x)$ and $N_{1-z}(x)$ are $o(x)$.

THEOREM 5.1. *Let* $N_{-1}(x) = O(x\exp\{-(\log x)^{a'}\})$ *for some* $a' \in (0,1)$ *and let* $\mathrm{li}(x) = c + \int_2^x (\log t)^{-1}dt$. *Then, for any fixed number* $\varrho < 1$,

$$\Pi(x) = \mathrm{li}(x) + O(x\exp\{-(\varrho\log x)^{a'}\}).$$

THEOREM 5.2. *Let c_{jz} be as in Theorem 4, and suppose that for some integer $\nu \geqslant 2$ the equation*

(5.1) $$N_z(x) = \sum_{j=1}^{\nu-1} c_{jz} x(\log x)^{z-j} + O\{x(\log x)^{z-\nu}\}$$

holds uniformly for all z in some real interval $(0, \varepsilon)$. Then, for any fixed positive number δ,

$$\Pi(x) = \mathrm{li}(x) + O\{x(\log x)^{\delta-\nu/2}\}.$$

THEOREM 5.3. *Suppose there exists one value of z in $(0, 1)$ for which (5.1) holds with $\nu = 3$. Then*

$$\Pi(x) \sim \mathrm{li}(x).$$

Proof of Theorem 5.1. Let

$$\psi(x) = \int_1^x \log t\, d\Pi(t), \qquad \psi(x) = \int_1^x L\,dn * dn^{-1} = n(x) - \gamma + R(x),$$

where

$$R(x) = \int_1^x (L\,dn - dn * dt + \gamma\,dn) * dn^{-1}.$$

To estimate R note that

$$|dn^{-1}| = |e^{-d\Pi}| \leqslant e^{d\Pi} = dn \quad \text{and} \quad \int_1^y L\,dn - dn * dt + \gamma\,dn = O(\log y).$$

Let $x_1 = x\exp\{-(\log x)^{\alpha'}\}$ and $x_2 = x/x_1$.

$$R(x) = \int_1^{x_1} \Big\{\int_1^{x/t} L\,dn - dn * dt + \gamma\,dn\Big\} dn^{-1}(t) +$$
$$+ \int_{t=1}^{x_2} \{N_{-1}(x/t) - N_{-1}(x_1)\}\{L\,dn - dn * dt + \gamma\,dt\}$$
$$= O\big(x\exp\{-(\log x_1)^{\alpha'}\}\log^2 x\big),$$

and this implies Theorem 5.1.

Proof of Theorem 5.2. For z any number satisfying $0 < |z| < 1$, we have

$$z^{-1}\big(N_z(x) - 1\big) = \Pi(x) + R(x),$$

with

$$R(x) = \int_1^x \sum_{n=2}^{\infty} z^{n-1} d\,\Pi^n/n! = O\Big(|z| \int_1^x \sum d\,\Pi^n/n!\Big) = O(|z|x).$$

On the other hand, assuming z lies in $(0, \varepsilon)$,

$$z^{-1}\big(N_z(x) - 1\big) = \sum_{j=1}^{\nu-1} z^{-1} c_{jz} x(\log x)^{z-j} - z^{-1} + O\{z^{-1} x(\log x)^{z-\nu}\}.$$

Take $z = (\log x)^{-\nu/2}$ and note that $z^{-1} c_{jz} \to \Gamma(j)$ as $z \to 0$. This gives the desired result. To obtain a better error term, one estimates the rapidity of convergence of $z^{-1} c_{jz}$ as $z \to 0$.

Proof of Theorem 5.3. We will show that $\psi(x) \sim x$, where $\psi(x) = \int_1^x \log t\, d\Pi(t)$. Since $dn^z = e^{z d\Pi}$, the following formula is a consequence of (2.1):

$$z\,d\psi = zL\,d\Pi = L\,dn^z * dn^{-z}.$$

The idea used here is to approximate $L\,dn^z$ by measures that are more easily convolved with dn^{-z}. In the classical case ($z = 1$) this procedure enables one to show that $\psi(x) = O(x)$, but no more than that. In the present case we can show that both $N_z(x)$ and $N_{1-z}(x)$ are $o(x)$, and with this fact deduce the prime number theorem.

LEMMA. *Let r be any fixed number in $(0, 1)$. Then*

$$N_r(x) = \int_1^x dn^r = o(x).$$

Proof. Let

$$A = \overline{\lim}\, N_r(x)/N_1(x), \qquad 0 \leqslant N_r(x) = \int_1^x e^{r d\Pi} \leqslant \int_1^x e^{d\Pi} = N_1(x) \Rightarrow 0 \leqslant A \leqslant 1.$$

By (2.1)

(5.2) $$\int_1^x L\,dn^r = r\int_1^x N_r(x/t)\, d\psi(t),$$

and we estimate the last integral in terms of A using the fact that

$$\int_1^x N_1(x/t)\, d\psi(t) = \int_1^x L\,dN_1 = N_1(x)\log x + O(x)$$

and Chebychev's estimate $\psi(x) = O(x)$. We integrate the first integral in (5.2) by parts and conclude that $A = \overline{\lim}\, N_r(x)/N_1(x) \leqslant rA$. Since A is finite and $1 - r > 0$, $A = 0$. This completes the proof of the lemma.

From the power series of the exponentials, we see that $|dn^{-z}| \leqslant dn^z$, for z positive. Let

$$b_1 = z\int_1^{\infty} \big(N_z(t) - c_{1z} t(\log t)^{z-1}\big) t^{-2}\, dt \quad \text{and} \quad b_2 = 1 - c_{2z}/c_{1z}(1 - z).$$

Then

$$\int_1^x (L\,dn^z - z\,dt * dn^z + b_1\,dn + b_2\,dn^z) = O\big(x(\log x)^{z-2}\big),$$

and

$$z\psi(x) = \int_1^x L\,dn^z * dn^{-z}$$

$$= \int_1^x (z\,dt * dn^z + b_1 dn + b_2 dn^z) * dn^{-z} + O\left\{\int_1^x \frac{x}{t}(\log ex/t)^{z-2} dn^z(t)\right\}.$$

Integration of the O-term by parts yields $O\{N_z(x)\}$. Also, $\int_1^x dn * dn^{-z} = N_{1-z}(x)$, and each of the last two expressions is of magnitude $o(x)$. Thus $\psi(x) = x + o(x)$, and the proof is complete.

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Approximation to real numbers by quadratic irrationals

by

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1. Introduction. It is well known that if ξ is any real number, not itself rational, there are infinitely many rational approximations p/q to ξ which satisfy

$$|\xi - p/q| < q^{-2}. \tag{1}$$

Many different proofs have been given (see [1], chapters 1-3).

In this paper we investigate the analogous problem of approximation to a real number ξ, not itself rational or a quadratic irrational, by rationals or quadratic irrationals. If α is rational or quadratic irrational, then α satisfies a unique equation

$$x\alpha^2 + y\alpha + z = 0 \tag{2}$$

with relatively prime integral coefficients x, y, z, not all zero, and with the polynomial

$$f(\theta) = x\theta^2 + y\theta + z$$

irreducible over the rationals. We define the *height* $H(\alpha)$ of α by

$$H(\alpha) = \max(|x|, |y|, |z|). \tag{3}$$

Our main result is as follows.

Theorem. *For any real ξ which is not rational or quadratic irrational, there are infinitely many rational or real quadratic irrational α which satisfy*

$$|\xi - \alpha| < CH(\alpha)^{-3}, \tag{4}$$

where

$$C = \begin{cases} C_0 & \text{if} \quad |\xi| < 1, \\ C_0\,\xi^2 & \text{if} \quad |\xi| > 1, \end{cases} \tag{5}$$

and C_0 is any fixed number greater than $\frac{160}{9} = 17 \cdot 77 \ldots$

The relation between the cases $|\xi| < 1$ and $|\xi| > 1$ is very simple. If $|\xi| < 1$ and $\xi_1 = 1/\xi$, and if $\alpha_1 = 1/\alpha$, then $H(\alpha_1) = H(\alpha)$ and

$$|\xi_1 - \alpha_1| = |(\xi - \alpha)\,\xi_1\alpha_1|.$$