A method in diophantine approximation

by

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The object of this paper is to present a theorem which gives a method for obtaining results about the diophantine approximation of some values of certain functions into $\mathbb{R}^n$ where $m \geq 1$. The method is then applied in two corollaries and a number of examples to functions satisfying linear differential equations, both scalar and vector with not necessarily (real) analytic coefficients. Two examples deal with applications to non-linear functional equations.

Section I. Suppose that:

(I) $y$ is a function from a set $S$ to $\mathbb{R}^n$ (the $m$ by 1 matrices over $R$);

(II) $U$ is a vector space of functions from $S$ to $\mathbb{R}^n$ over the field $R$;

(III) $T$ is a linear operator and $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_l$ (l $\geq 2$) are subspaces of $U$ such that $T^*$ is defined from $U_i$ to $U$ ($1 \leq i \leq l$);

(IV) $y$ belongs to $U_1$;

(V) $M$ is a vector space over $\mathbb{R}$ of functions from $S$ to the $m$ by $m$ matrices over $R$;

(VI) $\Phi$ is a function from $M$ to $M$;

(VII) if $f$ belongs to $U_1$ and $g$ belongs to $M$, then $gf$ belongs to $U_1$ and

$$Tgf = gTf + \Phi(g)f;$$

(VIII) we have

$$y = \sum_{i=1}^{l} y_i T^i y,$$

where the $y_i$ belong to $M$ and each $\Phi^k(y_i) = 0$;

(IX) there exists a subspace $W$ of $U_1$ and a linear operator $T^{-1}$, defined from $TW$ to $U$ such that $T^{-1}f|W = f|W$;

(X) $\Phi^l(y) T^{l-1} y$ is defined and belongs to $W$ for each $1 \leq i \leq l$, $j \geq 0$, and $k \geq 0$, as does each $T^{-n}y$ for $n \geq 0$;

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(XI) there exists a $\delta > 0$ such that

$$|T^{-n}y(x_0)| \leq \left(\frac{C(x_0)}{\pi}\right)^{-n}\delta^n \quad (a = 1, 2, \ldots),$$

where $C(x_0)$ is positive and independent of $n$, for each $x_0$ belonging to $S$.

(XII) we are given $x_0$ belonging to $S$ such that each $\phi^j(y)(x_0)$ belongs to $Q^n$ ($Q$ denotes the rationales) for each $1 \leq i \leq I$, $0 \leq j < i$ and

(XIII) $g(x_0)$ is nonsingular.

Now set

$$\deg g_i = \min\{j | \phi^{j+1}(y)(x_0) = 0\}$$

and

$$d = \max_{1 \leq i \leq I} \deg g_i > 0.$$

**Definition.** By the absolute value of a matrix we mean the maximum of the absolute values of its entries.

**Theorem.** Under conditions (I)-(XIII) either

(a) each $T^iy(x_0) = 0$ $(0 \leq i \leq l - 1)$

or

(b) for each $e > 0$ there exists $C(e) > 0$ such that

$$\max_{0 \leq i \leq l - 1} |T^iy(x_0) - P_i/e| \geq C(e)e^{-d/2},$$

for all $m$ by $1$ matrices of integers $P_i$ and positive integers $q$.

**Proof.** We apply the theorem:

(I) $S = (0, e_1)$ and $y: (0, e_2) \rightarrow R$;

(II) $U$ is the space of all continuous functions from $(0, e_1)$ to $R$;

(III) $T = y_1 \frac{d}{dx} + y_2$ and $U_k = C^k$ on $(0, e_1)$;

(IV) $y$ belongs to $U_k$;

(V) $M$ is the vector space (over $R$) spanned by the $\frac{d}{dx}$

(1 $\leq i \leq I, j > 0$);

(VI) $\Phi = y_1 \frac{d}{dx}$

(VII) $Tgf = gTf + \left(y_1 \frac{d}{dx}\right)f$;

(VIII) $y = \sum_{i=1}^{t} g_i T^iy$;

(IX) $T^{-k} = \exp \left(-\frac{\int \frac{d}{dx}}{y_1} t \right) \exp \left[\int \frac{d}{dx} \right] \exp \left[\int \frac{d}{dx} \right].$
for all \( k \) in \( TW \), where \( W \) is the space of \( C^k \) functions \( f \) on \((0, \varepsilon)\) satisfying \( \lim_{x \to 0} f(x) = 0 \).

If \( f \) belongs to \( W \), then \( T^{-1}f = f \). Hence \( T^{-1}T/W = I/W \).

(X) By assumption, \( \lim x^{a} f(t) T^{-i-1-x} = 0 \) for each \( 1 \leq i \leq 1 \), \( 0 \leq j < i \), \( k \geq 0 \), and \( j < -i - k \geq 0 \). It follows that under these conditions each \( \phi^{a}(g_{i}) T^{-i-1-x} g_{i} \) belongs to \( W \). If \( i-j-1-k < 0 \), we see that the limit is, again, zero since \( \phi^{a}(g_{i}) \) belongs to \( C^{0}(0, \varepsilon) \) and \( \lim x^{a} f(t) = 0 \) for each \( a \geq 1 \). Each \( T^{-x} g(t) \) \((n \geq 1)\) belongs to \( C^{0}(0, \varepsilon) \); hence \( \phi^{a}(g_{i}) T^{-i-1-x} g_{i} \) belongs to \( W \) for \( 1 \leq i \leq 1 \), \( 0 \leq j < 1 \), and \( k \geq 0 \), as does each \( T^{-x} g(t) \) for \( n \geq 1 \).

\[
\left| x^{a} f(t) x^{a} g(t) \right| \leq \left| \exp \left( -\int x^{a} f(t) \, dt \right) \right| \left( \max_{t \in (0, \varepsilon)} \left| x^{a} g(t) \right| \right) t^{a} \left( \max_{t \in (0, \varepsilon)} \left| x^{a} f(t) \right| \right)
\]

By hypothesis both (XII) and (XIII) hold. The corollary then follows immediately from the theorem.

**Example I.** We wish to show that the hypotheses of Corollary I are satisfied under many circumstances. Let \( \int x^{a} f(t) \, dt = t \). Then

\[
T = \frac{d}{dt} x^{a} f(t) + \frac{d}{dt} x^{a} g(t)
\]

where we now regard \( x^{a} f(t) \) and \( x^{a} g(t) \) as functions of \( t \). (Since \( x^{a} f(t) \) did not vanish on \([0, \varepsilon)\), we see that this change of variables is an \( I \) times differentiable homeomorphism.) We see that

\[
\exp \left( \int x^{a} f(t) \, dt \right) T y = \frac{d}{dt} \exp \left( \int x^{a} f(t) \, dt \right) y.
\]

The identity \( \phi^{a}(g_{i}) = 0 \) says that \( d^{a} f/dx^{a} = 0 \), hence each \( g_{i} \) is a polynomial in \( t \) of degree less than \( i \). We obtain from (1) under these circumstances

\[
\exp \left( \int x^{a} f(t) \, dt \right) y = \sum_{i=0}^{\infty} g_{i}(t) \frac{d^{i}}{d t^{i}} \exp \left( \int x^{a} f(t) \, dt \right) y.
\]

Suppose that zero is a regular singular point of the above analytic linear differential equation in

\[Y = \exp \left( \int x^{a} f(t) \, dt \right) y.
\]

Suppose further that \( r \) of the roots of the indicial equation associated with the differential equation (2) in \( Y \) have real parts greater than \( l-1 \). By the construction of solutions about a regular point, \( (2) \) when the coefficient functions are polynomials there exists an \( r \) dimensional solution space over \( R \) of functions \( Y \) which satisfy (3), belong to \( C^{0} \) on \((0, \varepsilon)\), and satisfy

\[
\lim_{t \to 0} \frac{d^{r} Y}{dt^{r}} = 0 \quad (0 \leq n \leq l-1).
\]

It follows that

\[0 = \lim_{t \to 0} \exp \left( \int x^{a} f(t) \, dt \right) T^{n} y = \lim_{t \to 0} T^{n} y = \lim_{t \to 0} T^{n} y.
\]

Thus

\[
\lim_{t \to 0} \phi^{a}(g_{i}) T^{-i-1-x} g_{i} = 0
\]

for \( 1 \leq i \leq 1 \), \( j < i \), \( k \geq 0 \), and \( j < i - 1 - k \geq 0 \). If we can find an \( x_{a} \) such that each \( \phi^{a}(g_{i}) \) belongs to \( Q \) and \( g_{a}(x_{a}) \) does not vanish, we may apply the Corollary.

**Example II.** We treat in detail a specific equation, consider

\[y = \frac{d}{dx} \left( x^{a} - x^{b} - 1 \right) y = \frac{dy}{dx} - 5x^{a} + x^{b} \frac{dy}{dx} + x^{a} \frac{dy}{dx}.
\]

Here \( T = d/dx \), \( g_{1} = 9 \), \( g_{2} = -5x \), and \( g_{3} = x^{a} \). Two solutions of this equation which are linearly independent are \( y_{1} = x^{a} + \ldots \), and \( y_{2} = \ln(x) y_{1} + y_{1} \), where \( y_{a} = x^{a} + \ldots \). Set \( y = C_{1} y_{1} + C_{2} y_{2} \), where \( C_{1} \) and \( C_{2} \) are arbitrary constants. Here

\[
\lim_{t \to 0} \phi^{a}(g_{i}) T^{-i-1-x} y = 0, \quad 1 \leq i < 3, \quad 0 < i < 1, \quad i-j-1-k \geq 0, \quad 0 \leq j \leq 1, \quad 1 \leq k \leq 3.
\]

If \( x_{a} \) is rational, then for every \( t > 0 \) there exists \( C(t) > 0 \) with

\[
\max_{t \to 0} \left| \frac{d^{a} y}{dx^{a}} - \frac{p_{a} q}{q} \right| = C(t) q^{1-a},
\]
by Corollary I. At any rational point \( x_0 \) larger than zero we have
\[
\frac{y_1(x_0)}{y_1'(x_0)} 
\neq \frac{y_2(x_0)}{y_2'(x_0)},
\]
since otherwise there would exist nonzero constants \( C_1 \) and \( C_2 \) with \( y = y' = 0 \) and \( y'' = 1 \). Thus we may define \( y \) by
\[
\begin{bmatrix}
  y_1(x) & y_2(x) \\
y_1'(x) & y_2'(x)
\end{bmatrix} 
= \begin{bmatrix}
y_1' y_2(x) - 1 \\
y_1(x) y_2'(x) - y_1 y_2'(x) + y_2 y_2'(x)
\end{bmatrix} \quad \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}.
\]
Now we have that \( y(x_1) = 1 \) and \( y'(x_1) = 0 \). Hence given \( \varepsilon > 0 \) there exists \( C(\varepsilon) > 0 \) with
\[
|y''(x_1) - p/q| > C(\varepsilon)q^{-1+\delta}p,
\]
for all integers \( p \) and positive integers \( q \), where
\[
y''(x_1) = \begin{bmatrix}
y_1''(x_1) & y_2''(x_1) \\
y_1(x_1) y_2'(x_1) - y_1'(x_1) y_2(x_1)
\end{bmatrix}.
\]

**Example III.** Consider the nonlinear differential equation \( y = ax^2y' + y^2 \). We shall show that there exists a solution
\[
y_1 = \sum_{n=0}^{\infty} a_n x^n
\]
which converges in an open disk of radius \( \frac{1}{4} \) about the origin. The recurrence formula for the coefficients is given by simplifying to \( y = x(y'' + 3yy' + y^2) \) and substituting \( y = \sum_{n=0}^{\infty} a_n x^n \) for \( y \). We obtain
\[
\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 3 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_m a_n x^{n+m}
\]
We must have that
\[
a_n = (n+1)a_{n+1} + 3 \sum_{m=0}^{n} a_m a_n x^{n+m} + \sum_{m=0}^{n} a_m a_{n-m} x^{n-m-1}.
\]

Letting \( n = 0 \), we obtain \( a_0 = 0 \). Now set \( a_1 = 1 \). If \( n > 0 \) we define \( a_{n+1} \) by
\[
a_{n+1} = \frac{1}{(n+1)(n)} \left[ a_n - 3 \sum_{m=0}^{n} a_m a_{n-m} - \sum_{m=0}^{n} a_m a_n a_{n-m-1} \right].
\]
Now \( |a_n| < 2^n \) and \( |a_1| = 1 < 2^1 \). We shall show by induction that \( |a_n| < 2^n \). If \( n > 0 \), then applying the induction hypothesis we obtain
\[
|a_{n+1}| \leq \frac{2^n}{(n+1)(n)} \left[ 1 + 3 \sum_{m=0}^{n} \frac{m+1}{2} \sum_{m=0}^{n} \frac{m}{2} \right]
\]
\[
\leq \frac{2^n}{(n+1)(n)} \left[ 1 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} \sum_{m=0}^{n} (n-1-m) \right]
\]
\[
= \frac{2^n}{(n+1)(n)} \left[ 1 + \frac{3}{2} n^2 + \frac{3}{2} n + \frac{1}{2} (n-1) \right] \leq 2^{n+1}.
\]

Now apply Corollary I to the linear equation \( y = ax^2y' \), where \( T = \frac{d}{dx} + y_1 \) and \( y = y_1 \). We verify that \( y_1 \) belongs to \( C^\infty \) on \((0, \frac{1}{2})\) and that
\[
\Phi(x) = a^2y_1/dx^2 = 0.
\]
Since \( y_1 \) vanishes at \( x = 0 \), we see that
\[
\lim_{x \to 0} \Phi(x) x^2 = y_1 = 0
\]
for \( j < 2, k > 0 \), and \( 1-j-k \geq 0 \). For each rational number in \((0, \frac{1}{2})\) it then follows that one of the two numbers \( y_1 \) and \( y_1' + y_1' \) is irrational.

**Example IV.** Consider the equation
\[
y = \int_0^x \frac{dt}{1+y} \left( \frac{(1+y)}{dx} \right)^1 y.
\]
If there exists a solution \( y_1 \) of (3) which belongs to \( C^2 \) on \((0, \varepsilon)\) for some \( \varepsilon > 0 \), satisfies \( y_1 = 0 \), and does not equal \(-1\) on \((0, \varepsilon)\), then we may apply Corollary I to the linear differential equation
\[
y = \left( \int_0^x \frac{dt}{1+y} \right) T^2 y,
\]
where \( T = (1+y_1) \frac{dy}{dx} \) and \( y = y_1 \). It will follow that one of
\[
\int_0^x \frac{dt}{1+y_1}, \quad y_1, \quad \text{and} \quad (1+y_1)y_1'
\]
is irrational at each point of \((0, \varepsilon)\).
We now construct a function \( y \), which satisfies the above requirements. Consider the function

\[
f(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!(s-1)^n}.
\]

Suppose that we can find a solution \( y \) of the equation

\[
f\left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right) = y
\]

which belongs to \( C^1 \) on \([0, \varepsilon]\) and does not equal \(-1\) on \([0, \varepsilon]\). Note that

\[f(s) = s \frac{d^2}{ds^2} f(s),\]

hence

\[
f\left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right) = \left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right) \left( \frac{d}{dx} f\right) \left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right)
\]

\[y = \left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right) \left( \frac{d}{dx} \right) y_{1}.\]

We know that the non-linear differential equation \( 1/Y' - 1 = f(Y) \) has an analytic solution \( Y \), in a neighborhood of the origin with \( Y'(0) = 0 \) (See [1]). Set \( y_1 = 1/Y' - 1 = f(Y), \) since \( Y'(0) = f(0) = 0 \), we have \( y_1(0) = 0 \). There exists \( \varepsilon_1 > 0 \) such that \( y_1 \) belongs to \( C^0 \) on \([0, \varepsilon] \) and \( y_1 \neq -1 \) on \([0, \varepsilon] \). Finally, since \( y_1 = 1/Y' - 1 \) and \( y_1(0) = 0 \), we have

\[y = \int_{1+y}^{1+y} \frac{dt}{1+y}.
\]

Therefore

\[y = \frac{1}{Y_1} - 1 = f(Y) = f\left( \int_{1+y}^{1+y} \frac{dt}{1+y} \right),\]

so \( y_1 \) satisfies (3) on \((0, \varepsilon)\). This completes example IV.

**Definition.** If \( g = (g_{ij}) \) is a function into the \( m \) by \( m \) matrices over \( R \), then by \( g \) belongs to \( C^{i-1} \) \((i \geq 2)\) on \([0, \varepsilon_1]\), \([0, \varepsilon_2]\) or \([0, \varepsilon_3]\) we mean that each \( g_{ij} \) belongs to \( C^{i-1} \) on the appropriate set.

Now we show that Corollary I can be extended to the case of matrix coefficients. As before

\[T = v_1 \frac{d}{dx} + v_1\]

where now \( v_1 \) and \( v_2 \) are matrix valued functions. We assume that \( v_1 \) takes on only nonsingular values and define \( v_2 \) by \( v_2(x) = (v_1(x))^{-1} \). Assume that \( v_2 \) and \( v_1 \) belong to \( C^{i-1} \) on \([0, \varepsilon] \) and that each \( g_i \) \((0 < i < l)\) is a matrix valued function which belongs to \( C^i \) on \([0, \varepsilon] \). Assume that each \( g_i \) satisfies \((v_2, d/dx)g_i = 0 \) and that \( v_1 \) and \( v_2 \) both commute with each \( (v_2, d/dx)g_i \). Define \( d(g_2)g \) and \( d \) as before. Choose \( a_i \) belonging to \([0, \varepsilon_i] \) such that each \((v_2, d/dx)g_i(a_i)\) belongs to the \( m \) by \( m \) matrices over \( Q \) and \( g_i(a_i) \) is nonsingular. With these assumptions we again consider the functional equation (1).

**Corollary II.** Under the above conditions, if \( g \) is a function from \((0, \varepsilon) \) to \( E^m \) (the \( m \) by \( m \) matrices over \( R \)), \( g \) belongs to \( C^0 \) on \([0, \varepsilon] \), \( g \) satisfies equation (1) on \([0, \varepsilon] \), and

\[
\lim_{x \to 0} \left[ \left( \frac{d}{dx} \right) g_1 \right] \left[ \left( \frac{d}{dx} \right) g_1 \right]^{-1} = 0
\]

(for each \( 0 \leq j < k \leq 0, g_1(j-k) \geq 0 \), and \( i-j-1-k \geq 0 \), then either

(a) each \( T^y g_j(x) \) is \( 0 \) \((0 \leq i < l-1)\),

or

(b) for every \( \varepsilon > 0 \) there exists \( C(\varepsilon) > 0 \) such that

\[
\max_{0<|x|<1} |T^y g_j(x)| - P_i(\varepsilon) > C(\varepsilon) |x|^{-l+i-k}
\]

for all \( m \) by \( 1 \) matrices \( P_i \) over the integers and positive integers \( g \).

**Proof.** We apply the theorem.

(I) \( y = (0, \varepsilon) \) and \( y : (0, \varepsilon) \to E^m; \)

(II) \( y \) is the space of all continuous functions from \((0, \varepsilon) \) to \( R^m; \)

(III) \( T = v_1 \frac{d}{dx} + v_2 \) and \( U_i = C^0 \) on \([0, \varepsilon] \);

(IV) \( y \) belongs to \( U_i \);

(V) \( M \) is the vector space spanned by the \( \phi_j(g_1) \) \((1 \leq i < l, j \geq 0)\);

(VI) \( \phi = v_1 \frac{d}{dx} \);

(VII) \( Tg = gTf + f(g)I \) for all \( g \) in \( M \) and \( f \) in \( U_1 \);

(VIII) \( y \) is \( \sum_{i=1}^{l} g_i T_i y \) and
\[(IX) \quad T^{-1}h = a(x) \int \int \varphi_i(t) a(t)^{-1} h(t) \, dt \quad \text{for all } h \text{ belonging to } TW,
\]

where \(W\) is the subspace of \(U\), consisting of all functions whose limit at \(x = 0\) is zero and \(a(x)\) is a constant \(m\) by \(n\) matrix valued function on \([0, \varepsilon_1]\). We choose \(a(x)\) to be any solution of
\[
\frac{d}{dx} + \psi_1(a(x)) \varphi_i(x) = 0 \quad \text{on } [0, \varepsilon_1],
\]

which assumes nonsingular values on \([0, \varepsilon_1]\). (Such an \(a(x)\) exists since \(\varphi_i\) is continuous on \([0, \varepsilon_1]\). See [1] where this result is proven.) If \(f\) belongs to \(W\) then \(TT^{-1}f\) is defined and belongs to \(W\). Note that \(TT^{-1} = f\) belongs to \(W\) and also to the kernel of \(T\). By the uniqueness of the solution of \(Ty = 0\) when \(y(0)\) is known we see that \(TT^{-1}f = f\) or \(T^{-1}T/W = I/W\).

(X) The argument of Corollary I holds here.

\[(XII) \quad |T^{-n}y(x_0)| \leq \left(\max |a(x)|\right)^n \left(\max |a(x)|\right)^{-1} \quad \text{on } [0, \varepsilon_1],
\]

where each maximum is taken on \([0, \varepsilon_1]\).

Conditions (XII) and (XIII) hold by assumption. Thus Corollary II follows.

The equation
\[
y = \sum_{i=1}^{t} g_i \left( \psi_i \frac{d}{dx} + \psi_1 \right) y
\]

may be transformed into the equation
\[
y = a(x) g_1 = a(x) \sum_{i=1}^{t} g_i \left( \psi_i \frac{d}{dx} \right) y_i
\]

where \(a(x)\) is as in the proof of Corollary II, if \(a(x)\) can be chosen so as to commute with \(\psi_1\) and each \(g_i\). In the general case there is no particular reason to believe that this is possible, or if possible, that it is easy to accomplish. It might be supposed that the \(g_i\) could be shown to be polynomials with matrix coefficients in \(t = \int \psi_1^{-1} \, ds\). This does not appear to hold, however. We know that \(\{\psi_i \frac{d}{ds}\} g_i = 0\), whence \(\{\psi_i \frac{d}{ds}\} g_i = c_1\). Since \(\psi_1\) commutes with each \(\Phi(g_i)\), we have
\[
\psi_1 \frac{d}{dx} g_0 = c_1 t + c_2
\]

and
\[
\frac{d}{dx} g_0 = c_1 \int \psi_1^{-1} + c_1 \psi_1^{-1} \quad \text{so } \quad g_0 = c_1 \int \psi_1^{-1} + c_1 t + c_2.
\]

Now \(\int \psi_1^{-1} \, ds = t/2\) if and only if \(\psi_1^{-1} = t^{-1}\). The most that we know is that \(\psi_1^{-1}(c_1 t + c_2) = (c_1 t + c_2) \psi_1^{-1}\) which is not good enough.

As we shall see we have reached the first place where the (possible) non-analyticity of our coefficients appears to represent a true generalization. For scalar equations of the type treated in Corollary I the change of dependent variable from \(y\) to \(Y\) given by
\[
Y = \exp \left( \int \frac{1}{\psi_1} \, ds \right) y
\]

and the change of independent variable from \(x\) to \(t\) given by
\[
t = \int \frac{1}{\psi_1} \, ds
\]

combine (see Example I of Corollary I) to yield an analytic differential equation in \(Y(t)\) to which we may apply Corollary I. Calculating the \(d^t Y/dt\) at
\[
t_i = \int \frac{1}{\psi_1} \, ds
\]

and the \(T^t y\) at \(t_i\), \(0 \leq i \leq n-1\), it becomes clear that each
\[
\frac{d^t Y}{dt^t} (t_i) = \exp \left( \int \frac{1}{\psi_1} \, ds \right) T^t y(t_i).
\]

Recalling that \(y\) is at most determined up to a multiplicative constant, it follows that we obtain the same number-theoretic information from the analytic differential equation for \(Y\) as from the original equation for \(y\). Certainly, however, an analogous argument does not go through in the case of vector differential equations, for the reasons outlined above.

We may treat scalar analytic differential equations in the plane by writing them as vector differential equations. Given the equation
\[
y = \sum_{i=1}^{t} g_i(x) d^t y d^t x,
\]
we investigate the behavior of \( y \) at the point \( s+i0 \neq 0 \) by substituting 
\[
\begin{pmatrix}
\alpha & -t \\
1 & \alpha
\end{pmatrix}
\begin{pmatrix}
v(u) \\
w(u)
\end{pmatrix}
\]
for \( z, \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} \) for \( y = u+i0 \). Then 
\[
\begin{pmatrix}
\alpha & -t \\
1 & \alpha
\end{pmatrix}
\begin{pmatrix}
\frac{d}{ds} \\
\frac{d}{ds}
\end{pmatrix}
\begin{pmatrix}
v(u) \\
v(u)
\end{pmatrix}
\]
for \( \frac{d}{ds} (0 < s < < \infty) \). This puts the equation in a form where it may be possible to apply Corollary II.

Several final remarks before proving the theorem. It seems very natural to generalize these results to functions of more than one variable and have \( T \) be, say, a partial differential operator. Suppose that
\[
T = \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial x_2}
\]
where \( \psi \) is an m by m matrix valued function of \( x_1 \) and \( y \) commutes with each \( T^1(g_i), \) if \( \psi \), then \( T^1(g_i) \), and the space \( U_1 \) are sufficiently “nice” one might hope to define \( T^{-1} \) by
\[
T^{-1} = \left( \sum_{i=1}^{\infty} \left( -E \psi \frac{\partial}{\partial x_2} \right)^i \right) E
\]
where
\[
Eh = \int_0^h \psi(t, \alpha x_2) dt.
\]
(Suppose that \( U \) consists of all functions \( f \) which are real analytic functions of \( x_1 \) and \( x_2 \) and which as analytic functions of \( x_1 \) and \( x_2 \) in \( D \times C \), where \( D \) is an open disk about zero, satisfy
\[
|f(x_1, x_2)| < \beta \exp(\beta |x_2|),
\]
for fixed \( \alpha > 0 \) and \( \beta > 0 \) depending on \( f \). If we define \( |f| = \inf \beta \), then \( U \) is a Banach space. Using the integral representation of \( \partial f/\partial x_2 \), we see that
\[
|\partial f/\partial x_2| < \alpha \beta \exp(\beta |x_2|),
\]
hence \( \partial/\partial x_2 \) is a bounded operator. Suppose that \( \psi = \psi(x_2) \) belongs to \( U \). Then \( T^{-1} \) is a bounded operator from \( U \). It is easy to show that \( ||T^{-1}|| < \alpha \beta \exp(\beta |x_2|) \) for a constant \( \beta > 0 \). \( W \) would consist of all \( f \) in \( U \) which vanish when \( x_2 \) is zero.

For the above choice of \( U \) and \( W \) one might also investigate the operator
\[
T = \frac{\partial}{\partial x} + \psi \partial_4
\]
where \( \psi \) is as before and
\[
\partial_4 f(x_1, x_2) = f(x_1, x_2+1)-f(x_1, x_2).
\]

If \( g(x_1, x_2) \) belongs to \( M \) implies \( g(x_1, x_2) = g(x_1) \) we have that
\[
Tg = gT + \frac{\partial g}{\partial x_1} f
\]
for all \( f \) in \( U \). Here we may set
\[
T^{-1} = \left( \sum_{i=0}^{\infty} \left( -E \psi \partial_4 \right)^i \right) E,
\]
as \( \partial_4 \) is a bounded operator on \( U \).

Section II

Proof of the theorem. From (X) we know that each \( \phi^j(g_1) T^{i-j-k-1} y \)
belongs to the vector space \( W \). Hence so does \( w \) where
\[
w = \sum_{i=0}^{\infty} \frac{\partial g}{\partial x_1} \phi^j(g_1) T^{i-j-k-1} y.
\]

Since \( T \) is defined on \( U_1 \supset W \), we may apply \( T \) to \( w \). Now each
\[
\phi^j(g_1)
\]
belongs to \( M \), each \( T^i y \) \( (0 \leq i \leq l-1) \) belongs to \( U_{i-1} \supset U_i \), and each \( T^{l-k} y \) \( (0 \leq k \leq \infty) \) belongs to \( W \supset U_l \). Thus we may apply the identity of (VII) to obtain
\[
T \phi^j(g_1) T^{i-j-k-a-1} y = \phi^j+1(g_1) T^{i-j-k-a-1} y + \phi^j(g_1) T^{i-j-k-a-1} y.
\]
If \( i-j-k-a-1 \geq 0 \), then \( T^{i-j-k-a-1} y = T^{i-j-k-a-1} y \) by definition. If \( i-j-k-a-1 < 0 \) then the function \( T^{i-j-k-a-1} y \) belongs to \( W \). It is easily verified that if \( T/W : W \to TW \) and \( T^{-1} : TW \to W \) satisfy
\[
T^{-1} W = I/W, \quad T/W T^{-1} = I/TW.
\]
Hence
\[
T^{i-j-k-a-1} y = T^{i-j-k-a-1} y = T^{i-j-k-a} y
\]
here also. It follows that we always have
\[
T \phi^j(g_1) T^{i-j-k-a-1} y = \phi^{j+1}(g_1) T^{i-j-k-a-1} y + \phi^j(g_1) T^{i-j-k-a-1} y.
\]

Thus
\[
Tw = \sum_{i=0}^{\infty} \frac{\partial g}{\partial x_1} (-1)^i \phi^j(g_1) T^{i-j-k-a-1} y + \sum_{i=0}^{\infty} (-1)^i \phi^j(g_1) T^{i-j-k-a-1} y = \phi(g_1) T^{i-j-k-a} y.
\]
Now using $T^{-1}Tw = w$, it follows that

$$T^{-1}\phi'(g)T^{-i-b}y = \sum_{a=0}^{\deg (i)} (-1)^{a}\phi^{i+a}(g)T^{-i-b-a-1}y. \tag{4}$$

We next show, by induction, that for each $n \geq 1$

$$T^{-n}y, T^{n}y = \sum_{a=0}^{\deg (n)} (-1)^{a}\phi^{n+a}(g)T^{-n+i-a}y. \tag{5}$$

for nonnegative integers $C_{a}$, satisfying $C_{1} \leq n$. If $n = 1$, then line (5) follows from (4) with $j = k = 0$. (Each $C_{n} = 1 - 1^{n}$.) We assume the induction statement for $n \geq 1$ and shall prove it for $n+1$. By (4) we see that

$$T^{-1}\phi'(g)T^{-n-1}y = \sum_{a=0}^{\deg (n+1)} (-1)^{a}\phi^{n+1+a}(g)T^{-n-1+i-a}y.$$

Hence we obtain from (5), by applying $T^{-1}$,

$$T^{-(n+1)}y, T^{n+1}y = \sum_{a=0}^{\deg (n+1)} (-1)^{a}\phi^{n+a}(g)T^{-n+1+i-a}y.$$

Setting $y = a+b$, we obtain

$$T^{-n+1}y, T^{n+y} = \sum_{a=0}^{\deg (a)} (-1)^{a}\phi^{n+a}(g)T^{-n+y}y.$$

Set

$$C_{n+1} = \sum_{a=0}^{\deg (a)} C_{a}.$$

Then

$$C_{n+1} \leq \sum_{a=0}^{\deg (a)} C_{a} \leq C_{n} \leq 1.$$

This proves (5). From above

$$C_{n} \leq n = \sum_{a=0}^{\deg (a)} (-1)^{a} \leq \sum_{a=0}^{\deg (i)} (-1)^{a} \leq n(-1)^{a}.$$

We now define $C_{a}$ for $\deg g < a < i-1$ to be zero. Hence our result (5) may be rewritten as

$$T^{-n}y, T^{n+y} = \sum_{a=0}^{\deg (n)} (-1)^{a}\phi^{n+a}(g)T^{-n+i-a}y.$$

where each $C_{a} \leq n(-1)^{a}$.

Apply the operator $T^{-n}$ now to the functional equation (1), term by term. We obtain

$$T^{-n}y = \frac{1}{1} \sum_{a=0}^{\deg (n)} (-1)^{a} C_{a} \phi^{n+a}(g)T^{-n+i-a}y,$$

which becomes, where $i-a = \beta$,

$$T^{-n}y = \sum_{a=0}^{\deg (n)} (-1)^{a} C_{a} \phi^{n+a}(g)T^{-n+i-a}y.$$

Set

$$d_{n}(x_{1}) = \sum_{a=0}^{\deg (n)} (-1)^{a} C_{a} \phi^{n+a}(g)T^{-n+i-a}y(x_{1}).$$

Then

$$T^{-n}y(x_{1}) = \sum_{a=0}^{\deg (n)} d_{n}(x_{1})T^{-n+i-a}y(x_{1}).$$

where

$$|d_{n}(x_{1})| = \sum_{a=0}^{\deg (n)} (-1)^{a} C_{a} \phi^{n+a}(g)T^{-n+i-a}y(x_{1}).$$

for some $K_{n}(x_{1})$. Now we may write for $n \geq 1$

$$\begin{bmatrix}
T^{-(n-1)}y(x_{1}) \\
\vdots \\
T^{-(n-i+1)}y(x_{1}) \\
T^{-(n-i)}y(x_{1}) \\
T^{-(n-i+1)}y(x_{1}) \\
\vdots \\
T^{-(n-i+1)}y(x_{1})
\end{bmatrix} = \delta_{n} \begin{bmatrix}
T^{-(n-1)}y(x_{1}) \\
\vdots \\
T^{-(n-i+1)}y(x_{1}) \\
T^{-(n-i+1)}y(x_{1}) \\
T^{-(n-i+1)}y(x_{1}) \\
\vdots \\
T^{-(n-i+1)}y(x_{1})
\end{bmatrix}$$

where $\delta_{n}$ is an $l$ by $l$ matrix with the identity (the entries are in $Q^{m}$, the $m$ by $m$ matrices over $Q$) in all of the $i-1$, $i$ positions (1 $\leq i \leq l-1$), a first row consisting of $d_{i}(x_{1})$, ..., $d_{i}(x_{1})$, and zeros elsewhere. Thus where $\delta_{n} = (\delta_{n}(l))$, we have

$$|d_{n}(x_{1})| \leq (K_{n}(x_{1})m^{1-i+1}. \tag{9}$$

We now show that an inverse matrix exists for $\delta_{n}$. This amounts to showing the existence of

$$(d_{n}(x_{1}))^{-1}.$$

By definition,

$$d_{n}(x_{1}) = C_{g}(x_{1})^{-1}.$$

Since by (XIII) $g(x_{1})$ is nonsingular, $g(x_{1}) \neq 0$ and we have $0 < \deg g(x_{1})$. Therefore $C_{g}(x_{1})$ is defined inductively by line (6). Using (6) we have

$$C_{g}(x_{1}) = C_{g}(x_{1})^{-1} = \ldots = C_{g}(x_{1}) = 1.$$
Then

\[ d_i(x) = g_i(x) \]

which as noted above is nonsingular. So \( d_i \) exists.

Since each \( \Phi_i' y_i(x) \) is rational (1 \( \leq i \leq l, j \geq 0 \)), we may choose a positive integer \( K_i(x) \) such that each \( K_i(x) d_i(x) \) is an integer. Hence each entry in \( K_i(x) d_i(x) \) belongs to \( \mathbb{Z}^m \), the \( m \) by \( m \) matrices over the integers. Set

\[
Y = \begin{bmatrix} y(x_1) \\ \vdots \\ T^{-1}y(x_1) \end{bmatrix}, \quad T^{-n}Y = \begin{bmatrix} T^{-n}y(x_1) \\ \vdots \\ T^{-n-1}y(x_1) \end{bmatrix}, \quad \frac{P}{q} = q^{-1} \begin{bmatrix} P_0 \\ \vdots \\ P_{n-1} \end{bmatrix}
\]

and

\[ d_n = (K_i(x))^{\ast} d_n d_{n-1} \ldots d_1. \]

Then we have

\[ (K_i(x))^{\ast} T^{-n}Y = d_n Y \]

where \( d_n \) exists and the entries of \( d_n \) belong to \( \mathbb{Z}^m \). We define the absolute value of a matrix with matrix entries to be the maximum of the absolute values of the entries. Now write

\[ (K_i(x))^{\ast} T^{-n}Y = d_n \left( Y - \frac{P}{q} \right) + d_n \frac{P}{q} \]

or

\[ d_n \left( Y - \frac{P}{q} \right) = - \left( d_n \frac{P}{q} - (K_i(x))^{\ast} T^{-n}Y \right), \]

which implies that

\[ d_n \left( Y - \frac{P}{q} \right) \geq \left| \frac{P}{q} \right| - \left| (K_i(x))^{\ast} T^{-n}Y \right|. \]

Since \( d_n \) exists, \( d_n P/q \) is the zero vector if and only if \( P \) is zero. We exclude this case. (There are two possibilities. If each \( T^iy(x_i) \) is zero, \( 0 \leq i \leq 1 - l \), we have nothing to prove. If some \( T^iy(x_i) \) is nonzero, it suffices to prove the theorem for nonzero \( P \).) Then \( |d_n P/q| \geq 1/q \) since each entry in \( d_n \) belongs to \( \mathbb{Z}^m \). Using also the bound for \( |T^{-n}y(x_i)| \) in (XI), we see that there exists \( K_i(x) \), a positive real number independent of \( n \), such that

\[ 1 |d_n| \max_i |T^iy(x_i) - \frac{P_i}{q}| \geq \frac{1}{q} - (K_i(x))^{\ast}. \]

Choose \( n \) sufficiently large that \( (K_i(x))^{\ast} \leq 1/2q \). It follows then that

\[ \max_i |T^iy(x_i) - \frac{P_i}{q}| \geq (2q|d_n|)^{-1}. \]

To be definite in our choice of \( n \) we take \( n \) to be the first positive integer such that \( (K_i(x))^{\ast} \leq 1/2q \). Since \( K_i(x) \geq 1 \), we must have \( n \geq 1 \) and therefore we may write

\[ (K_i(x))^{\ast} < 1/2q \leq (K_i(x)/(n-1))^{\ast}. \]

Observe that \( \log (K_i(x))^{\ast} \) is asymptotic to \( \log (K_i(x)/(n-1))^{\ast} \). Thus given \( \epsilon > 0 \), there exists \( N \) such that if \( n \geq N > 1 \) we have

\[ (K_i(x))^{\ast} \geq (K_i(x)/(n-1))^{\ast}. \]

Therefore, if \( n \geq N \)

\[ (K_i(x))^{\ast} \geq (K_i(x)/(n-1))^{\ast}, \]

If \( n \leq K_i(x) \) then (11) cannot hold. Thus \( n > K_i(x) \). For \( n > K_i(x) \) the extreme left-hand side of (11) decreases in a strictly monotone manner to zero as \( n \to \infty \). Therefore, if we restrict ourselves to values of \( q \) which satisfy \( (K_i(x))^{\ast} > 1/2q \), we must have \( n > N \) and we may use (12). We restrict ourselves to values of \( q \) in this range in what follows.

At this point an estimate of \( |d_n| \) in terms of \( n \) is needed. The desired inequality is

\[ |d_n| \leq (K_i(x))^{\ast} \]

for some positive number \( K_i(x) \) independent of \( n \). We shall demonstrate that the theorem follows from (13) and end the proof of the theorem by showing (13). We have that

\[ |d_n| \leq (K_i(x))(2q)^{\ast}, \]

so there exists \( K_i(x) \) independent of \( n \), such that

\[ |d_n| \leq K_i(x) \]

as may be seen by consideration of the orders of growth of the functions involved. Now apply (12) to obtain

\[ |d_n| \leq (K_i(x)(2q)^{\ast})^{1/(2q)^{\ast}}, \]

which we use in (10). Then

\[ \max_i |T^iy(x_i) - \frac{P_i}{q}| \geq (K_i(x)(2q)^{\ast})^{1/(2q)^{\ast}}. \]
if \(q\) is sufficiently large. We see that there exists \(K(x_1) > 0\) such that
\[
\max_{t} |T^t y(x_1) - P_t/q| \geq K(x_1) q^{-\frac{t(t+1)}{2t+3}}.
\]
Choose \(s_1\) such that \((d+2)(1+e_2) = d+2s_1\). Then
\[
\max_{t} |T^t y(x_1) - P_t/q| \geq K(x_1) q^{-\frac{t(t+1)}{2t+3}}.
\]
Corresponding to the finite number of \(q\)'s for which (12) does not hold find the minimal value of
\[
\max_{t} |T^t y(x_1) - P_t/q|
\]
and call it \(K(x_1)\). If \(K(x_1)\) is not zero let \(C(x)\) be the smaller of \(K(x)\) and \(K(x_1)\). Then
\[
\max_{t} |T^t y(x_1) - P_t/q| \geq C(x) q^{-\frac{t(t+1)}{2t+3}}
\]
for all \(P/q\) with \(q > 0\). If \(K(x_1)\) is zero, then \(Y = P/q\), for some \(q_1\) small enough that (12) and, hence, (15) does not apply. But then obviously
\[
Y = \frac{m_P}{q_0}
\]
for \(m = 1, 2, \ldots\). At some point (15) applies, and we have a contradiction. Thus (16) holds, and except for (13) the theorem has been proven.

Now to establish (13). Recall that
\[
\Delta_n = (K(x_1))^{\theta_n} \theta_{n-1} \ldots \theta_1.
\]
Hence we would be through if we could show \(\theta_n \ldots \theta_1 < (K(x_1))^{\theta_n}\), for some positive constant \(K(x_1)\) independent of \(n\). We shall establish by induction a much stronger statement that the term in the \(i\)th row and \(j\)th column of \(\theta_n \ldots \theta_1\) is less in absolute value than
\[
K(x_1) \{K(x_1) \ln m\}^{d+2i-j-1}
\]
for some positive \(K(x_1)\) independent of \(n\). This would imply that
\[
\theta_n \ldots \theta_1 \leq K(x_1) \{K(x_1) \ln m\}^{d+2i-j-1} \leq K(x_1)^{\theta_n},
\]
for some positive \(K(x_1)\) independent of \(n\). We choose \(K(x_1)\) large enough that our statement is true for \(n = 1, 2, \ldots, l-1\). Assuming the induction hypothesis for \(n \geq l-1\), we shall show it for \(n + 1\). Recall line (9) now that
\[
|\theta_n| < (K(x_1) \ln m)^{i+2i-1}
\]
where \(\theta_n = (\theta_n^B)\). Then the \(i\), \(j\)th term of \(\theta_{n+1} \ldots \theta_1\) has absolute value less than or equal to
\[
\sum_{k=1}^{l} \{K(x_1)(n+1)^{\theta_n} \ln m\}^{d+2i-j-1} \leq K(x_1) \{K(x_1) \ln m\}^{n+1}.
\]
This completes the proof (13) and of the theorem.

References


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