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Further developments in the comparative prime-number theory VI

Accumulation theorems for residue-classes representing quadratic residues mod k

by

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1. In this paper we return to the "modified Abelian means", introduced in paper [2] and further studied throughout [3] and [4]. Our present aim is to compare, in the sense of this means, the number of primes belonging to progressions $\equiv l_1 \pmod{k}$ resp. $\equiv l_2 \pmod{k}$, where *both* l_1 and l_2 are quadratic residues mod k. As before, we have to assume the Haselgrove-condition: there is an $E = E(k) > 0$ such that none of the $L(s, \chi)$-functions mod k vanishes in

$$\sigma \geqslant \tfrac{1}{2}, \quad |t| \leqslant E(k), \quad s = \sigma + it. \tag{1.1}$$

In addition to (1.1), we have to assume what we call "a finite Riemann-Piltz hypothesis": with a suitable η satisfying (¹)

$$0 < \eta < \min\left(c_1, \left(\frac{E(k)}{8\pi}\right)^2\right) \tag{1.2}$$

none of the $L(s, \chi)$-functions mod k vanishes in

$$\sigma > \frac{1}{2}, \quad |t| \leqslant \frac{3}{\sqrt{\eta}}. \tag{1.3}$$

There is no loss of generality in supposing

$$E(k) \leqslant k^{-15}; \tag{1.4}$$

this and (1.2) give automatically

$$\eta < k^{-30}. \tag{1.5}$$

With these provisions we can state the following:

(¹) c_1 and later $c_2, \ldots$ stand for positive numerical constants.

THEOREM 1. *If l_1, l_2, satisfying $(l_1, k) = (l_2, k) = 1$, $l_1 \not\equiv l_2 \pmod k$, are both quadratic residues* mod k, *and* (1.1), (1.2), (1.3), (1.4) *hold, then for every* (2)

$$T > e_2(\eta^{-3}) \tag{1.6}$$

there are x_1, x_2 and ν_1, ν_2 with

$$T^{1-\sqrt{\eta}} \leqslant x_1, x_2 \leqslant T\log T, \tag{1.7}$$

$$2\eta\log T \leqslant \nu_1, \nu_2 \leqslant 2\eta\log T + \log\log T \tag{1.8}$$

such that

$$\sum_{p\equiv l_2 (\mathrm{mod}\, k)} \log p \cdot e^{-\frac{1}{\nu_1}\log^2\frac{p}{x_1}} - \sum_{p\equiv l_1 (\mathrm{mod}\, k)} \log p \cdot e^{-\frac{1}{\nu_1}\log^2\frac{p}{x_1}} > T^{\frac{1}{2}-2\sqrt{\eta}}$$

and

$$\sum_{p\equiv l_2 (\mathrm{mod}\, k)} \log p \cdot e^{-\frac{1}{\nu_2}\log^2\frac{p}{x_2}} - \sum_{p\equiv l_1 (\mathrm{mod}\, k)} \log p \cdot e^{-\frac{1}{\nu_2}\log^2\frac{p}{x_2}} < -T^{\frac{1}{2}-2\sqrt{\eta}}.$$

Again, following the pattern of our paper [2], we can derive from Theorem 1 a direct comparison of the distribution of primes $\equiv l_1 \pmod k$ resp. $\equiv l_2 \pmod k$ in relatively short intervals.

This is given by

THEOREM 2. *Under the conditions of Theorem 1 there are numbers U_1, U_2, U_3, U_4 satisfying*

$$T^{1-4\sqrt{\eta}} \leqslant U_1 < U_2 \leqslant T^{1+4\sqrt{\eta}},$$

$$T^{1-4\sqrt{\eta}} \leqslant U_3 < U_4 \leqslant T^{1+4\sqrt{\eta}}$$

such that

$$\sum_{\substack{p\equiv l_2 (\mathrm{mod}\, k)\\ U_1\leqslant p\leqslant U_2}} 1 - \sum_{\substack{p\equiv l_1 (\mathrm{mod}\, k)\\ U_1\leqslant p\leqslant U_2}} 1 > T^{\frac{1}{2}-3\sqrt{\eta}}$$

and

$$\sum_{\substack{p\equiv l_2 (\mathrm{mod}\, k)\\ U_3\leqslant p\leqslant U_4}} 1 - \sum_{\substack{p\equiv l_1 (\mathrm{mod}\, k)\\ U_3\leqslant p\leqslant U_4}} 1 < -T^{\frac{1}{2}-3\sqrt{\eta}}.$$

We wish to emphasize once more that we have not been able to prove a similar result in case where *exactly one* of the l_j's is a quadratic residue and none of l_1, l_2 is $\equiv 1 \pmod k$.

The simplest case in which our present methods fail is that of $k = 5$, $l_1 = 2$ (or 3), $l_2 = 4$.

(2) $e_2(\tau)$ stands for $\exp\{\exp\tau\}$.

2. Proof of Theorem 1 will be based on a number of lemmas. The first of them is a combination of Lemmas 1 and 6 of our paper [4].

LEMMA 1. *Under the conditions of Theorem 1 there exists a prime $P_0 \equiv l_2 \pmod k$, satisfying*

$$c_2\varphi(k)^{5/2} \leqslant P_0 \leqslant c_3\varphi(k)^{5/2}, \tag{2.1}$$

such that

$$\frac{1}{\varphi(k)}\mathrm{Re}\sum_{\chi}\{\bar\chi(l_1)-\bar\chi(l_2)\}\sum_{\varrho(\chi)}{}' e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)} > c_4P_0\log^2P_0, \tag{2.2}$$

where

$$\xi_0 = 2P_0^2\log^3P_0, \quad r_0 = P_0^{-2}\log^{-2}P_0, \tag{2.3}$$

and $\sum'_{\varrho(\chi)}$ means that the summation is to be extended over the ϱ's with $|\mathrm{Im}\,\varrho| \leqslant c_5k^5$ (3).

Before formulating Lemma 2, we wish to explain our notation. We consider n complex numbers $z_1, z_2, \ldots, z_n$ with

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$$

and such that

$$\varkappa \leqslant |\arg z_j| \leqslant \pi \quad (j = 1, 2, \ldots, n) \tag{2.4}$$

is true with a certain $0 < \varkappa \leqslant \pi/2$; m being a non-negative integer, we suppose that for an h

$$|z_h| > \frac{4n}{m+n(3+\pi/\varkappa)}. \tag{2.5}$$

Further, we fix an h_1 with

$$|z_{h_1}| < |z_h| - \frac{2n}{m+n(3+\pi/\varkappa)} \tag{2.6}$$

and, given numbers $b_1, b_2, \ldots, b_n$, define

$$B = \min_{h\leqslant w<h_1} \mathrm{Re}\sum_{j=1}^{w} b_j; \tag{2.7}$$

(3) c_5 can be thought of as being as large as we please, however, fixed; clearly, making c_1 in (1.2) sufficiently small (in dependence of c_5), we conclude from (1.2)-(1.3)-(1.4) that $|\mathrm{Im}\,\varrho| < c_5k^5$ implies $\mathrm{Re}\,\varrho = \frac{1}{2}$. c_5 will be properly chosen in section 7.

if there is no h_1 satisfying (2.6), we put

$$B = \min_{h\leqslant w\leqslant n} \operatorname{Re} \sum_{j=1}^{w} b_j. \tag{2.8}$$

We assert (proof to be found in [1]):

Lemma 2. *If $B>0$, there exist integer ν_1, ν_2 with*

$$m+1 \leqslant \nu_1, \nu_2 \leqslant m+n(3+\pi/\varkappa) \tag{2.9}$$

such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant \frac{B}{2n+1}\left\{\frac{n}{24(m+n(3+\pi/\varkappa))}\right\}^{2n}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)} \tag{2.10}$$

and

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -\frac{B}{2n+1}\left\{\frac{n}{24(m+n(3+\pi/\varkappa))}\right\}^{2n}\left(\frac{z_h}{2}\right)^{m+n(3+\pi/\varkappa)}. \tag{2.11}$$

Next we have (proof to be found in [2]):

Lemma 3. *If $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2 \ldots$ are real numbers with*

$$|\alpha_\nu| \geqslant U \; (>0),$$

further, if with a $\gamma>1$

$$\sum_\nu \frac{1}{1+|\alpha_\nu|^\gamma} \leqslant V \; (<\infty),$$

then every real interval of length $>1/U$ contains a ξ-value such that

$$\|\alpha_\nu \xi+\beta_\nu\| \geqslant \frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma}$$

for every ν; here, as usually, $\|x\|$ stands for the distance of x from the nearest integer.

We shall need two more lemmas.

Lemma 4. *There exists a broken line W in the vertical strip $\frac{1}{5}\leqslant\sigma\leqslant\frac{1}{4}$, consisting of horizontal and vertical segments alternately, each horizontal strip of width 1 containing at most one horizontal segment, such that the inequality*

$$\left|\frac{L'}{L}(s,\chi)\right| < c_6\varphi(k)\log^2 k(2+|t|)$$

holds along W for every L-function mod k.

This lemma can be proved following mutatis mutandis the Appendix III of [6]. The last lemma, which we need, is a simple consequence of a theorem of Siegel [5].

Lemma 5. *Any L-function, modulo any $k\geqslant 1$ has at least one zero in*

$$\tfrac{1}{2}\leqslant\sigma<1, \qquad \tau\leqslant t\leqslant\tau+c_7$$

where real τ is arbitrary and c_7 numerical.

3. We pass over to the proof of Theorem 1. Let $\lambda_1, \lambda_2, \ldots, \lambda_g$ resp. $\mu_1, \mu_2, \ldots, \mu_g$ denote the solutions of $x^2\equiv l_1 \pmod k$ resp. $x^2\equiv l_2 \pmod k$. We consider the function

$$f(s) = \frac{1}{\varphi(k)}\left\{\sum_\chi \big(\bar\chi(l_1)-\bar\chi(l_2)\big)\frac{L'}{L}(s,\chi) - \sum_{j=1}^{g}\sum_\chi \big(\bar\chi(\lambda_j)-\bar\chi(\mu_j)\big)\frac{L'}{L}(2s,\chi)\right\}.$$

Setting

$$\varepsilon_k(n, l_2, l_1) = \begin{cases} -1 & \text{if } n\equiv l_1 \pmod k,\\ +1 & \text{if } n\equiv l_2 \pmod k,\\ 0 & \text{otherwise,}\end{cases}$$

we can write (p standing for primes)

$$f(s) = \sum_p \frac{\log p}{p^s}\varepsilon_k(p, l_2, l_1) + \sum_p \frac{\log p}{p^{2s}}\varepsilon_k(p^2, l_2, l_1) + $$
$$+f_1(s) - \sum_p \frac{\log p}{p^{2s}}\sum_{j=1}^{g}\varepsilon_k(p,\mu_j,\lambda_j)+f_2(s)$$
$$= \sum_p \frac{\log p}{p^s}\varepsilon_k(p, l_2, l_1)+f_3(s),$$

where $f_1(s), f_2(s), f_3(s)$ are regular and bounded in $\sigma\geqslant\frac{2}{5}$. In particular,

$$\left|f(s)-\sum_p\frac{\log p}{p^s}\varepsilon_k(p,l_2,l_1)\right|\leqslant c_8, \quad \text{if} \quad \sigma\geqslant\frac{2}{5}. \tag{3.1}$$

4. Let us apply Lemma 3 for $\alpha_\nu = \frac{1}{4\pi}t_\varrho$ and $\alpha_\nu = \frac{1}{8\pi}t_\varrho$, $\beta_\nu = \frac{1}{4\pi}\sigma_\varrho t_\varrho$ and $\beta_\nu = \frac{1}{16\pi}\sigma_\varrho t_\varrho$, where $\varrho = \sigma_\varrho+it_\varrho$ runs through all non-trivial L-zeros mod k, further

$$\gamma = \frac{11}{10}, \qquad U = \frac{E(k)}{8\pi}.$$

Since we can evidently put

$$V = c_9 k\log k,$$

Lemma 3 insures the existence of a ξ with

$$\eta^{-1}-\eta^{-1/2}\leqslant \xi \leqslant \eta^{-1} \tag{4.1}$$

(owing to (1.2), the condition $\eta^{-1/2}>1/U$ of Lemma 3 is satisfied) such that for all ϱ's

$$\left\|\frac{1}{2\pi}\cdot\frac{1}{2}(\xi t_\varrho+\sigma_\varrho t_\varrho)\right\|\geqslant\frac{c_{10}}{k\log k}\cdot\frac{1}{1+|t_\varrho|^{11/10}} \tag{4.2}$$

and

$$\left\|\frac{1}{2\pi}\cdot\frac{1}{4}\left(\xi t_\varrho+\frac{1}{2}\sigma_\varrho t_\varrho\right)\right\|\geqslant\frac{c_{10}}{k\log k}\cdot\frac{1}{1+|t_\varrho|^{11/10}}. \tag{4.3}$$

We choose, further,

$$m=2\eta\log T, \tag{4.4}$$

and restrict integer r by

$$m\leqslant r\leqslant m+\eta^{-6/5}. \tag{4.5}$$

Next, we consider the integral

$$H(r)=\frac{1}{2\pi i}\int_{(2)} e^{\frac{r_0}{4}(s+\xi_0)^2+\frac{r}{4}(s+\xi)^2} f(s)\,ds. \tag{4.6}$$

Owing to (3.1) and the integral formula

$$\int_{(2)} e^{s^2y+s\delta}\,ds=i\sqrt{\frac{\pi}{y}}\,e^{-\frac{1}{4y}\delta^2}\qquad (y>0),$$

we get

$$H(r)=\frac{1}{2\pi i}\int_{(2)} e^{\frac{r_0}{4}(s+\xi_0)^2+\frac{r}{4}(s+\xi)^2}\sum_p\frac{\log p}{p^s}\varepsilon_k(p,l_2,l_1)\,ds+ \tag{4.7}$$

$$+\frac{1}{2\pi i}\int_{(2/5)} e^{\frac{r_0}{4}(s+\xi_0)^2+\frac{r}{4}(s+\xi)^2}O(1)\,ds$$

$$=\frac{e^{(r_0\xi_0^2+r\xi^2)/4}}{\sqrt{\pi(r_0+r)}}\sum_p\varepsilon_k(p,l_2,l_1)\log p\cdot e^{-(\log p-(r_0\xi_0+r\xi)/2)^2/(r_0+r)}+$$

$$+O\left(e^{\frac{r_0}{4}\left(\xi_0+\frac{2}{5}\right)^2+\frac{r}{4}\left(\xi+\frac{2}{5}\right)^2}\right).$$

5. We get another expression for $H(r)$ on shifting the line of integration to the polygonal line W (resp. $W/2$) of Lemma 4. The main term is a sum of residues

$$\frac{1}{\varphi(k)}\sum_\chi\{\bar\chi(l_1)-\bar\chi(l_2)\}\sum_{\substack{\varrho=\varrho(\chi)\\ (\text{right to } W)}}{}' e^{\frac{r_0}{4}(\varrho+\xi_0)^2+\frac{r}{4}(\varrho+\xi)^2}+$$

$$+\frac{1}{\varphi(k)}\sum_{j=1}^{g}\sum_\chi\{\bar\chi(\lambda_j)-\bar\chi(\mu_j)\}\sum_{\varrho\text{ right to } W}{}' e^{\frac{r_0}{4}\left(\frac{\varrho}{2}+\xi_0\right)^2+\frac{r}{4}\left(\frac{\varrho}{2}+\xi\right)^2},$$

and the remainder can be estimated, using Lemma 4, by

$$c_{11}k\log^2 k\,e^{\frac{r_0}{4}\left(\xi_0+\frac{1}{4}\right)^2+\frac{r}{4}\left(\xi+\frac{1}{4}\right)^2}.$$

Combining this with (4.7), and dividing by

$$\frac{e^{(r_0\xi_0^2+r\xi^2)/4}}{\sqrt{\pi(r_0+r)}},$$

we obtain

$$\left|\sum_p\varepsilon_k(p,l_2,l_1)\log p\cdot e^{-\frac{(\log p-(r_0\xi_0+r\xi)/2)^2}{r_0+r}}-\right. \tag{5.1}$$

$$-\sqrt{\pi(r_0+r)}\left\{\sum_\chi\frac{\bar\chi(l_1)-\bar\chi(l_2)}{\varphi(k)}\sum_{\varrho\text{ right to } W}{}' e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)+\frac{r}{4}(\varrho^2+2\xi\varrho)}-\right.$$

$$\left.\left.-\sum_{j=1}^{g}\sum_\chi\frac{\bar\chi(\lambda_j)-\bar\chi(\mu_j)}{\varphi(k)}\sum_{\varrho\text{ right to } W}{}' e^{\frac{r_0}{4}(\varrho^2/4+\xi_0\varrho)+\frac{r}{4}(\varrho^2/4+\xi\varrho)}\right\}\right|$$

$$\leqslant c_{12}(k\log^2 k)\sqrt{r_0+r}\,e^{\frac{r_0}{4}\left(\frac{4}{5}\xi_0+\frac{4}{25}\right)+\frac{r}{4}\left(\frac{4}{5}\xi+\frac{4}{25}\right)}.$$

We can easily estimate the contribution of ϱ's with $|t_\varrho|>3/\sqrt{\eta}$ to the sums $\sum'$ in (5.1). Using (1.2), (1.4), (1.5), (2.1), (2.3), (4.1), (4.4) and (4.5), we get for it the upper bound

$$c_{13}\sqrt{r_0+r}\left\{e^{\frac{r_0}{4}(2\xi_0+1)+\frac{r}{4}(2\xi+1)}\sum_{n\geqslant 3/\sqrt{\eta}-1} e^{-\frac{r_0+r}{4}n^2}\log kn+\right.$$

$$\left.+ke^{\frac{r_0}{4}\left(\xi_0+\frac{1}{4}\right)+\frac{r}{4}\left(\xi+\frac{1}{4}\right)}\sum_{n\geqslant 3/\sqrt{\eta}-1} e^{-\frac{r_0+r}{4}\cdot\frac{n^2}{4}}\log kn\right\}$$

$$<c_{14}\sqrt{r_0+r}\left\{e^{\frac{r_0}{4}(2\xi_0+1)+\frac{r}{4}(2\xi+1)}\left(\log\frac{k}{\sqrt{\eta}}\right)e^{-\frac{r_0+r}{4}\cdot\frac{8}{\eta}}+\right.$$

$$\left.+ke^{\frac{r_0}{4}\left(\xi_0+\frac{1}{4}\right)+\frac{r}{4}\left(\xi+\frac{1}{4}\right)}\left(\log\frac{k}{\sqrt{\eta}}\right)e^{-\frac{r_0+r}{2}\cdot\frac{1}{\eta}}\right\}.$$

Since

$$e^{-(r_0+r)/\eta} < e^{-r/\eta} \leqslant e^{-2\log T} = T^{-2},$$

we estimate it further by

$$c_{15}(\log T)^{1/2}\big(P_0 T^{1+\eta/2} e^{\eta^{-11/5}}(\log\log T)T^{-4} + $$
$$+P_0^{1/2}(\log T)T^{1/2+\eta/8}e^{\eta^{-11/5}}(\log\log T)T^{-1}\big) < c_{16}.$$

A similar estimation gives the bound of $T^{2/5+\eta}$ for the error-term in (5.1). Together, we obtain

$$(5.2)\qquad \Big|\sum_p \varepsilon_k(p, l_2, l_1)\log p\cdot e^{-\frac{(\log p-(r_0\xi_0+r\xi)/2)^2}{r_0+r}}$$
$$-\sqrt{\pi(r_0+r)}\Big\{\sum_\chi \frac{\bar\chi(l_1)-\bar\chi(l_2)}{\varphi(k)} \sum_{\substack{\varrho(\chi)\\ \operatorname{Im}\varrho\leqslant 3\eta^{-1/2}}} e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)+\frac{r}{4}(\varrho^2+2\xi\varrho)} -$$
$$-\sum_{j=1}^{g}\sum_\chi \frac{\bar\chi(\lambda_j)-\bar\chi(\mu_j)}{\varphi(k)} \sum_{\substack{\varrho(\chi)\\ \operatorname{Im}\varrho\leqslant 3\eta^{-1/2}}} e^{\frac{r_0}{4}\left(\frac{\varrho^2}{4}+\xi_0\varrho\right)+\frac{r}{4}\left(\frac{\varrho^2}{4}+\xi\varrho\right)}\Big\}\Big| \leqslant c_{17}T^{\frac{2}{5}+\eta}.$$

6. Let $\varrho_1 = u_1+iv_1$ be one of the non-trivial $L(s,\chi)$-zeros, χ modulo k and $\chi(l_1)\neq\chi(l_2)$, such that $|e^{\frac{1}{4}(\varrho^2+2\xi\varrho)}|$, considered for all $\varrho=\sigma_\varrho+it_\varrho$ with $|t_\varrho|\leqslant 3\eta^{-1/2}$, attains maximum at $\varrho=\varrho_1$. We put (5.2) in the form

$$(6.1)\qquad \Big|\sum_p \varepsilon_k(p, l_2, l_1)\log p\cdot e^{-\frac{(\log p-(r_0\xi_0+r\xi)/2)^2}{r_0+r}}$$
$$-\sqrt{\pi(r_0+r)}\,\big|e^{\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}\big|^r \operatorname{Re}\Big\{\sum_\chi \frac{\bar\chi(l_1)-\bar\chi(l_2)}{\varphi(k)}\times$$
$$\times\sum_{\substack{\varrho(\chi)\\ |t_\varrho|\leqslant 3\eta^{-1/2}}} e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)}\big(e^{\frac{1}{4}(\varrho^2+2\xi\varrho)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}\big)^r$$
$$-\sum_{j=1}^{g}\sum_\chi \frac{\bar\chi(\lambda_j)-\bar\chi(\mu_j)}{\varphi(k)} \sum_{\substack{\varrho(\chi)\\ |t_\varrho|\leqslant 3\eta^{-1/2}}} e^{\frac{r_0}{4}\left(\frac{\varrho^2}{4}+\xi_0\varrho\right)}\big(e^{\frac{1}{4}\left(\frac{\varrho^2}{4}+\xi\varrho\right)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}\big)^r\Big\}\Big| \leqslant c_{17}T^{\frac{2}{5}+\eta}.$$

In order to apply Lemma 2, we have to introduce numbers z_j and b_j. With ϱ's occurring in (6.1), we shall call numbers

$$z_{j'} = e^{\frac{1}{4}(\varrho^2+2\xi\varrho)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}$$

z_j-numbers of the first class, and numbers

$$z_{j''} = e^{\frac{1}{4}(\varrho^2/4+\xi\varrho)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}$$

z_j-numbers of the second class; accordingly,

$$b_{j'} = \frac{\bar\chi(l_1)-\bar\chi(l_2)}{\varphi(k)}\, e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)}$$

and

$$b_{j''} = \frac{\bar\chi(\mu_i)-\bar\chi(\lambda_i)}{\varphi(k)}\, e^{\frac{r_0}{4}(\varrho^2/4+\xi_0\varrho)}$$

will be called b_j-numbers of the first resp. second class. It is clear, after (4.2), (4.3), and (1.5), that for all z_j's

$$(6.2)\qquad |\arg z_j| > \varkappa \overset{\text{def}}{=} \eta^{3/5}.$$

In addition to ϱ_1, we will introduce two more special ϱ's. We define $\varrho_2 = u_2+iv_2$ as the $\varrho(\chi)\bmod k$, with $\chi(l_1)\neq\chi(l_2)$, for which v_2 is maximal and $\leqslant c_5k^5$. Hence c_5 is the constant occurring in Lemma 1; obviously, as pointed out in the footnote in Lemma 1, we have $u_2=\frac{1}{2}$. Then, similarly, $\varrho_3 = u_3+iv_3$ is defined as the $\varrho(\chi)\bmod k$, with $\chi(l_1)\neq\chi(l_2)$, for which v_3 is minimal and $\geqslant c_5k^5+1$. Again, noting Lemma 5, we may assume $u_3=\frac{1}{2}$.

Making c_5 sufficiently large, we may assert

$$(6.3)\qquad 2\leqslant v_2\leqslant c_5k^5$$

and

$$(6.4)\qquad c_5k^5+1\leqslant v_3\leqslant 2c_5k^5.$$

Next we define h and h_1 by putting

$$z_h = e^{\frac{1}{4}(\varrho_2^2+2\xi\varrho_2)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)},$$
$$z_{h_1} = e^{\frac{1}{4}(\varrho_3^2+2\xi\varrho_3)-\operatorname{Re}\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}.$$

We assert that all z_j's of the second class are absolutely $\leqslant |z_{h_1}|$. This, however, reduces to the inequality

$$(6.5)\qquad |e^{\varrho^2/16+\xi\varrho/4}| \leqslant |e^{\varrho_3^2/4+\xi\varrho_3/2}|$$

for $|\operatorname{Im}\varrho|\leqslant 3\eta^{-1/2}$.

In fact, it is enough to show

$$e^{\xi} > e^{2v_3^2};$$

this inequality, however, follows from (1.2), (1.5), (4.1) and (6.4).

In our notation (6.1) may be simply put as follows

$$(6.8)\qquad \Big|\sum_p \varepsilon_k(p,l_2,l_1)\log p\cdot e^{-\frac{(\log p-(r_0\xi_0+r\xi)/2)^2}{r_0+r}} - \sqrt{\pi(r_0+r)}\,|e^{\frac14(\varrho_1^2+2\xi\varrho_1)}|^r\,\mathrm{Re}\sum_{j=1}^n b_j z_j^r\Big| \leqslant c_{17}T^{\frac25+\eta},$$

and we will proceed to estimate

$$(6.9)\qquad Z(r)=\mathrm{Re}\sum_{j=1}^n b_j z_j^r.$$

First of all, we observe that

$$(6.10)\qquad n\leqslant c_{20}k\eta^{-1/2}\log(k\eta^{-1})<\eta^{-4/7}.$$

Further, (4.5), (6.2), (6.10) imply that we may restrict r to

$$(6.11)\qquad m+1\leqslant r\leqslant m+(3+\pi/\varkappa)n.$$

7. In order to apply Lemma 2, and get reasonable bounds for $Z(r)$, we have to check (2.5) and (2.6). Since

$$u_1=u_2=u_3=\tfrac12,$$

$$|z_h|=e^{\frac14(v_1^2-v_2^2)}>e^{-c_5^2k^{10}},$$

and also, using (4.4) and (6.10),

$$\frac{4n}{m+n(3+\pi/\varkappa)}<\frac{4\eta^{-4/7}}{2\eta\log T}<\frac{1}{\eta^2\log T}.$$

Making c_1 in (1.2) small enough and using (1.5),

$$e^{-c_5^2k^{10}}>e^{-\eta^{-1}};$$

further, by (1.6),

$$e^{-\eta^{-1}}>\eta^{-2}(\log T)^{-1}$$

and (2.5) follows. As to (2.6), we have

$$|z_h|-|z_{h_1}|=e^{\frac14(v_1^2-v_2^2)}-e^{\frac14(v_1^2-v_3^2)}>e^{-v_2^2}(1-e^{v_2^2-v_3^2})>e^{-v_2^2}(1-e^{-1})>\tfrac12 e^{-v_2^2},$$

which clearly reduces our problem to the previous one. We also observe that for $h\leqslant w<h_1$ $\mathrm{Re}\sum_{j=1}^{w}b_j$ differs from the series in (2.2) by at most

$$(7.1)\qquad \frac{1}{\varphi(k)}\sum_\chi|\bar\chi(l_1)-\bar\chi(l_2)|\sum_{\mathrm{Im}\varrho\geqslant v_2}|e^{\frac{r_0}{4}(\varrho^2+2\xi_0\varrho)}|;$$

here we made use of our remark that all z_j's of the second class are absolutely $\leqslant|z_{h_1}|$. The sum in (7.1) is easily estimated by

$$c_{21}P_0\log(c_5k^5)e^{-r_0c_5^2k^{10}/20}$$

and this in turn is made

$$<\tfrac12 c_4P_0\log^2P_0,$$

on choosing c_5 large enough. It follows that

$$(7.2)\qquad B\overset{\mathrm{def}}{=}\min_{h\leqslant w<h_1}\mathrm{Re}\sum_{j=1}^{w}b_j>\tfrac12 c_4P_0\log^2P_0.$$

8. We proceed to estimate the sum $Z(r)$ in (6.9). Lemma 2 says that for a suitable $r=r_1$, satisfying (6.11), we have

$$Z(r_1)>\frac{B}{2n+1}\Big\{\frac{n}{24(m+n(3+\pi/\varkappa))}\Big\}^{2n}\Big(\frac{|z_h|}{2}\Big)^{m+n(3+\pi/\varkappa)}.$$

It follows that

$$(8.1)\qquad |e^{\frac14(\varrho_1^2+2\xi\varrho_1)}|^{r_1}Z(r_1)>\frac{B}{2n+1}\Big\{\frac{n}{24(m+n(3+\pi/\varkappa))}\Big\}^{2n}\times$$
$$\times 2^{-m-n\left(3+\frac{\pi}{\varkappa}\right)}e^{\frac14\mathrm{Re}(\varrho_2^2+2\xi\varrho_2)\left(m+n\left(3+\frac{\pi}{\varkappa}\right)\right)}e^{\frac14\mathrm{Re}(\varrho_1^2+2\xi\varrho_1)\left(r_1-m-n\left(3+\frac{\pi}{\varkappa}\right)\right)}.$$

Since, by (4.1)

$$\mathrm{Re}(\varrho_1^2+2\xi\varrho_1)=\tfrac14-v_1^2+\xi<\tfrac14+\eta^{-1}$$

and $r_1-m-n(3+\pi/\varkappa)$ is non-positive, we can estimate from below the last term in (8.1) by

$$e^{-\left(\frac1{16}+\frac14\eta^{-1}\right)n\left(3+\frac{\pi}{\varkappa}\right)}>e^{-\eta^{-3}}.$$

Further, by (1.5), (4.1), (4.4), (6.3),

$$e^{\frac14\mathrm{Re}(\varrho_2^2+2\xi\varrho_2)(m+n(3+\pi/\varkappa))}>e^{\frac14\left(\frac14-v_2^2+\xi\right)(m+n(3+\pi/\varkappa))}$$
$$>e^{\frac14(\eta^{-1}-\eta^{-1/2}-c_{22}k^{10})m}>e^{\frac14\left(\eta^{-1}-2\eta^{-1/2}\right)2\eta\log T}=T^{1/2-\sqrt{\eta}}.$$

Since roughly

$$\Big(\frac{n}{24(m+n(3+\pi/\varkappa))}\Big)^{2n}>e^{-\frac1\eta\log\log T}>T^{-\sqrt{\eta}/3}$$

and

$$2^{-m-n(3+\pi/\varkappa)}>T^{-2\eta}e^{-\eta^{-6/5}}>T^{-\sqrt{\eta}/3},$$

we get finally from (8.1), noting also (7.2),

$$|e^{\frac{1}{4}(\varrho_1^2+2\xi\varrho_1)}|^{r_1} Z(r_1) > T^{\frac{1}{2}-\frac{7}{4}\sqrt{\eta}}.$$

Setting

$$\nu_1 = r_0 + r_1, \qquad x_1 = e^{(r_0\xi_0+r_1\xi)/2}, \tag{8.2}$$

we deduce thus from (6.8)

$$\sum_p \varepsilon_k(p, l_2, l_1)\log p \cdot e^{-\frac{1}{\nu_1}\left(\log\frac{p}{x_1}\right)^2} > T^{\frac{1}{2}-2\sqrt{\eta}},$$

i.e. the first statement of Theorem 1. The second statement follows mutatis mutandis.

9. To complete the proof, it remains to show (1.7) and (1.8). By (8.2), (2.1), (2.3), (4.1), (4.4), (4.5)

$$x_1 \leqslant c_3 k^{5/2} e^{\eta^{-11/5}} T$$

and

$$x_1 \geqslant T^{1-\sqrt{\eta}},$$

so that (1.7) follows in view of (1.5) and (1.6). As to (1.8), (8.2), (4.4), (4.5) yield

$$\nu_1 \leqslant 1 + 2\eta\log T + \eta^{-6/5}$$

and

$$\nu_1 \geqslant 2\eta\log T,$$

which give the result.

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A refinement of a theorem of Schur on primes in arithmetic progressions II

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I. Schur [6] gave purely algebraic proofs of the existence of infinitely many primes in the following special arithmetic progressions:

$$2^\nu z + 2^{\nu-1} \pm 1 \quad \text{where} \quad \nu \geqslant 1,$$

$$8nz+2n+1, \quad 8nz+4n+1, \quad 8nz+6n+1,$$

where n is an odd square-free integer > 0 and

$$p^\nu nz + l_\nu,$$

where

$$l_\nu \equiv \begin{cases} 1 \bmod n, \\ -1 \bmod p^\nu \end{cases}$$

and p is an odd prime.

In the last case Schur assumed the existence of an integer c such that $\left(\frac{F_n(c)}{p}\right) = -1$, where F_n is the nth cyclotomic polynomial.

A. S. Bang [1] gave proofs similar to those of Schur for the existence of infinitely many primes in the following progressions:

$$4p^n z + 2p^n + 1, \qquad p \equiv 3 \bmod 4,$$

$$6p^{2n+1} z + 2p^{2n+1} + 1, \qquad p \equiv 2 \bmod 3,$$

$$6p^{2n} z + 4p^{2n} + 1, \qquad p \equiv 2 \bmod 3.$$

The main aim of the present paper is to prove on the same way a theorem which comprises all the above results as special cases and covers several new cases, e.g. the progressions:

$$48x+7, \quad 48x+25, \quad 48x+31, \quad 105x+64, \quad 105x+71, \quad 105x+76\,(^1).$$

(1) The last three progressions correspond to the case $p_\nu nz + l_\nu$ considered by Schur. However, it is impossible to find here an integer c satisfying $\left(\frac{F_n(c)}{p}\right) = -1$.