

## On the approximation of real numbers by roots of integers

by

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*To J. G. van der Corput  
on his 75th birthday*

Let  $A$  be the set of all numbers  $a > 1$  for which none of the powers  $a, a^2, a^3, \dots$  is an integer. For every positive integer  $n$  there exists at least one integer  $g_n$  which is closest to  $a^n$  and thus satisfies the inequality

$$|a^n - g_n| \leq 1/2.$$

We are here concerned with the lower limit

$$P(a) = \liminf_{n \rightarrow \infty} |a^n - g_n|^{1/n}$$

which trivially has the property

$$0 \leq P(a) \leq 1 \quad \text{for all } a \in A.$$

A few years ago, one of us (Mahler, 1957) proved that

$$P(a) = 1 \text{ if } a \text{ is any rational number in } A.$$

One can further show that there are irrational algebraic numbers  $a \in A$  for which  $P(a) = 1$ ; e.g. the number  $\frac{1}{2}(2 + \sqrt{3} + \sqrt{3 + 4\sqrt{3}})$  is of this kind. It is also well known that there exist algebraic numbers  $a$  in  $A$  for which

$$0 < P(a) < 1;$$

e.g. the number  $\frac{1}{2}(1 + \sqrt{5})$  has this property.

In the present note, the following three results will be proved.

- (a) *If  $P(a) = 0$ , then  $a$  is transcendental.*
- (b) *In every neighbourhood of every number  $x > 1$  there exist non-countably many  $a \in A$  for which  $P(a) = 0$ .*
- (c) *For almost all  $a$  in  $A$ ,  $P(a) = 1$ ; thus there are transcendental numbers with this property.*

Proof of (a). Let  $a$  be an algebraic number of degree  $m$ , and let

$$f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m = a_0 \prod_{\mu=0}^{m-1} (x - \alpha^{(\mu)})$$

be a primitive irreducible polynomial with integral coefficients of which  $a = \alpha^{(0)}$  is a zero. For each  $n$  the product

$$a_0^n \prod_{\mu=0}^{m-1} (\alpha^{(\mu)n} - g_n) = p_n$$

say, is an integer. This integer is distinct from zero because  $a^n \neq g_n$  and hence also

$$\alpha^{(\mu)n} - g_n \neq 0 \quad (\mu = 0, 1, \dots, m-1).$$

Therefore

$$|p_n| \geq 1.$$

Next  $a > 1$ , hence  $g_n \geq 1$ , and therefore

$$|\alpha^{(\mu)n} - g_n| \leq g_n (|\alpha^{(\mu)}| + 1)^n,$$

so that

$$|a_0|^n \prod_{\mu=1}^{m-1} |\alpha^{(\mu)n} - g_n| \leq g_n^{m-1} \left\{ |a_0| \prod_{\mu=1}^{m-1} (|\alpha^{(\mu)}| + 1) \right\}^n.$$

Here

$$g_n \leq a^n + \frac{1}{2} < (2a)^n$$

and so finally

$$1 \leq |p_n| \leq |a^n - g_n| \left\{ (2a)^{m-1} |a_0| \prod_{\mu=1}^{m-1} (|\alpha^{(\mu)}| + 1) \right\}^n.$$

There exists then a constant  $c > 1$  depending only on  $a$  such that

$$|a^n - g_n| \geq c^{-n} \quad \text{for all } n,$$

proving that  $P(a) > 0$ . Conversely, if  $P(a) = 0$ , then  $a$  necessarily is transcendental.

Proof of (b). Let  $x > 1$ ,  $0 < \varepsilon < \frac{1}{2}(x-1)$ . We show that there is a sequence of positive integers

$$1 = n_0 < n_1 < n_2 < \dots < n_k < \dots$$

depending on  $x$ , but not on  $\varepsilon$ , with the following property:

Given any sequence  $\{\eta_k\}$  with  $\eta_k$  equal to either 0 or 1 (briefly, an  $\eta$ -sequence), there is a real number  $a$ , where

$$(1) \quad 0 < x - a < \varepsilon,$$

such that

$$(2) \quad \lim_{k \rightarrow \infty} \{ |\alpha^{n_k} - g_{n_k}|^{1/n_k} - \eta_k \} = 0.$$

Clearly if  $\eta_k = 0$  for infinitely many values of  $k$ , then  $P(a) = 0$ ; and if  $\{\eta_k\}, \{\eta'_k\}$  are two essentially different  $\eta$ -sequences, i.e. such that  $\eta_k \neq \eta'_k$  for infinitely many  $k$ , then the corresponding real numbers  $a$  and  $a'$  are distinct. Since there are non-countably many essentially different  $\eta$ -sequences, we obtain non-countably many  $a$  with  $P(a) = 0$  in the (left)  $\varepsilon$ -neighbourhood of  $x$ , hence also non-countably many  $a \in A$  with this property.

For the proof take any increasing sequence  $n_k$  which satisfies the condition

$$(3) \quad -\frac{1}{n_k} \log \left( 1 - 2 \left( \frac{1+x}{2} \right)^{-n_k} \right) \leq \frac{x-1}{x+1} 2^{-kn_k-1} x^{-n_k-1} / n_{k-1} \quad \text{for } k > 0.$$

The condition is clearly satisfied if  $n_k$  increases sufficiently rapidly.

Let  $\{\eta_k\}$  be an arbitrary  $\eta$ -sequence and  $\varepsilon > 0$ ; we may assume  $\varepsilon < \frac{1}{2}$ . Determine  $K > 1$  so that

$$(4) \quad \prod_{k=K}^{\infty} (1 - 2^{-k}) > 1 - \frac{\varepsilon}{x}.$$

We define now  $x = x_0 \geq x_1 \geq x_2 \geq \dots$  as follows: For  $0 \leq k < K$  we set  $x_k = x$ . Suppose that for some  $k \geq K$ ,  $x_{k-1}$  has already been determined so that

$$(5) \quad \frac{1}{2}(1+x) + 2^{-k}(x-1) \leq x_{k-1} \leq x.$$

Set

$$(6) \quad x_{k-1}^{n_k} = a_k + \lambda_k, \quad a_k \text{ integer, } 1 \leq \lambda_k < 2.$$

We then define

$$(7) \quad x_k = (a_k + \frac{1}{2} \eta_k + 2^{-kn_k})^{1/n_k}.$$

Clearly  $\frac{1}{2} \eta_k + 2^{-kn_k} \leq 1 \leq \lambda_k$ ,  $x_k \leq x_{k-1}$ , and so the second half of inequality (5) is satisfied. We now show that also

$$x_k \geq \frac{1}{2}(1+x) + 2^{-k-1}(x-1).$$

Set

$$(8) \quad a_k = x_{k-1} \left( 1 - \frac{\delta_k}{n_k} \right).$$

By (6) and (7) we have for  $k \geq K$

$$a_k + \frac{1}{2} \eta_k + 2^{-kn_k} = (a_k + \lambda_k) \left( 1 - \frac{\delta_k}{n_k} \right)^{n_k},$$

or setting

$$\zeta_k = \lambda_k - \frac{1}{2} \eta_k - 2^{-kn_k}, \quad 0 \leq \zeta_k < 2,$$

$$1 - \zeta_k \alpha_k^{-n_k} = \left(1 - \frac{\delta_k}{n_k}\right)^{n_k} < e^{-\delta_k}.$$

By (5) therefore

$$(9) \quad 0 \leq \delta_k \leq -\log \left(1 - \zeta_k \left(\frac{1+x}{2}\right)^{-n_k}\right) < -\log \left(1 - 2 \left(\frac{1+x}{2}\right)^{-n_k}\right),$$

hence by (5) and (8),

$$\begin{aligned} \alpha_k &\geq \frac{1}{2}(x+1) + 2^{-k}(x-1) \left(1 + \frac{1}{n_k} \log \left(1 - 2 \left(\frac{1+x}{2}\right)^{-n_k}\right)\right) \\ &\geq \frac{1}{2}(x+1) + 2^{-k-1}(x-1), \end{aligned}$$

since

$$(10) \quad \frac{1}{n_k} \log \left(1 - 2 \left(\frac{1+x}{2}\right)^{-n_k}\right) \geq -\frac{x-1}{x+1} 2^{-kn_k-1} x^{-n_k-1} / n_{k-1} > -\frac{x-1}{x+1} 2^{-k-1} \geq -\left(2 + \frac{x+1}{x-1} 2^k\right)^{-1}$$

by (3). Thus (5) is proved for  $\alpha_k$ .

From (5) and the monotony of  $\alpha_k$  it follows that  $\alpha = \lim_{k \rightarrow \infty} \alpha_k$  exists and  $\frac{1}{2}(1+x) \leq \alpha \leq x$ . From (8) and (9) we find, since  $\delta_k = 0$  for  $1 \leq k < K$ , that

$$\begin{aligned} \alpha &= x \prod_{k=K}^{\infty} \left(1 - \frac{\delta_k}{n_k}\right) \geq x \prod_{k=K}^{\infty} \left(1 + \frac{1}{n_k} \log \left(1 - 2 \left(\frac{1+x}{2}\right)^{-n_k}\right)\right) \\ &\geq x \prod_{k=K}^{\infty} \left(1 + \frac{x+1}{x-1} 2^k\right) \left(2 + \frac{x+1}{x-1} 2^k\right)^{-1} \\ &\geq x \prod_{k=K}^{\infty} (1 - 2^{-k}) > x \left(1 - \frac{\varepsilon}{x}\right) = x - \varepsilon \end{aligned}$$

by (10) and (4). Hence (1) is proved.

It only remains to verify (2). We shall first prove that

$$(11) \quad \alpha_m^{n_k} \geq a_k + \frac{1}{2} \eta_k + 2^{-mn_k} \quad \text{for} \quad m \geq k > 0.$$

Equality obviously holds for  $m = k$ , by (7); suppose therefore that the inequality is true for  $m-1$ . We then have for  $m > k$

$$\alpha_m^{n_k} = \alpha_{m-1}^{n_k} \left(1 - \frac{\delta_m}{n_m}\right)^{n_k} \tag{by (8)}$$

$$= (a_k + \frac{1}{2} \eta_k + 2^{-(m-1)n_k}) \left(1 + \frac{1}{n_m} \log \left(1 - 2 \left(\frac{1+x}{2}\right)^{-n_m}\right)\right)^{n_k} \tag{by (9) and (11)}$$

$$\geq (a_k + \frac{1}{2} \eta_k + 2^{-(m-1)n_k}) (1 - 2^{-mn_k} x^{-n_k}) \tag{by (3)}$$

$$\geq (a_k + \frac{1}{2} \eta_k + 2^{-(m-1)n_k}) (1 - 2^{-mn_k} \alpha_k^{-n_k}) \tag{by (5)}$$

$$\geq a_k + \frac{1}{2} \eta_k + 2^{-(m-1)n_k} - 2^{-mn_k} \tag{by (6)}$$

$$\geq a_k + \frac{1}{2} \eta_k + 2^{-mn_k},$$

and (11) is proved. But (11) implies

$$\alpha^{n_k} = \lim_{m \rightarrow \infty} \alpha_m^{n_k} \geq a_k + \frac{1}{2} \eta_k.$$

On the other hand

$$\alpha^{n_k} \leq \alpha_k^{n_k} = a_k + \frac{1}{2} \eta_k + 2^{-kn_k}$$

by (7), therefore

$$|\alpha^{n_k} - a_k|^{1/n_k} \leq 2^{-k} \quad \text{if} \quad \eta_k = 0,$$

$$2^{-1/n_k} \leq |\alpha^{n_k} - a_k|^{1/n_k} \leq (\frac{1}{2} + 2^{-kn_k})^{1/n_k} \quad \text{if} \quad \eta_k = 1,$$

and (2) holds with  $g_{n_k} = a_k$ .

Proof of (c). Let  $\varepsilon, a$  and  $b$  be real numbers satisfying

$$0 \leq \varepsilon \leq 1 \leq a < b,$$

and denote by  $A(\varepsilon)$  the set of all  $\alpha \in A$  satisfying  $P(\alpha) \leq 1 - \varepsilon$ , and by  $A(\varepsilon, a, b)$  the subset of those  $\alpha \in A(\varepsilon)$  for which

$$a \leq \alpha \leq b.$$

The upper bound for  $P(\alpha)$  means that there exists to  $\alpha$  an infinite set  $N$  of positive integers  $n$  satisfying

$$(12) \quad |\alpha^n - g_n| \leq (1 - \frac{1}{2} \varepsilon)^n, \quad g_n \geq 2.$$

Therefore, if  $\alpha \in A(\varepsilon, a, b)$ , then for each such  $n$ ,

$$\frac{1}{2} \alpha^n \leq \frac{1}{2} \alpha^n \leq g_n \leq 2 \alpha^n \leq 2b^n,$$

because

$$(1 - \frac{1}{2} \varepsilon)^n \leq 1 \leq \frac{1}{2} g_n.$$

This further implies that, if  $n \in N$  is given, the integer  $g_n$  has not more than  $2b^n$  possibilities. Next, if both  $n \in N$  and  $g_n$  are given, then  $a$  is by (12) restricted to the interval

$$I_n(g_n): \{g_n - (1 - \frac{1}{2}\varepsilon)^{n,1/m} \leq a \leq \{g_n + (1 - \frac{1}{2}\varepsilon)^{n,1/m}$$

of length

$$\{g_n + (1 - \frac{1}{2}\varepsilon)^{n,1/m} - \{g_n - (1 - \frac{1}{2}\varepsilon)^{n,1/m} \sim \frac{2(1 - \frac{1}{2}\varepsilon)^n}{n g_n^{(n-1)/n}}$$

For large  $n$  this is less than  $(1 - \frac{1}{2}\varepsilon)^n / a^{n-1}$  because

$$g_n^{(n-1)/n} \geq 2^{-(n-1)/n} a^{n-1} > \frac{2}{n} a^{n-1}.$$

Therefore, for each sufficiently large element  $n$  of  $N$ , the total length of all the intervals  $I_n(g_n)$  corresponding to possible values of  $g_n$  is less than

$$2b^n \frac{(1 - \frac{1}{2}\varepsilon)^n}{a^{n-1}} = 2a \left( \frac{(1 - \frac{1}{2}\varepsilon)b}{a} \right)^n.$$

This again implies that every point  $a$  of  $A(\varepsilon, a, b)$  lies in the union of a countable set of intervals of total length not exceeding

$$S_m = \sum_{n=m}^{\infty} 2a \left( \frac{(1 - \frac{1}{2}\varepsilon)b}{a} \right)^n,$$

where  $m$  can be chosen as large as we please.

If now

$$b < \frac{a}{1 - \frac{1}{2}\varepsilon},$$

then  $S_1$  converges, and hence  $A(\varepsilon, a, b)$  has the Lebesgue measure zero.

Since the set  $A(\varepsilon)$  can be written as

$$A(\varepsilon) = \bigcup_{n=1}^{\infty} A(\varepsilon, (1 - \frac{1}{2}\varepsilon)^{-(n-1)}, (1 - \frac{1}{2}\varepsilon)^{-n}),$$

it evidently is a union of countably many sets all of measure zero. Therefore  $A(\varepsilon)$  and hence also  $\bigcup_{n=1}^{\infty} A(1/n)$  have the measure zero, which proves the assertion.

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