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# On the approximation of real numbers by roots of integers

by

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*To J. G. van der Corput on his 75th birthday*

Let $A$ be the set of all numbers $a > 1$ for which none of the powers $a, a^2, a^3, \ldots$ is an integer. For every positive integer $n$ there exists at least one integer $g_n$ which is closest to $a^n$ and thus satisfies the inequality

$$|a^n - g_n| \leqslant 1/2.$$

We are here concerned with the lower limit

$$P(a) = \liminf_{n\to\infty} |a^n - g_n|^{1/n}$$

which trivially has the property

$$0 \leqslant P(a) \leqslant 1 \quad \text{for all } a \in A.$$

A few years ago, one of us (Mahler, 1957) proved that

$$P(a) = 1 \text{ if } a \text{ is any rational number in } A.$$

One can further show that there are irrational algebraic numbers $a \in A$ for which $P(a) = 1$; e.g. the number $\frac{1}{2}\left(2+\sqrt{3}+\sqrt{3+4\sqrt{3}}\right)$ is of this kind. It is also well known that there exist algebraic numbers $a$ in $A$ for which

$$0 < P(a) < 1;$$

e.g. the number $\frac{1}{2}(1+\sqrt{5})$ has this property.

In the present note, the following three results will be proved.

(a) *If $P(a) = 0$, then $a$ is transcendental.*

(b) *In every neighbourhood of every number $x > 1$ there exist non-countably many $a \in A$ for which $P(a) = 0$.*

(c) *For almost all $a$ in $A$, $P(a) = 1$; thus there are transcendental numbers with this property.*

Proof of (a). Let $a$ be an algebraic number of degree $m$, and let

$$f(x) = a_0x^m+a_1x^{m-1}+\ldots+a_m = a_0\prod_{\mu=0}^{m-1}(x-a^{(\mu)})$$

be a primitive irreducible polynomial with integral coefficients of which $a = a^{(0)}$ is a zero. For each $n$ the product

$$a_0^n\prod_{\mu=0}^{m-1}(a^{(\mu)n}-g_n) = p_n$$

say, is an integer. This integer is distinct from zero because $a^n \neq g_n$ and hence also

$$a^{(\mu)n}-g_n \neq 0 \quad (\mu = 0, 1, \ldots, m-1).$$

Therefore

$$|p_n| \geqslant 1.$$

Next $a > 1$, hence $g_n \geqslant 1$, and therefore

$$|a^{(\mu)n}-g_n| \leqslant g_n(|a^{(\mu)}|+1)^n,$$

so that

$$|a_0|^n\prod_{\mu=1}^{m-1}|a^{(\mu)n}-g_n| \leqslant g_n^{m-1}\Big\{|a_0|\prod_{\mu=1}^{m-1}(|a^{(\mu)}|+1)\Big\}^n.$$

Here

$$g_n \leqslant a^n+\tfrac{1}{2} < (2a)^n$$

and so finally

$$1 \leqslant |p_n| \leqslant |a^n-g_n|\Big\{(2a)^{m-1}|a_0|\prod_{\mu=1}^{m-1}(|a^{(\mu)}|+1)\Big\}^n.$$

There exists then a constant $c > 1$ depending only on $a$ such that

$$|a^n-g_n| \geqslant c^{-n} \quad \text{for all } n,$$

proving that $P(a) > 0$. Conversely, if $P(a) = 0$, then $a$ necessarily is transcendental.

Proof of (b). Let $x > 1$, $0 < \varepsilon < \frac{1}{2}(x-1)$. We show that there is a sequence of positive integers

$$1 = n_0 < n_1 < n_2 < \ldots < n_k < \ldots$$

depending on $x$, but not on $\varepsilon$, with the following property:

Given any sequence $\{\eta_k\}$ with $\eta_k$ equal to either 0 or 1 (briefly, an $\eta$-sequence), there is a real number $a$, where

$$0 < x-a < \varepsilon, \tag{1}$$

such that

$$\lim_{k\to\infty}\{|a^{n_k}-g_{n_k}|^{1/n_k}-\eta_k\} = 0. \tag{2}$$



Clearly if $\eta_k = 0$ for infinitely many values of $k$, then $P(a) = 0$; and if $\{\eta_k\}$, $\{\eta_k'\}$ are two essentially different $\eta$-sequences, i.e. such that $\eta_k \neq \eta_k'$ for infinitely many $k$, then the corresponding real numbers $a$ and $a'$ are distinct. Since there are non-countably many essentially different $\eta$-sequences, we obtain non-countably many $a$ with $P(a) = 0$ in the (left) $\varepsilon$-neighbourhood of $x$, hence also non-countably many $a \in A$ with this property.

For the proof take any increasing sequence $n_k$ which satisfies the condition

$$-\frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right) \leqslant \frac{x-1}{x+1}2^{-kn_{k-1}}x^{-n_{k-1}}/n_{k-1} \quad \text{for} \quad k > 0. \tag{3}$$

The condition is clearly satisfied if $n_k$ increases sufficiently rapidly.

Let $\{\eta_k\}$ be an arbitrary $\eta$-sequence and $\varepsilon > 0$; we may assume $\varepsilon < \frac{1}{2}$. Determine $K > 1$ so that

$$\prod_{k=K}^{\infty}(1-2^{-k}) > 1-\frac{\varepsilon}{x}. \tag{4}$$

We define now $x = x_0 \geqslant x_1 \geqslant x_2 \geqslant \ldots$ as follows: For $0 \leqslant k < K$ we set $x_k = x$. Suppose that for some $k \geqslant K$, $x_{k-1}$ has already been determined so that

$$\tfrac{1}{2}(1+x)+2^{-k}(x-1) \leqslant x_{k-1} \leqslant x. \tag{5}$$

Set

$$x_{k-1}^{n_k} = a_k+\lambda_k, \quad a_k \text{ integer, } 1 \leqslant \lambda_k < 2. \tag{6}$$

We then define

$$x_k = (a_k+\tfrac{1}{2}\eta_k+2^{-kn_k})^{1/n_k}. \tag{7}$$

Clearly $\frac{1}{2}\eta_k+2^{-kn_k} \leqslant 1 \leqslant \lambda_k$, $x_k \leqslant x_{k-1}$, and so the second half of inequality (5) is satisfied. We now show that also

$$x_k \geqslant \tfrac{1}{2}(1+x)+2^{-k-1}(x-1).$$

Set

$$x_k = x_{k-1}\left(1-\frac{\delta_k}{n_k}\right). \tag{8}$$

By (6) and (7) we have for $k \geqslant K$

$$a_k+\tfrac{1}{2}\eta_k+2^{-kn_k} = (a_k+\lambda_k)\left(1-\frac{\delta_k}{n_k}\right)^{n_k},$$

or setting

$$\zeta_k = \lambda_k - \tfrac{1}{2}\eta_k - 2^{-kn_k}, \qquad 0 \leqslant \zeta_k < 2,$$

$$1-\zeta_k x_{k-1}^{-n_k} = \left(1-\frac{\delta_k}{n_k}\right)^{n_k} < e^{-\delta_k}.$$

By (5) therefore

$$(9) \qquad 0 \leqslant \delta_k \leqslant -\log\left(1-\zeta_k\left(\frac{1+x}{2}\right)^{-n_k}\right) < -\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right),$$

hence by (5) and (8),

$$x_k \geqslant \tfrac{1}{2}(x+1)+2^{-k}(x-1)\left(1+\frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right)\right)$$

$$\geqslant \tfrac{1}{2}(x+1)+2^{-k-1}(x-1),$$

since

$$(10) \qquad \frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right) \geqslant -\frac{x-1}{x+1}2^{-kn_{k-1}}x^{-n_{k-1}}/n_{k-1}$$

$$> -\frac{x-1}{x+1}2^{-k-1} \geqslant -\left(2+\frac{x+1}{x-1}2^k\right)^{-1}$$

by (3). Thus (5) is proved for $x_k$.

From (5) and the monotonity of $x_k$ it follows that $\alpha = \lim_{k\to\infty} x_k$ exists and $\tfrac{1}{2}(1+x) \leqslant \alpha \leqslant x$. From (8) and (9) we find, since $\delta_k = 0$ for $1 \leqslant k < K$, that

$$\alpha = x\prod_{k=K}^{\infty}\left(1-\frac{\delta_k}{n_k}\right) \geqslant x\prod_{k=K}^{\infty}\left(1+\frac{1}{n_k}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_k}\right)\right)$$

$$\geqslant x\prod_{k=K}^{\infty}\left(1+\frac{x+1}{x-1}2^k\right)\left(2+\frac{x+1}{x-1}2^k\right)^{-1}$$

$$\geqslant x\prod_{k=K}^{\infty}(1-2^{-k}) > x\left(1-\frac{\varepsilon}{x}\right) = x-\varepsilon$$

by (10) and (4). Hence (1) is proved.

It only remains to verify (2). We shall first prove that

$$(11) \qquad x_m^{n_k} \geqslant a_k+\tfrac{1}{2}\eta_k+2^{-mn_k} \quad \text{for} \quad m \geqslant k > 0.$$



Equality obviously holds for $m = k$, by (7); suppose therefore that the inequality is true for $m-1$. We then have for $m > k$

$$x_m^{n_k} = x_{m-1}^{n_k}\left(1-\frac{\delta_m}{n_m}\right)^{n_k} \qquad \text{(by (8))}$$

$$= (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})\left(1+\frac{1}{n_m}\log\left(1-2\left(\frac{1+x}{2}\right)^{-n_m}\right)\right)^{n_k}$$

(by (9) and (11))

$$\geqslant (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})(1-2^{-mn_k}x^{-n_k}) \qquad \text{(by (3))}$$

$$\geqslant (a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k})(1-2^{-mn_k}x_{k-1}^{-n_k}) \qquad \text{(by (5))}$$

$$\geqslant a_k+\tfrac{1}{2}\eta_k+2^{-(m-1)n_k}-2^{-mn_k} \qquad \text{(by (6))}$$

$$\geqslant a_k+\tfrac{1}{2}\eta_k+2^{-mn_k},$$

and (11) is proved. But (11) implies

$$\alpha^{n_k} = \lim_{m\to\infty} x_m^{n_k} \geqslant a_k+\tfrac{1}{2}\eta_k.$$

On the other hand

$$\alpha^{n_k} \leqslant x_k^{n_k} = a_k+\tfrac{1}{2}\eta_k+2^{-kn_k}$$

by (7), therefore

$$|\alpha^{n_k}-a_k|^{1/n_k} \leqslant 2^{-k} \quad \text{if} \quad \eta_k = 0,$$

$$2^{-1/n_k} \leqslant |\alpha^{n_k}-a_k|^{1/n_k} \leqslant (\tfrac{1}{2}+2^{-kn_k})^{1/n_k} \quad \text{if} \quad \eta_k = 1,$$

and (2) holds with $g_{n_k} = a_k$.

Proof of (c). Let $\varepsilon$, $a$ and $b$ be real numbers satisfying

$$0 \leqslant \varepsilon \leqslant 1 \leqslant a < b,$$

and denote by $A(\varepsilon)$ the set of all $\alpha \in A$ satisfying $P(\alpha) \leqslant 1-\varepsilon$, and by $A(\varepsilon, a, b)$ the subset of those $\alpha \in A(\varepsilon)$ for which

$$a \leqslant \alpha \leqslant b.$$

The upper bound for $P(\alpha)$ means that there exists to $\alpha$ an infinite set $N$ of positive integers $n$ satisfying

$$(12) \qquad |\alpha^n-g_n| \leqslant (1-\tfrac{1}{2}\varepsilon)^n, \quad g_n \geqslant 2.$$

Therefore, if $\alpha \in A(\varepsilon, a, b)$, then for each such $n$,

$$\tfrac{1}{2}a^n \leqslant \tfrac{1}{2}\alpha^n \leqslant g_n \leqslant 2\alpha^n \leqslant 2b^n,$$

because

$$(1-\tfrac{1}{2}\varepsilon)^n \leqslant 1 \leqslant \tfrac{1}{2}g_n.$$

This further implies that, if $n \in N$ is given, the integer $g_n$ has not more than $2b^n$ possibilities. Next, if both $n \in N$ and $g_n$ are given, then $a$ is by (12) restricted to the interval

$$I_n(g_n):\ \{g_n-(1-\tfrac{1}{2}\varepsilon)^n\}^{1/n} \leqslant a \leqslant \{g_n+(1-\tfrac{1}{2}\varepsilon)^n\}^{1/n}$$

of length

$$\{g_n+(1-\tfrac{1}{2}\varepsilon)^n\}^{1/n}-\{g_n-(1-\tfrac{1}{2}\varepsilon)^n\}^{1/n} \sim \frac{2(1-\frac{1}{2}\varepsilon)^n}{n g_n^{(n-1)/n}}.$$

For large $n$ this is less than $(1-\frac{1}{2}\varepsilon)^n/a^{n-1}$ because

$$g_n^{(n-1)/n} \geqslant 2^{-(n-1)/n} a^{n-1} > \frac{2}{n} a^{n-1}.$$

Therefore, for each sufficiently large element $n$ of $N$, the total length of all the intervals $I_n(g_n)$ corresponding to possible values of $g_n$ is less than

$$2b^n \frac{(1-\frac{1}{2}\varepsilon)^n}{a^{n-1}} = 2a\left(\frac{(1-\frac{1}{2}\varepsilon)b}{a}\right)^n.$$

This again implies that every point $a$ of $A(\varepsilon, a, b)$ lies in the union of a countable set of intervals of total length not exceeding

$$S_m = \sum_{n=m}^{\infty} 2a\left(\frac{(1-\frac{1}{2}\varepsilon)b}{a}\right)^n,$$

where $m$ can be chosen as large as we please.

If now

$$b < \frac{a}{1-\frac{1}{2}\varepsilon},$$

then $S_1$ converges, and hence $A(\varepsilon, a, b)$ has the Lebesgue measure zero.

Since the set $A(\varepsilon)$ can be written as

$$A(\varepsilon) = \bigcup_{n=1}^{\infty} A\left(\varepsilon, (1-\tfrac{1}{3}\varepsilon)^{-(n-1)}, (1-\tfrac{1}{3}\varepsilon)^{-n}\right),$$

it evidently is a union of countably many sets all of measure zero. Therefore $A(\varepsilon)$ and hence also $\bigcup_{n=1}^{\infty} A(1/n)$ have the measure zero, which proves the assertion.

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