fast überall, wo γ eine Konstante ist. Es ist nämlich $\lim_m \sqrt[m]{a_n^{(1)} \dots a_n^{(m)}}$ fast überall endlich, wie schon aus Satz 7 in [5] hervorgeht. Der Vollständigkeit halber sei in diesem Zusammenhang erwähnt, daß schon aus Satz 5 in [5] und Satz 2 in [6] folgt

$$a_i^{(\nu)} > \nu \ln \nu$$

für endlich viele Werte ν fast überall und demnach

$$\frac{1}{N} \sum_{\mu=1}^{N} a_i^{(\mu)} \to \infty$$

fast überall.

Die noch offene Frage ist, wie das invariante Maß μ aussieht, oder anders ausgedrückt, welche Funktion $f(x)$ die Eigenschaft

$$\mu(E) = \int_E f(x)\,dx$$

besitzt.

Literaturverzeichnis

[1] N. Dunford and D. S. Miller, *On the ergodic theorem*, Trans. Amer. Math. Soc. 60 (1946), S. 538-549.

[2] S. Hartman, E. Marczewski et Ryll-Nardzewski, *Théorèmes ergodiques et leurs applications*, Colloq. Math. 2 (1950), S. 109-123.

[3] O. Perron, *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann. 64 (1907), S. 1-76.

[4] C. Ryll-Nardzewski, *On the ergodic theorems (II)*, Studia Math. 12 (1951), S. 74-79.

[5] F. Schweiger, *Geometrische und elementare metrische Sätze über den Jacobischen Algorithmus*, Sitzungsber. Öst. Akademie Wiss., Math.-naturw. Klasse, Abt. II, 173 (1964), S. 59-92.

[6] — *Metrische Sätze über den Jacobischen Algorithmus*, Monatsh. Math. 69 (1965), S. 243-255.

[7] — *Ergodische Theorie des Jacobischen Algorithmus*, Acta Arith. 11 (1966), S. 451-460.

Reçu par la Rédaction le 16. 3. 1966

ACTA ARITHMETICA
XII (1967)

On distribution of values of multiplicative functions in residue classes

by

W. Narkiewicz (Wrocław)

1. The following notion of uniform distribution of sequences of integers was introduced by I. Niven ([4]):

Let $N \geqslant 2$ be an integer. A sequence $a_1, a_2, \dots$ of integers is uniformly distributed $(\bmod N)$ if and only if for every j, $N\big(n \leqslant x \mid a_n \equiv j \pmod N\big) \sim x/N$ for x tending to infinity. (Here $N(n \leqslant x \mid \mathrm{P})$ denotes the number of positive integers $n \leqslant x$ with the property P.)

In this note we shall consider a similar, but weaker notion of uniform distribution:

Let $N \geqslant 3$ be an integer. A sequence $a_1, a_2, \dots$ of integers is *weakly uniformly distributed* $(\bmod N)$ if and only if for every pair of integers j_1, j_2 with $(j_1, N) = (j_2, N) = 1$

$$N\big(n \leqslant x \mid a_n \equiv j_1 \pmod N\big) \sim N\big(n \leqslant x \mid a_n \equiv j_2 \pmod N\big)$$

for x tending to infinity, provided that the set $\{j \mid (a_j, N) = 1\}$ is infinite.

For shortness we shall write that such sequence is WUD $(\bmod N)$.

It is easy to see that a necessary and sufficient condition for a sequence $a_1, a_2, \dots$ of integers to be WUD$(\bmod N)$ is that for all characters $\chi(\bmod N)$ which are not equal to χ_0 — the principal character, the following evaluation holds

$$\sum_{n \leqslant x} \chi(a_n) = o\Big(\sum_{n \leqslant x} \chi_0(a_n)\Big). \tag{1}$$

In fact, assuming (1) we get

$$\sum_{\substack{n \leqslant x \\ a_n \equiv j \pmod N}} 1 = \varphi^{-1}(N) \sum_{\chi} \overline{\chi(j)} \sum_{n \leqslant x} \chi(a_n)$$

$$= \varphi^{-1}(N) \sum_{n \leqslant x} \chi_0(a_n) + \varphi^{-1}(N) \sum_{\chi \neq \chi_0} \overline{\chi(j)} \sum_{n \leqslant x} \chi(a_n) = \big(\varphi^{-1}(N) + o(1)\big) \sum_{n \leqslant x} \chi_0(a_n)$$

and conversely, from

$$\sum_{\substack{n\leqslant x\\ a_n\equiv j \pmod N}} 1 = \big(\varphi^{-1}(N)+o(1)\big)\sum_{n\leqslant x}\chi_0(a_n) \qquad ((j,N)=1)$$

we get readily that

$$R(j) = \sum_{\chi}\overline{\chi(j)}\sum_{n\leqslant x}\chi(a_n) = \sum_{n\leqslant x}\chi_0(a_n)+o\Big(\sum_{n\leqslant x}\chi_0(a_n)\Big)$$

for all j with $(j,N)=1$ and this implies for $\chi\neq\chi_0$

$$\sum_{n\leqslant x}\chi(a_n) = \varphi^{-1}(N)\sum_{\substack{1\leqslant j\leqslant N\\ (j,N)=1}}\chi(j)R(j)$$

$$= \varphi^{-1}(N)\sum_{\substack{1\leqslant j\leqslant N\\ (j,N)=1}}\chi(j)\sum_{n\leqslant x}\chi_0(a_n)+o\Big(\sum_{n\leqslant x}\chi_0(a_n)\Big) = o\Big(\sum_{n\leqslant x}\chi_0(a_n)\Big).$$

This criterion is analogous to one proved by S. Uchiyama ([7]) in the case of uniform distribution in the sense of Niven.

Now let k be a positive integer and let C_k be the class of all multiplicative, integer valued functions $f(n)$ satisfying the following condition:

There exist polynomials $W_1(x),\ldots,W_k(x)$ with integral coefficients such that for all primes p and $j=1,2,\ldots,k$ one has $f(p^j)=W_j(p)$.

The aim of this note is to give necessary and sufficient conditions for $f\in C_k$ to be WUD(mod N), provided that the set $\{n|\ (f(n),N)=1\}$ is not too small in a sense to be explained below. Let G_N be the multiplicative group of residue classes (mod N) relatively prime to N and let for f in C_k and $j=1,2,\ldots,k$, $A_j=A_j(f,N)$ be the subgroup of G_N generated by the set $R_j=R_j(f,N)$ consisting of all residue classes r in G_N for which the congruence $W_j(x)\equiv r \pmod N$ has a solution $x\in G_N$. Finally let $K(f)$ be the largest number k such that $f\in C_k$, if such a number exists, and $K(f)=\infty$ if $f\in C_k$ for all k. Then we have the following

THEOREM I. *Let $f\in C_k$ for some k. If $R_1(f,N)=\ldots=R_{m-1}(f,N)=\emptyset$ and $R_m(f,N)\neq\emptyset$ for some m not exceeding $K(f)$, then the sequence $f(1)$, $f(2),\ldots$ is WUD(mod N) if and only if for every nonprincipal character χ of G_N which is trivial on $A_m(f,N)$ there exists a prime p such that*

$$1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-j/m}=0.$$

(Note that such a p must necessarily either divide N or be at most equal to 2^m.)

In the case $m=1$ the sufficiency of this condition can be easily inferred from a result of E. Wirsing ([8], Satz 2).

As an application we shall derive the following corollaries:

COROLLARY 1. *The divisor function $d(n)$ is WUD(mod N) if and only if one of the following conditions hold:*

(i) $N=4$,

(ii) $N=2\cdot 3^a$ $(a\geqslant 1)$,

(iii) $N=p^a$, *p is an odd prime, $a\geqslant 1$ and 2 is a primitive root* mod p^a,

(iv) $N=2p^a$, *$p\geqslant 5$ is a prime, $a\geqslant 1$ and 3 is a primitive root* mod p^a.

In all these cases

$$N\big(n\leqslant x|\ d(n)\equiv j \pmod N\big)\sim Cx^{1/m}$$

holds for $(j,N)=1$ with $C>0$ independent on j and $m=\min_{p\nmid N} p-1$.

(A part of this corollary, namely the result that $d(n)$ is WUD(mod p^a) provided that 2 is a primitive root mod p^a is due to L. G. Sathe ([6]). Note, however, that his remark made after Theorem 5 in [6] that in this case the values of $d(n)$ are uniformly distributed in arithmetical progressions kp^a+j $(j=1,2,\ldots,p^a-1)$ is not correct for $a>1$.)

COROLLARY 2. *The Euler's function $\varphi(n)$ is WUD(mod N) if and only if $(N,6)=1$. If this condition is satisfied then we have for $(j,N)=1$*

$$N\big(n\leqslant x|\ \varphi(n)\equiv j \pmod N\big)\sim Cx(\log x)^{A}$$

with $A=\prod_{p|N}(p-2)/(p-1)-1$ and $C>0$ dependent on N only.

If $f\in C_k$ for some k, but for all $m\leqslant K(f)$ the set $R_m(f,N)$ is void, then we show that the set $\{n|\ (f(n),N)=1\}$ must be small in some sense. In fact we prove

THEOREM II. *Let $f\in C_k$ for some k.*

(i) *If $K(f)$ is finite and the sets $R_i(f,N)$ are void for $i=1,2,\ldots,K=K(f)$, then*

$$N\big(n\leqslant x|\ (N,f(n))=1\big)=O(x^{1/(K+1)+\varepsilon})$$

for every positive ε.

(ii) *If $K(f)$ is infinite and the sets $R_i(f,N)$ are void for all i, then*

$$N\big(n\leqslant x|\ (N,f(n))=1\big)=O\big((\log x)^r\big),$$

where r is the number of distinct primes dividing N.

2. For the proof we need a tauberian theorem due to H. Delange, which we state as

LEMMA 1. (See [1], th. III). *If a_n are nonnegative real numbers, and for* $\operatorname{re} s>a>0$

$$\sum_{n=1}^{\infty}a_n n^{-s}=g_0(s)(s-a)^{-b}+\sum_{j=1}^{w}g_j(s)(s-a)^{-b_j}+h(s)$$

where b is a real number not equal to zero or a negative integer, $g_0(s), g_1(s), \ldots, g_w(s), h(s)$ *are regular in the closed half-plane* $\mathrm{re}\, s \geqslant a$, $g_0(a) \neq 0$, $\mathrm{re}\, b_j < b$ $(j = 1, 2, \ldots, w)$ *and* $b_j \neq 0, -1, -2, \ldots$ *then for x tending to infinity one has*

$$\sum_{n\leqslant x} a_n \sim a^{-1}\Gamma(b)^{-1}x^a(\log x)^{b-1}.$$

We shall need also a corollary to this theorem, which we state as

Lemma 2. *The assumptions are the same as in Lemma 1, except that the condition* $g_0(a) \neq 0$ *is replaced by* $g_0(s) = 0$ *for all s,* $a = 1/m$ *and* $0 \leqslant b \leqslant 1$. *Then for x tending to infinity one has*

$$a_n = o\big(x^{1/m}(\log x)^{b-1}\big).$$

Proof of the lemma. Let

$$F(s) = \prod_p (1+bp^{-s}) = \sum_{n=1}^{\infty} B_n n^{-s}.$$

Clearly $B_n \geqslant 0$ and in view of $0 \leqslant b \leqslant 1$ we have for $\mathrm{re}\, s \geqslant 1$ the equality $F(s) = G(s)(s-1)^{-b}$ with $G(s)$ regular for $\mathrm{re}\, s \geqslant 1$ and $G(1) \neq 0$. Consider

$$F(ms) + \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} A_n n^{-s}.$$

Lemma 1 implies

$$\sum_{n\leqslant x} B_n n^{-m} \sim \sum_{n\leqslant x} A_n \sim Cx^{1/m}(\log x)^{b-1}$$

which is clearly equivalent to the assertion of our lemma.

Now let $N \geqslant 3$, let $\chi(n)$ be a character of the group G_N, treated as a function defined for all integers. (For $(d, N) \neq 1$ we put $\chi(d) = 0$.)

Let λ_j be the number of solutions of the congruence $W_m(x) \equiv j \pmod N$ in $x \in G_N$ if $(j, N) = 1$ and $\lambda_j = 0$ if $(j, N) \neq 1$. (Here m is the number occurring in the statement of Theorem I. Note that $R_m(f, N) = \{j|\ 1 \leqslant j \leqslant N,\ \lambda_j \neq 0\}$.) Let $F(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(f(n))}{n^s}$ Now we prove

Lemma 3. *For* $\mathrm{re}\, s > 1/m$

$$F(\chi, s) = H(\chi, s)\Big(s - \frac{1}{m}\Big)^{-A(\chi)}$$

where $H(\chi, s)$ *is regular in the closed half-plane* $\mathrm{re}\, s \geqslant 1/m$ *and does not vanish at* $s = 1/m$, *and* $A(\chi) = \varphi^{-1}(N)\sum_{j=1}^{N}\chi(j)\lambda_j$, *if for every prime p*

$$1 + \sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-j/m} \neq 0.$$

If the last condition is not satisfied, then $A(\chi)$ *is a complex number of the form*

$$\varphi^{-1}(N)\sum_{j=1}^{N}\chi(j)\lambda_j - l \quad (l \geqslant 1,\ \textit{integer}).$$

Proof of the lemma. Observe first that for $j = 1, 2, \ldots, m-1$, $(N, f(P^j)) = 1$ can hold with a prime P only if $P|N$, and so $\chi(f(n)) \neq 0$ implies that every prime divisor of n either divides N or occurs in n with an exponent at least equal to m. It follows that the series defining $F(\chi, s)$ converges absolutely for $\mathrm{re}\, s > 1/m$. Indeed, it is majorized by the product $\sum_{n\in A} n^{-\sigma}\sum_{n\in B} n^{-\sigma}$ (where $\sigma = \mathrm{re}\, s$, $A = \{n|\ n = p_1^{a_1}\ldots p_u^{a_u},\ p_i|N\}$ and B is the set of all m-full numbers, i.e. such numbers n for which $p|n$ implies $p^m|n$) of two series, the first of which is convergent for $\sigma > 0$, and the second can be written in the form

$$\sum_{t=1}^{\infty}\frac{|\mu(t)|}{t^{\sigma m}}\prod_{p|t}(1-p^{-\sigma})^{-1}$$

which implies its convergence for $\mathrm{re}\, s > 1/m$ in view of the evaluation

$$\prod_{p|t}(1-p^{-\sigma})^{-1} = O(t^{\varepsilon})$$

valid for every fixed positive $\sigma < 1$ and arbitrary positive ε. (This evaluation results immediately from

$$\sum_{p\leqslant y} p^{-\sigma} = O(y^{1-\sigma})$$

(see [5]).)

It follows that

$$F(\chi, s) = \prod_p\Big(1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big) \quad \text{for} \quad \mathrm{re}\, s > 1/m. \tag{2}$$

Let P_1 be the set of all primes which either divide N or are less than 2^m+1, and let P_2 be the set of all remaining primes. For $p \in P_2$ and $\mathrm{re}\, s > 1/m$ we have (in view of $\chi(f(p)) = \ldots = \chi(f(p^{m-1})) = 0$)

$$\Big|\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big| \leqslant \sum_{j=m}^{\infty}p^{-j/m} = p^{-1}(1-p^{-1/m}) < 2/p \leqslant 1$$

and so

$$\prod_{p\in P_2}\Big(1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big) = \exp\Big\{\sum_{p\in P_2}\sum_{l=1}^{\infty}(-1)^{l+1}l^{-1}\Big(\sum_{j=m}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big)^l$$

$$= \exp\Big\{\sum_{p\in P_2}\chi\big(f(p^m)\big)p^{-ms}\Big\}\exp\Big\{\sum_{p\in P_2}\sum_{j=1+m}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big\}\times$$

$$\times\exp\Big\{\sum_{p\in P_2}\sum_{l=2}^{\infty}(-1)^{l+1}l^{-1}\Big(\sum_{j=m}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big)\Big\}.$$

Now for $0 < \varepsilon < (m+1)^{-2}$, $\operatorname{re} s \geqslant 1/m - \varepsilon$

$$\Big|\sum_{j=1+m}^{\infty} \chi\big(f(p^j)\big)p^{-js}\Big| \leqslant \sum_{j=1+m}^{\infty} p^{-j/(1/m-\varepsilon)} \leqslant (p^{1+1/m(m-1)} - p^{1-\varepsilon})^{-1}$$

hence the function

$$\exp\Big\{\sum_{p\in P_2}\sum_{j=1+m}^{\infty} \chi\big(f(p^j)\big)p^{-js}\Big\}$$

is regular for $\operatorname{re} s \geqslant 1/m$ and does not vanish at $s = 1/m$, and moreover for $\operatorname{re} s \geqslant 3/4m$

$$\Big|\sum_{l=2}^{\infty}(-1)^{l+1}l^{-1}\Big(\sum_{j=m}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big)^l\Big| \leqslant \sum_{l=2}^{\infty}\Big(\sum_{j=m}^{\infty}p^{-3j/4m}\Big)^l \leqslant Bp^{-3/2}$$

with a suitable $B > 0$. Consequently the function

$$\exp\Big\{\sum_{p\in P_2}\sum_{l=2}^{\infty}(-1)^{l+1}l^{-1}\Big(\sum_{j=m}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big)^l\Big\}$$

s regular for $\operatorname{re} s \geqslant 1/m$ and does not vanish at $s = 1/m$.

Finally

$$\sum_{p\in P_2}\chi\big(f(p^m)\big)p^{-ms} = \sum_{j=1}^{N}\chi(j)\sum_{\substack{p\\ W_m(p)\equiv j \,(\mathrm{mod}\, N)}} p^{-ms}$$
$$= \Big(\varphi^{-1}(N)\sum_{j=1}^{N}\chi(j)\lambda_j\Big)\log\big(1/(s-1/m)\big) + g(s)$$

with $g(s)$ regular for $\operatorname{re} s \geqslant 1/m$, and we get for $\operatorname{re} s > 1/m$

(3) $$\prod_{p\in P_2}\Big(1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big) = g(\chi, s)(s-1/m)^{-\varphi^{-1}(N)\sum_{j=1}^{N}\chi(j)\lambda_j}$$

with $g(\chi, s)$ regular for $\operatorname{re} s \geqslant 1/m$ and $g(\chi, 1/m) \neq 0$.

The product

$$\prod_{p\in P_1}\Big(1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-js}\Big)$$

defines obviously a function regular for $\operatorname{re} s > 0$, which does not vanish identically and may thus be written in the form $g_1(\chi, s)(s-1/m)^M$ where M is an integer $\geqslant 0$ and $g_1(\chi, 1/m) \neq 0$.

Observe now that $M \neq 0$ holds if and only if for some prime p in P_1

$$1+\sum_{j=1}^{\infty}\chi(p^j)p^{-j/m} = 0$$

and consequently (2) and (3) imply the lemma.

Proof of Theorem I. For $(j, N) = 1$ we get by Lemma 3

$$\sum_{\substack{n\\ f(n)\equiv j \,(\mathrm{mod}\, N)}} n^{-s} = \varphi^{-1}(N)\sum_{\chi}\overline{\chi(j)}F(\chi, s)$$
$$= \varphi^{-1}(N)\sum_{\chi}\overline{\chi(j)}H(\chi, s)(s-1/m)^{-A(\chi)}.$$

As obviously

$$\operatorname{re} A(\chi) \leqslant \operatorname{re} A(\chi_0) = \Big(\sum_{j=1}^{N}\lambda_j\Big)\varphi^{-1}(N),$$

we can write

(4) $$\sum_{\substack{n\\ f(n)\equiv j \,(\mathrm{mod}\, N)}} n^{-s} = \Big\{\varphi^{-1}(N)\sum_{\chi\in X}\overline{\chi(j)}H(\chi, s)\Big\}(s-1/m)^{-A(\chi_0)} + $$
$$+\sum_{j=1}^{t} g_j(s)(s-1/m)^{-a_j} + h(s)$$

where X is the set of all characters of G_N with $A(\chi) = A(\chi_0)$, $g_1(s), \ldots, \ldots, g_t(s)$, $h(s)$ are regular for $\operatorname{re} s \geqslant 1/m$ and $\operatorname{re} a_i < A(\chi_0)$.

Note that $\chi \in X$ if and only if $\lambda_j \neq 0$ implies $\chi(j) = 1$ and moreover (by Lemma 2) for all primes p

$$1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-j/m} \neq 0.$$

If for every nonprincipal character χ of G_N, which is trivial on A_m (and a fortiori on R_m) there exists a prime p with

$$1+\sum_{j=1}^{\infty}\chi\big(f(p^j)\big)p^{-j/m} = 0,$$

then X consists of the principal character exclusively, and by (4) and Lemma 1 we get for $(j, N) = 1$

$$N\big(n \leqslant x \mid f(n) \equiv j \,(\mathrm{mod}\, N)\big) \sim Cx^{1/m}(\log x)^{A(\chi_0)-1}$$

where C does not depend on j, and this means that the sequence $f(1), f(2), \ldots$ is WUD(mod N). The first part of Theorem I is thus proved.

Now assume that the sequence $f(1), f(2), \ldots$ is WUD(mod N). Applying first Lemma 3 to the principal character χ_0 of G_N, and then Lemma 1, we get for $(j, N) = 1$

$$N_j(x) = N\big(n \leqslant x \mid f(n) \equiv j \,(\mathrm{mod}\, N)\big) \sim \varphi^{-1}(N) N\big(n \leqslant x \mid (f(n), N) = 1\big)$$
$$\sim \Gamma^{-1}\big(A(\chi_0)\big)\varphi^{-1}(N) m H(\chi_0, 1/m) x^{1/m} (\log x)^{A(\chi_0)-1}.$$

Note now that for $(j, N) = 1$

$$\sum_{\chi \in X} \overline{\chi(j)} H(\chi, 1/m) \neq 0$$

as otherwise we would get by Lemma 2 from (4)

$$N_j(x) = o\big(x^{1/m} (\log x)^{A(\chi_0)-1}\big)$$

contrary to the evaluation just obtained. Consequently (4) and Lemma 1 lead us to

$$N_j(x) \sim m\Gamma^{-1}\big(A(\chi_0)\big)\varphi^{-1}(N) \sum_{\chi \in X} \overline{\chi(j)} H(\chi, 1/m) x^{1/m} (\log x)^{A(\chi_0)-1}$$

whence

$$\sum_{\chi \in X} \overline{\chi(j)} H(\chi, 1/m) = H(\chi_0, 1/m)$$

if $(j, N) = 1$.

Let now the set $\{j_1, \ldots, j_t\}$ be a set of representatives of G_N/Λ_m in G_N. Then in view of the last equality the system

$$\sum_{\chi \in X} \overline{\chi(j_k)} x(\chi) = H(\chi_0, 1/m) \qquad (k = 1, 2, \ldots, t)$$

has in case $|X| \geqslant 2$ at least two distinct solutions: $x(\chi_0) = H(\chi_0, 1/m)$, $x(\chi) = 0$ for $\chi \neq \chi_0$ and $x(\chi) = H(\chi, 1/m)$ for $\chi \in X$. But the matrix

$$\|\overline{\chi(j_k)}\|_{\substack{k=1,\ldots,t\\ \chi \in X}}$$

is of rank $|X|$ which gives a contradiction. Consequently X consists solely of the principal character and this means that for every nonprincipal character of G_N which is trivial on Λ_m there exists a prime p such that

$$1 + \sum_{j=1}^{\infty} \chi\big(f(p^j)\big) p^{-j/m} = 0.$$

This proves the second part of Theorem I.

Proof of Theorem II. If $K(f)$ if finite and $R_i(f, N)$ is void for $i = 1, 2, \ldots, K(f) = K$, then as in proof of Lemma 2 we conclude that the function $F(\chi_0, s)$ is regular for $\mathrm{re}\, s > 1/(K+1)$ and by Ikehara's theorem one gets easily (see e.g. [3], Lemma 4) that

$$N\big(n \leqslant x \mid (f(n), N) = 1\big) = O\big(x^{1/(K+1)+\varepsilon}\big)$$

for every positive ε, thus proving part (i) of Theorem II.

To prove part (ii) note that if $K(f)$ is infinite and all sets $R_i(f, N)$ are void, then $(f(n), N) = 1$ implies that all prime divisors of n must divide N and consequently

$$N\big(n \leqslant x \mid (f(n), N) = 1\big) \leqslant N\Big(n \leqslant x \mid n = \prod_{i=1}^{r} p_i^{a_i},\ p_i \mid N\Big) = O\big((\log x)^r\big).$$

Theorem II is thus proved.

3. In this section we prove the corollaries.

Proof of Corollary 1. For $f(n) = d(n)$ we have obviously $K(f) = \infty$, $W_i(x) = 1+i$, and for odd N, $R_1 = \{2\}$, whereas for N even $R_1 = R_2 = \ldots = R_{m-1} = \emptyset$, $R_m = \{1+m\}$ if $m+1$ is the least prime not dividing N. Observe that for $|z| < 1$, $z \neq 0$

$$1 + \sum_{j=1}^{\infty} \chi\big(d(p^j)\big) z^j = 1 + z^{-1} \sum_{j=2}^{\infty} \chi(j) z^j$$
$$= 1 + z^{-1} \sum_{k=1}^{N-1} \chi(k) \sum_{\substack{j \geqslant 2\\ j \equiv k \,(\mathrm{mod}\, N)}} z^j = z^{-1}(1-z^N)^{-1} \sum_{i=1}^{N-1} \chi(i) z^i.$$

If thus for some character χ of G_N and for some prime p

$$1 + \sum_{j=1}^{\infty} \chi\big(d(p^j)\big) p^{-j/m} = 0$$

then the polynomial $M(z) = \sum_{i=1}^{N-1} \chi(i) z^i$ has $z = p^{-1/m}$ as a root, but all coefficients of $M(z)$ are units of the field $Q\big(\exp(2\pi i/N)\big)$ and so all its roots are algebraic integers, whereas $z = p^{-1/m}$ is not, a contradiction.

It follows that the sequence of values of $d(n)$ will be WUD(mod N) if and only if Λ_m coincides with G_N and as Λ_m is a cyclic group generated by the prime $1+m$, this means the same as the fact that the least prime not dividing N is a primitive root mod N. It is easy to check that all the numbers $N \geqslant 3$ satisfying this condition are those listed in the statement of Corollary 1. The evaluation given there is immediate, as $A(\chi_0) = 1$.

Proof of Corollary 2. For $f(n) = \varphi(n)$ we have $K(f) = \infty$ and $W(x) = x^{i-1}(x-1)$. As $\varphi(n)$ is even for $n \geqslant 3$, the sequence $\varphi(n)$ cannot be WUD(mod N) for N even. Let thus N be odd. In this case

$$R_1 = \{r \mid 1 \leqslant r \leqslant N-1,\ (r, N) = (r+1, N) = 1\}$$

and is not void as $1 \epsilon R_1$. Observe that for $|z| < 1$

$$1+\sum_{j=1}^{\infty}\chi(\varphi(p^j))z^j = 1+\sum_{j=1}^{\infty}\chi(p-1)\chi(p)^{j-1}z^j = 1+\chi(p-1)(1-\chi(p)z)^{-1}z.$$

If this is zero for $z = p^{-1}$, then $\chi(p)-\chi(p-1) = p$ which implies $p = 2$, but then $3 = \chi(2)$ which is impossible. It results, that $\varphi(n)$ is WUD(mod N) if and only if A_1 coincides with G_N. Let $N = p_1^{a_1}\dots p_r^{a_r}$ ($p_i \neq 2$). Then the group G_N is a product of $G_{p_1^{a_1}}, \dots, G_{p_r^{a_r}}$ and so we may represent every element y of G_N in the form $[y_1, \dots, y_r]$ with $1 \leqslant y_i < p_i^{a_i}$, $p_i \nmid y_i$ and $y \equiv y_i \pmod{p_i^{a_i}}$. In this notation A_1 is the group generated by the set

$$\{[y_1, \dots, y_r] \mid y_i \epsilon G_{p_i^{a_i}},\ y_i \not\equiv -1 \pmod{p_i}\}.$$

If N is divisible by 3, say $p_1 = 3$, then $A_1 \neq G_N$, as $[2, 1, 1, \dots, 1] \epsilon G_N$ but is not contained in A_1.

Now let $(N, 6) = 1$. We have to prove that $A_1 = G_N$. Let $y = [y_1, \dots, y_r] \epsilon G_N$, and let w_i be a solution of the congruence $2w_i \equiv y_i \pmod{p_i^{a_i}}$. Put

$$v_i = \begin{cases} y_i & \text{if} \quad y_i \not\equiv -1 \pmod{p_i}, \\ 2 & \text{if} \quad y_i \equiv -1 \pmod{p_i} \end{cases}$$

and

$$z_i = \begin{cases} 1 & \text{if} \quad y_i \not\equiv -1 \pmod{p_i}, \\ w_i & \text{if} \quad y_i \equiv -1 \pmod{p_i}. \end{cases}$$

Evidently $[v_1, v_2, \dots, v_r] \epsilon R_1$ as $p_i > 3$ and so $2 \not\equiv -1 \pmod{p_i}$. If $w_i \equiv -1 \pmod{p_i}$ then $-1 \equiv y_i \equiv 2w_i \equiv -2 \pmod{p_i}$ a nonsense, consequently $[z_1, \dots, z_r] \epsilon R_1$. But obviously $y = [v_1, \dots, v_r][z_1, \dots, z_r] \epsilon A_1$ and so $G_N = A_1$. The Corollary 2 is thus proved because the evaluation stated in it follows immediately from the fact that

$$A(\chi_0) = \varphi^{-1}(N)\sum_{j=1}^{N}\lambda_j = \varphi^{-1}(N)|R_1| = \prod_{p|N}(p-2)/(p-1).$$

(Cf. [2], vol. I, p. 147).

References

[1] H. Delange, *Généralisation du théorème de Ikehara*, Ann. Scient. de l'École Norm. Sup. 71 (1954), pp. 213-242.

[2] L. E. Dickson, *History of the theory of numbers*, Chelsea 1952.

[3] W. Narkiewicz, *Divisibility properties of a class of multiplicative functions*, Colloq. Math., to appear.

[4] I. Niven, *Uniform distribution of sequences of integers*, Trans. Amer. Math. Soc. 98 (1961), pp. 52-61.

[5] G. Pólya, *Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen*, Gött. Nachr. 1917, pp. 149-159.

[6] L. G. Sathe, *On a congruence property of the divisor function*, Amer. Journ. Math. 67 (1945), pp. 397-406.

[7] S. Uchiyama, *On the uniform distribution of sequences of integers*, Proc. Jap. Acad. 37 (1961), pp. 605-609.

[8] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. 143 (1961), pp. 75-102.

INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY

Reçu par la Rédaction le 26. 3. 1966