fast überall, wo $\nu$ eine Konstante ist. Es ist nämlich $lim_{n \to \infty} d_n^{(n)} > 0$ für alle $n$.


$$d_n^{(n)} \sim \nu^{(n)}$$

für endlich viele Werte $n$ fast überall und demnach

$$\frac{1}{N} \sum_{n=1}^{N} a(n)^n \sim \infty$$

fast überall.

Die noch offene Frage ist, wie das unveränderte Maß $\mu$ aussehst, oder anders ausgedrückt, welche Funktion $f(x)$ die Eigenschaft

$$\mu(E) = \int_E f(x) \, dx$$

besitzt.

Literaturverzeichnis


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On distribution of values of multiplicative functions in residue classes

by

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1. The following notion of uniform distribution of sequences of integers was introduced by I. Niven ([4]): Let $N \geq 2$ be an integer. A sequence $a_1, a_2, \ldots$ of integers is uniformly distributed $(\text{mod } N)$ if and only if for every $j, N \mid a_n - j(\text{mod } N)$ for $n$ tending to infinity. (Here $N \mid a, a \equiv j(\text{mod } N)$ denotes the number of positive integers $n \leq x$ with the property $P$).

In this note we shall consider a similar, but weaker notion of uniform distribution: Let $N \geq 3$ be an integer. A sequence $a_1, a_2, \ldots$ of integers is weakly uniformly distributed $(\text{mod } N)$ if and only if for every pair of integers $j_1, j_2$ with $j_1, j_2 \equiv j_2(\text{mod } N)$

$$N \mid a_n - j_1(\text{mod } N) \sim N \mid a_n - j_2(\text{mod } N)$$

for $x$ tending to infinity, provided that the set $\{j : (a_n, N) = 1\}$ is infinite.

For shortness we shall write that such sequence is WUD $(\text{mod } N)$.

It is easy to see that a necessary and sufficient condition for a sequence $a_1, a_2, \ldots$ of integers to be WUD $(\text{mod } N)$ is that for all characters $\chi(\text{mod } N)$ which are not equal to $\chi_0$ — the principal character, the following evaluation holds

$$\sum_{n=0}^{N-1} \chi(a_n) = \phi(\chi) \sum_{n=0}^{N-1} \chi_0(a_n).$$

In fact, assuming (1) we get

$$\sum_{n \equiv j(\text{mod } N)} 1 = \phi^{-1}(N) \sum_{x \equiv j(\text{mod } N)} \sum_{n \equiv j(\text{mod } N)} \chi(a_n)$$

$$= \phi^{-1}(N) \sum_{n \equiv j(\text{mod } N)} \sum_{n \equiv j(\text{mod } N)} \chi(a_n) = \phi^{-1}(N) \sum_{n \equiv j(\text{mod } N)} \sum_{n \equiv j(\text{mod } N)} \chi(a_n)$$
and conversely, from
\[ \sum_{a_n \equiv a \pmod{N}} 1 = \left( \phi^{-1}(N) + o(1) \right) \sum_{n \leq x} \chi(n) \left( \frac{a}{n} \right) \] (j, N) = 1
we get readily that
\[ R(j) = \sum_{n} \frac{n}{\phi(n)} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) = \sum_{n} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) + o \left( \sum_{n} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) \right) \]
for all j with (j, N) = 1 and this implies for \( \chi / \chi_n \)
\[ \sum_{n} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) = \sum_{n} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) = o \left( \sum_{n} \sum_{a_n \equiv a \pmod{N}} \chi(n) \left( \frac{a}{n} \right) \right) \]
This criterion is analogous to one proved by S. Uchiyama ([7]) in the case of uniform distribution in the sense of Niven.

Now let k be a positive integer and let \( \mathfrak{C}_k \) be the class of all multiplicative, integer valued functions \( f(n) \) satisfying the following condition:

There exist polynomials \( W_k(x), \ldots, W_k(x) \) with integral coefficients such that for all primes \( p \) and \( j = 1, 2, \ldots, k \) one has \( f(p^j) = W_k(p) \).

The aim of this note is to give necessary and sufficient conditions for \( f \) \( \mathfrak{C}_k \) to be WUD\( (\mod{N}) \), provided that the set \( \{ n \mid (f(n), N) = 1 \} \) is not too small in a sense to be explained below. Let \( G_N \) be the multiplicative group of residue classes \( (\mod{N}) \) relatively prime to \( N \) and let for \( f \) \( \mathfrak{C}_k \) and \( j = 1, 2, \ldots, k \), \( A_j = A_j(f, N) \) be the subgroups of \( G_N \) generated by the set \( B_j = B_j(f, N) \) consisting of all residue classes \( r \) in \( G_N \) for which the congruence \( W_k(r) = r(\mod{N}) \) has a solution \( x \in G_N \). Finally let \( K(f) \) be the largest number \( k \) such that \( f \) \( \mathfrak{C}_k \), if such a number exists, and \( K(f) = \infty \) if \( f \) \( \mathfrak{C}_k \) for all \( k \). Then we have the following

Theorem I. Let \( f \) \( \mathfrak{C}_k \) for some \( k \). If \( R_1(f, N) = \ldots = R_k(f, N) = 0 \) and \( R_{k+1}(f, N) \neq 0 \) for some \( m \), the sequence \( f(k) \), \( f(2) \), \ldots is WUD\( (\mod{N}) \) if and only if for every nonprincipal character \( \chi \) of \( G_N \) which is trivial on \( A_m(f, N) \), there exists a prime \( p \) such that
\[ 1 + \sum_{n \leq x} \chi(n) \left( \frac{p}{n} \right) n^{-\epsilon} = 0. \]
(Note that such a \( p \) must necessarily divide \( N \) or be at most equal to \( \delta^m \).)

In the case \( m = 1 \) the sufficiency of this condition can be easily inferred from a result of E. Wirsing ([8], Satz 2).

As an application we shall derive the following corollaries:

**Corollary 1.** The divisor function \( d(n) \) is WUD\( (\mod{N}) \) if and only if one of the following conditions hold:

(i) \( N = 4 \),

(ii) \( N = 2 \cdot 3^a \) (\( a \geq 1 \)),

(iii) \( N = p^a \), \( p \) is an odd prime, \( a \geq 1 \) and 2 is a primitive root \( \mod{p^a} \),

(iv) \( N = 2p^a \), \( p \geq 5 \) is a prime, \( a \geq 1 \) and 3 is a primitive root \( \mod{p^a} \).

In all these cases \( N \) holds for \( j, N = 1 \) with \( C > 0 \) independent on \( j \) and \( m = \min p - 1 \).

(A part of this corollary, namely the result that \( d(n) \) is WUD\( (\mod{p^a}) \) provided that \( 2 \) is a primitive root \( \mod{p^a} \), is due to L. G. Sathe ([6]). Note, however, that his remark makes after Theorem 5 in ([6]) that in this case the values of \( d(n) \) are uniformly distributed in arithmetical progressions \( \delta^a + j \) (\( a = 1, 2, \ldots, p - 1 \)) is not correct for \( a > 1 \).)

**Corollary 2.** Euler's function \( \varphi(n) \) is WUD\( (\mod{N}) \) if and only if \( (N, 6) = 1 \). If this condition is satisfied then we have for \( j, N = 1 \)
\[ N \equiv \varphi(n) \equiv j(\mod{N}) \sim o(\log log N)^A \]
with \( A = \prod (p - 2)(p - 1) - 1 \) and \( C > 0 \) dependent on \( N \) only.

Let \( f \) \( \mathfrak{C}_k \) for some \( k \), but for all \( m \leq K(f) \) the set \( R_m(f, N) \) is void, then we show that the set \( \{ n \mid (f(n), N) = 1 \} \) must be small in some sense. In fact we prove

**Theorem II.** Let \( f \) \( \mathfrak{C}_k \) for some \( k \).

(i) If \( K(f) \) is finite and the sets \( R_i(f, N) \) are void for \( i = 1, 2, \ldots, K = K(f) \), then
\[ N \equiv \varphi(n) \equiv j(\mod{N}) = \Omega(x^{\delta^{K+1}}) \]
for every positive \( x \).

(ii) If \( K(f) \) is infinite and the sets \( R_i(f, N) \) are void for all \( i \), then
\[ N \equiv \varphi(n) \equiv j(\mod{N}) = \Omega(\log\log x) \]
where \( r \) is the number of distinct prime divisors of \( N \).

2. For the proof we need a theorem due to H. Delange, which we state as

**Lemma 1.** ([8], th. III). If \( a_n \) are nonnegative real numbers, and for every \( \epsilon > 0 \),
\[ \sum_{n \leq x} a_n n^{-s} = g_s(x)(s - a)^{-1} + \sum_{n \leq x} g_s(x)(s - x)^{-1}= h(s) \]
where \( b \) is a real number not equal to zero or a negative integer, \( g_0(s), g_1(s), \ldots, g_n(s), h(k) \) are regular in the closed half-plane \( \Re s \geq \sigma \), \( \Re a \neq 0 \), \( \Re b \neq 0 \) if \( j, k = 1, 2, \ldots, n \) and \( \Re k \neq 0, -1, -2, \ldots \) then for \( x \) tending to infinity one has

\[ \sum_{d \leq x} a_d \sim x^{1-1}(\log x)^{\frac{1}{k} - 1}. \]

We shall need also a corollary to this theorem, which we state as

**Lemma 2.** The assumptions are the same as in Lemma 1, except that the condition \( g_n(s) \neq 0 \) is replaced by \( g_k(s) = 0 \) for all \( s, n = 1/m \) and \( 0 \leq k \leq 1 \). Then for \( x \) tending to infinity one has

\[ a_n \sim \alpha \left( \frac{x}{\log x} \right)^{\frac{1}{k} - 1}. \]

**Proof of the lemma.** Let

\[ F(s) = \sum_{p \leq x} B_p \ll n^{-s}. \]

Clearly \( B_p \gg 0 \) and in view of \( 0 \leq b \leq 1 \) we have for \( s \geq 1 \) the equality \( F(s) = G(s)(s-1)^{-b} \) with \( G(s) \) regular for \( s \geq 1 \) and \( G(1) \neq 0 \). Consider

\[ F'(s) + \sum_{k=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} A_n n^{-s}. \]

Lemma 1 implies

\[ \sum_{n=1}^{\infty} B_p n^{-s} \sim \sum_{n=1}^{\infty} A_n n^{-s}. \]

which is clearly equivalent to the assertion of our lemma.

Now let \( N \geq 3 \), let \( \chi(n) \) be a character of the group \( (Z/N)^{\times} \), treated as a function defined for all integers. (For \( (d, N) = 1 \) we put \( \chi(d) = 0 \).)

Let \( \lambda_j \) be the number of solutions of the congruence \( W_a(x) = j (\text{mod } N) \) in \( x \in (Z/N)^{\times} \) if \( (j, N) = 1 \) and \( \lambda_0 = 0 \) if \( (j, N) = 1 \). (Here \( m \) is the number occurring in the statement of Theorem 1. Note that \( \lambda_j = \chi(j) \).

**Lemma 3.** For \( s > 1/m \)

\[ F(s, \chi) = H(s, \chi) \left( s - \frac{1}{m} \right)^{-\lambda_0} \]

where \( H(s, \chi) \) is regular in the closed half-plane \( \Re s > 1/m \) and does not vanish at \( s = 1/m \), and \( A(s) = \sum_{j=1}^{\infty} \chi(j) \lambda_j \), if for every prime \( p \)

\[ 1 + \sum_{f=1}^{\infty} \chi(f(p^f)) p^{-m} = \neq 0. \]

If the last condition is not satisfied, then \( A(s) \) is a complex number of the form

\[ \nu^{-1}(N) \sum_{j=1}^{N} \chi(j) \lambda_j = 1 \]

(1 \( \geq 1 \), integer).

**Proof of the lemma.** Notice first that for \( j = 1, 2, \ldots, m-1 \), \( [x, f^{(p^f)}] = 1 \) can hold with a prime \( P \) only if \( P \mid N \) and so \( \chi(f(n)) \neq 0 \) implies that every prime divisor of \( n \) either divides \( N \) or occurs in \( n \) with an exponent at least equal to \( m \). It follows that the series defining \( F(s, \chi) \) converges absolutely for \( \Re s > 1/m \). Indeed, it is majorized by the product

\[ \sum_{k=1}^{\infty} n^{-s} \sum_{m=1}^{\infty} \chi(m) \]

(\( \sigma = \Re s \) and \( A = \{ n \mid \chi(n) = p_{1}^{\sigma_1} \cdots p_{r}^{\sigma_r} \} \)).

Let \( B \) be the set of all \( m \)-full numbers, i.e., such numbers \( n \) for which \( p \mid n \) implies \( p^m \mid n \) of two series, the first of which is convergent for \( s > 0 \), and the second can be written in the form

\[ \sum_{m=1}^{\infty} \frac{\sum_{n \in B} \left( \frac{n}{\chi(n)} \right)^{s}}{n^{s}} \]

which implies its convergence for \( s > 1/m \) in view of the evaluation

\[ \sum_{n \in B} \frac{\left( \frac{n}{\chi(n)} \right)^{s}}{n^{s}} = O(s) \]

valid for every fixed positive \( s \) and arbitrarily positive \( s \). (This evaluation results immediately from

\[ \sum_{n \in B} \left( \frac{n}{\chi(n)} \right)^{s} = O(s) \]

(see [3]).

It follows that

\[ F(s, \chi) = P_{\rho} \left( \frac{1}{1 - \sum_{f=1}^{\infty} \chi(f(p^{f})) p^{-m}} \right) \text{ for } \Re s > 1/m. \]

Let \( P_{\rho} \) be the set of all primes which either divide \( N \) or are less than \( 2^{m+1} \), and let \( P_{\rho} \) be the set of all remaining primes. For \( p \notin P_{\rho} \) and \( s > 1/m \) we have (in view of \( \chi(f(p^f)) = \cdots = \chi(f(p^m)) = 0 \))

\[ \left| \sum_{n \in B} \chi(f(p^f)) p^{-m} \right| \leq \sum_{n \in B} p^{-m} = p^{-1}(1 - p^{-1/m}) < 2/p \leq 1 \]

and

\[ \sum_{n \in B} \chi(f(p^f)) p^{-m} \]

\[ \leq \exp \left[ \sum_{n \in B} \chi(f(p^f)) p^{-m} \right] \times \exp \left[ \sum_{n \in B} \chi(f(p^f)) p^{-m} \right] \]

\[ \times \exp \left[ \sum_{n \in B} \chi(f(p^f)) p^{-m} \right]. \]

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Now for \( 0 < \varepsilon < (m+1)^{-2} \), \( r_m \geq 1 \)\( / m - \varepsilon \)

\[
\left| \sum_{s=1}^{n} \frac{1}{s} \right| \leq \sum_{s=1}^{n} \frac{1}{s} \leq \sum_{s=1}^{\infty} \frac{1}{s^{(m+1)}} \leq \frac{1}{(m+1)^m - 1}
\]

hence the function

\[
\exp \left\{ \sum_{p \equiv 2 \mod{5}} \sum_{s=1}^{n} \frac{1}{s} \right\}
\]

is regular for \( r_m \geq 1 \) and does not vanish at \( s = 1 / m \), and moreover for \( r_m \geq 3 / 4 m \)

\[
\left| \sum_{s=1}^{n} \frac{1}{s} \right| \leq \sum_{s=1}^{\infty} \frac{1}{s^{4/3m}} \leq 2 m^{-1/2}
\]

with a suitable \( B > 0 \). Consequently the function

\[
\exp \left\{ \sum_{p \equiv 2 \mod{5}} \sum_{s=1}^{n} \frac{1}{s} \right\}
\]

is regular for \( r_m \geq 1 \) and does not vanish at \( s = 1 / m \).

Finally

\[
\sum_{p \equiv 2 \mod{5}} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right) \leq \sum_{p \equiv 2 \mod{5}} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right)
\]

a regular for \( r_m \geq 1 \) and does not vanish at \( s = 1 / m \).

Finally

\[
\sum_{p \equiv 2 \mod{5}} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right) \leq \sum_{p \equiv 2 \mod{5}} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right)
\]

is regular for \( r_m \geq 1 \) and does not vanish at \( s = 1 / m \).

\[
\prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right) \leq \prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

with \( g(s) \) regular for \( r_m \geq 1 \), we get for \( r_m \geq 1 

(3)

\[
\prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right) \leq \prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

with \( g(s) \) regular for \( r_m \geq 1 \) and \( g(s) \) at \( s = 1 / m \). The product

\[
\prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

defines obviously a function regular for \( r_m \geq 0 \), which does not vanish identically and may thus be written in the form \( g(s) \) at \( s = 1 / m \).

On distribution of values of multiplicative functions

Observe now that \( M \neq 0 \) holds if and only if for some \( p \) in \( P_1 \)

\[
1 + \sum_{p \equiv 2 \mod{5}} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right) = 0
\]

and consequently (2) and (3) imply the lemma.

Proof of Theorem I. For \( j, N \) = 1 we get by Lemma 3

\[
\sum_{j=1}^{N} \frac{1}{s} \left( \sum_{s=1}^{n} \frac{1}{s} \right) \leq \prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

As obviously

\[
\prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

we can write

\[
\prod_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

where \( X \) is the set of all characters of \( G_N \) with \( A(\chi) = A(x_0) \), \( g_1(s), \ldots, g_L(s), h(S) \) are regular for \( r_m \geq 1 \) and \( r_m \geq 1 

Note that \( X \) at \( X \) if and only if \( g_1(s) \neq 0 \) implies \( g_1(s) = 1 \) and moreover (by Lemma 2) for all primes \( p \)

\[
1 + \sum_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

If for every nonprincipal character \( \chi \) of \( G_N \), which is trivial on \( A_m \) (and a fortiori on \( R_m \) there exists a prime \( p \) with

\[
1 + \sum_{p \equiv 2 \mod{5}} \left( 1 + \sum_{s=1}^{n} \frac{1}{s} \right)
\]

then \( X \) consists of the principal character exclusively, and by (4) and Lemma 1 we get for \( j, N \) = 1

\[
N(x \leq s) \sim C x^{(m+1)} (\log x)^{\alpha - 1}
\]

where \( C \) does not depend on \( j \), and this means that the sequence \( f(1), f(2), \ldots \) is WUD(mod N). The first part of Theorem I is thus proved.
Now assume that the sequence \( f(1), f(2), \ldots \) is \( \text{WUD}(\text{mod}, N) \). Applying first Lemma 3 to the principal character \( \chi_0 \) of \( G_N \), and then

\[
N_{\alpha}(x) = \sum_{n \leq x} \varphi(n) \alpha(n) = 1
\]

we get for \((j, N) = 1\)

\[
N_{\alpha}(x) = \sum_{n \leq x} \varphi(n) \alpha(n) = 1
\]

as otherwise we would get by Lemma 2 from (4)

\[
N_{\alpha}(x) = o\left(\frac{\omega(m) \log \omega}{\omega(m) \log N}\right)
\]

contrary to the evaluation just obtained. Consequently (4) and Lemma 1 lead us to

\[
N_{\alpha}(x) \sim m \sum_{n \leq x} \varphi(n) \alpha(n) \sim \sum_{n \leq x} \alpha(n) = \sum_{n \leq x} \varphi(n)
\]

whence

\[
\sum_{n \leq x} \varphi(n) \alpha(n) = H(x, 1/m)
\]

if \((j, N) = 1\).

Let now the set \( \{j_1, \ldots, j_t\} \) be a set of representatives of \( \mathbb{Z}/\mathbb{Z}\) in \( G_N \). Then in view of the last equality the system

\[
\sum_{n \leq x} \varphi(n) \alpha(n) = H(x, 1/m)
\]

has in case \(|X| \geq 2\) at least two distinct solutions: \( \alpha(x) = H(x, 1/m) \), \( \alpha(x) = 0 \) for \( x \neq x \), and \( \alpha(x) = H(x, 1/m) \) for \( x \neq x \). But the matrix

\[
\begin{bmatrix}
\chi(j_1) & \cdots & \chi(j_t)
\end{bmatrix} \begin{bmatrix}
\varphi(1) & \cdots & \varphi(t)
\end{bmatrix}
\]

is of rank \(|X|\) which gives a contradiction. Consequently \( X \) consists solely of the principal character and this means that for every nonprincipal character of \( G_N \) which is trivial on \( A_m \) there exists a prime \( p \) such that

\[
1 + \sum_{n \leq x} \varphi(n) p^{-nm} = 0
\]

This proves the second part of Theorem I.

Proof of Theorem II. If \( K(f) \) is finite and \( R_i(f, N) \) is void for \( i = 1, 2, \ldots, K(f) = K \), then as in proof of Lemma 2 we conclude that the function \( F(x, s) \) is regular for \( s > 1/(K + 1) \) and by Ikehara's theorem one gets easily (see e.g. [3], Lemma 4) that

\[
N \{ n \leq x; f(n), N \} = 1 \}

for every positive \( s \), thus proving part (i) of Theorem II.

To prove part (ii) note that if \( K(f) \) is infinite and all sets \( R_i(f, N) \) are void, then \( f(n), N \} = 1 \) implies that all prime divisors of \( N \) must divide \( N \) and consequently

\[
N \{ n \leq x; f(n), N \} = 1 \}

Theorem II is thus proved.

3. In this section we prove the corollaries.

Proof of Corollary 1. Let \( f(n) = d(n) \) we have obviously

\[
K(f) = \infty, \quad \mathbb{W}(d) = \pm 1, \quad \text{and for odd } N, R_i = \{2\}, \quad \text{whereas for } N \text{ even } R_i = \ldots = R_{m-1} = \{2\}, \quad R_m = \{1 + m\} \quad \text{if } m + 1 \text{ is the least prime not dividing } N. \text{ Observe that for } |x| < 1, x \neq 0
\]

\[
1 + \sum_{m \leq x} \varphi(p)^k = 1 + x - 1 + \sum_{m \leq x} \varphi(p)^k = 1 + x - 1 + \sum_{m \leq x} \varphi(p)^k
\]

\[
= 1 + x - 1 + \sum_{m \leq x} \varphi(p)^k = 1 + x - 1 + \sum_{m \leq x} \varphi(p)^k
\]

If thus for some character \( \chi \) of \( G_N \) and for some prime \( p \)

\[
1 + \sum_{m \leq x} \varphi(p)^k = 0
\]

then the polynomial \( M(z) = \sum_{m \leq x} \varphi(p)^k \) has \( z = p^{-km} \) as a root, but all coefficients of \( M(z) \) are units of the field \( \mathbb{Q}(\exp(2\pi \sqrt{N})) \) and so all its roots are algebraic integers, whereas \( z = p^{-km} \) is not, a contradiction.

It follows that the sequence of values of \( d(n) \) will be \( \text{WUD}(\text{mod}, N) \) if and only if \( A_m \) coincides with \( G_N \) and as \( A_m \) is a cyclic group generated by the prime \( 1 + m \), this means the same as the fact that the least prime not dividing \( N \) is a primitive root \( \text{mod} N \). It is easy to check that all the numbers \( N \geq 3 \) satisfying this condition are those listed in the statement of Corollary 1. The evaluation given there is immediate, as \( A(x) = 1 \).

Proof of Corollary 2. Let \( f(n) = \varphi(n) \) we have \( K(f) = \infty \) and \( W(x) = x^{-\frac{\omega(x)}{2}}(x-1) \). As \( \varphi(n) \) is even for \( n \geq 3 \), the sequence \( \varphi(n) \) cannot be \( \text{WUD}(\text{mod}, N) \) for \( N \) even. Let thus \( N \) be odd. In this case

\[
R_i = \{r \mid 1 \leq r \leq N-1, \ (r, N) = (r+1, N) = 1\}
\]
and is not void as $1_B$. Observe that for $|z| < 1$

$$1 + \sum_{z=1}^\infty x(y(p))z^d = 1 + \sum_{z=1}^\infty x(p-1)x(p-1)z = 1 + x(p-1)(1-x(p))z^{-1}z.$$ 

If this is zero for $z = p^{-1}$, then $x(p) - x(p-1) = 1$ which implies $p = 2$, but then $3 = \chi(2)$ which is impossible. It results, that $\varphi(n)$ is WUD(mod $N$) if and only if $A_i$ coincides with $G_n$. Let $N = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ ($p_i \neq 2$). Then the group $G_N$ is a product of $G_{p_i}$ for $p_i \neq 2$. Now we may represent every element $y$ of $G_N$ in the form $(y_1, \ldots, y_r)$ with $1 \leq y_i < p_i^\alpha_i$, $p_i | y_i$ and $y = y_i (\text{mod } p_i^\alpha_i)$. In this notation $A_i$ is the group generated by the set

$$\{(y_1, \ldots, y_r) \mid y_i \in G_{p_i}, \ y_i \neq -1(\text{mod } p_i)\}.$$ 

If $N$ is divisible by $3$, say $p_2 = 3$, then $A_2 \neq G_N$, as $[2, 1, 1, \ldots, 1] \neq G_N$ but is not contained in $A_1$.

Now let $(N, 6) = 1$. We have to prove that $A_2 = G_2$. Let $y = (y_1, \ldots, y_r) \\ G_2$, and let $x_i$ be a solution of the congruence $2x_i = y_i (\text{mod } p_i^\alpha_i)$. Put

$$x_i = \begin{cases} y_i & \text{if } y_i \neq -1(\text{mod } p_i) \\ 2 & \text{if } y_i \equiv -1(\text{mod } p_i) \end{cases}$$

and

$$x_i = \begin{cases} 1 & \text{if } y_i \neq -1(\text{mod } p_i) \\ x_i & \text{if } y_i \equiv -1(\text{mod } p_i). \end{cases}$$

Evidently $[x_1, x_2, \ldots, x_r] \in R_1$ as $p_2 > 3$ and so $2 \neq -1(\text{mod } p_2)$. If $x_i = -1(\text{mod } p_i)$ then $-1 = y_i = 2x_i = -2 (\text{mod } p_i)$ a nonsense, consequently $[x_1, \ldots, x_r] \in R_2$. But obviously $y = [x_1, \ldots, x_r] (\text{mod } A_1)$ and so $G_2 = A_2$. The Corollary 2 is thus proved because the evaluation stated in it follows immediately from the fact that

$$A(x_0) = \varphi^{-1}(N) \sum_{y_1} \mathbb{N} \frac{x}{(y_0, y_1) = \prod_{p \mid y_0} (p-1)/(p-1).}$$

(Cf. [2], vol. I, p. 147).

References

