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A generalization of a theorem of Landau

by

Indar S. Luthar (Urbana, Ill.)

In 1908 Landau [1] proved that the number $B(x)$ of integers $\leqslant x$ which are representable as the sum of two squares is asymptotic to $\frac{bx}{\sqrt{\log x}}$ with

$$b = \frac{1}{\sqrt{2}} \prod_r (1-r^{-2})^{-1/2}$$

the product being taken over all primes congruent to 3 mod 4. In his book on Ramanujan, Hardy remarks that by pushing the analysis a little harder this result can be improved to obtain an asymptotic expansion for $B(x)$ in the sense of Poincaré, namely

$$B(x) = \frac{bx}{\sqrt{\log x}}\left[1+\frac{c}{\log x}+\frac{d}{\log^2 x}+\ldots\right]. \tag{1}$$

An incorrect value for c was given by Stanley [3] and the correct one was later given by Shanks [2].

We now notice that a rational integer is representable as the sum of two squares if and only if it is the norm of a Gaussian integer. It is natural to ask whether we can obtain a result similar to (1) for the number of rational integers $\leqslant x$ which are norms of integers of an algebraic number field k. In the present paper, we restrict ourselves to the case when k is a quadratic field and obtain a formula for the number $B_k(x)$ of integers $\leqslant x$ which are norms of totally positive numbers of k. It may be remarked that our result does generalize the results of Landau and Shanks, since, being totally positive is no restriction in imaginary quadratic field and, since, a rational integer which is the norm of a number of a quadratic field of class number (in the restricted sense) 1 or 2 is, in fact, the norm of an integer of that field. To prove the second assertion, let

$$\alpha = \mathfrak{p}^{a}\mathfrak{p}'^{a'}\ldots$$

have integral norm; then $a+a', \ldots$ are all non-negative, so that

$$\mathfrak{a} = \mathfrak{p}^{a+a'}\ldots$$

is an integral ideal having the same norm as α. Moreover $\mathfrak{a}$ is principal since

$$\alpha\mathfrak{a} = \mathfrak{p}^{2a}(\mathfrak{p}\mathfrak{p}')^{a'}\ldots$$

is principal.

Let then k be a quadratic field with discriminant d and let $d = d_1\ldots d_g$ be the decomposition of d into prime discriminants. Let χ be a real character of the group of ideal classes (in the restricted sense) of k; then, as is well-known, χ corresponds in a 1-1 manner to a factorization

$$d = uv \tag{2}$$

of d into two discriminants u and v (we fix the order of the factorization by requiring that d_g does not occur in u). The relation between χ and the factorization (2) is expressed by

$$\chi(\mathfrak{p}) = \begin{cases} \left(\dfrac{u}{N(\mathfrak{p})}\right) & \text{if} \quad (u, \mathfrak{p}) = 1, \\ \left(\dfrac{v}{N(\mathfrak{p})}\right) & \text{if} \quad (v, \mathfrak{p}) = 1, \end{cases} \tag{3}$$

$\mathfrak{p}$ being any prime ideal of k. Obviously $\chi(\mathfrak{a})$ depends only on $N(\mathfrak{a})$. We let

$$f(s, \chi) = \sum_{n=1}^{\infty} b(n, \chi) n^{-s} \qquad (\sigma > 1) \tag{4}$$

where

$$b(n, \chi) = \begin{cases} 0 & \text{if } n \text{ is not the norm of any ideal,} \\ \chi(\mathfrak{a}) & \text{if } n = N(\mathfrak{a}). \end{cases} \tag{5}$$

Clearly

$$f(s, \chi) = \prod_{l|d} (1-\chi(\mathfrak{l})l^{-s})^{-1} \prod_{\left(\frac{d}{q}\right)=+1} (1-\chi(\mathfrak{q})q^{-s})^{-1} \prod_{\left(\frac{d}{r}\right)=-1} (1-\chi(\mathfrak{r})r^{-2s})^{-1}.$$

Here l runs through all the primes which divide d, q through those primes for which $\left(\dfrac{d}{q}\right) = +1$ and r through those primes for which $\left(\dfrac{d}{r}\right) = -1$; moreover $\mathfrak{l}$, $\mathfrak{q}$ and $\mathfrak{r}$ denote prime divisors in k of l, q and r respectively [1]. Taking account of (3) we obtain:

$$f(s, \chi) = \prod_{l|d} \left(1-\left(\frac{u}{l}\right)l^{-s}\right)^{-1}\left(1-\left(\frac{v}{l}\right)l^{-s}\right)^{-1} \prod_{\left(\frac{d}{q}\right)=+1} \left(1-\left(\frac{u}{q}\right)q^{-s}\right)^{-1/2} \times$$

$$\times \prod_{\left(\frac{d}{q}\right)=+1} \left(1-\left(\frac{v}{q}\right)q^{-s}\right)^{-1/2} \prod_{\left(\frac{d}{r}\right)=-1} (1-r^{-2s})^{-1}.$$

[1] There are two choices for $\mathfrak{q}$, but it does not matter which one we take.

Setting

$$\begin{cases} L_u(s) = \prod_p \left(1-\left(\dfrac{u}{p}\right)p^{-s}\right)^{-1} \\ L_v(s) = \prod_p \left(1-\left(\dfrac{v}{p}\right)p^{-s}\right)^{-1} \end{cases} \qquad (\sigma > 1), \tag{6}$$

we get

$$\frac{f^2(s, \chi)}{L_u(s)L_v(s)}$$
$$= \prod_{l|d} \left(1-\left(\frac{u}{l}\right)l^{-s}\right)^{-1}\left(1-\left(\frac{v}{l}\right)l^{-s}\right)^{-1} \prod_{\left(\frac{d}{r}\right)=-1} \frac{(1-r^{-2s})^{-2}}{\left(1-\left(\frac{u}{r}\right)r^{-s}\right)^{-1}\left(1-\left(\frac{v}{r}\right)r^{-s}\right)^{-1}}.$$

Since

$$\left(\frac{u}{r}\right)\left(\frac{v}{r}\right) = \left(\frac{d}{r}\right) = -1,$$

so that

$$\left(1-\left(\frac{u}{r}\right)r^{-s}\right)^{-1}\left(1-\left(\frac{v}{r}\right)r^{-s}\right)^{-1} = (1-r^{-2s})^{-1},$$

we have

$$f^2(s, \chi) \tag{7}$$
$$= \prod_{l|d} \left(1-\left(\frac{u}{l}\right)l^{-s}\right)^{-1}\left(1-\left(\frac{v}{l}\right)l^{-s}\right)^{-1} \prod_{\left(\frac{d}{r}\right)=-1} (1-r^{-2s})^{-1} L_u(s)L_v(s).$$

Let $\bar{G}$ be the region:

$$\sigma \geqslant 1 - \frac{c}{\log|t|} \ (\geqslant 3/4) \quad \text{for} \quad |t| \geqslant 3,$$

$$\sigma \geqslant 1 - \frac{c}{\log 3} \ (\geqslant 3/4) \quad \text{for} \quad |t| \leqslant 3$$

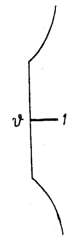

Fig. 1

and let G be the region $\bar{G}$ cut along the real axis from $\vartheta = 1 - \dfrac{c}{\log 3}$ to 1. Then, as is well-known, for a sufficiently small $c > 0$, the function $L_u(s)L_v(s)$ is regular in $\bar{G}$ except when $u = 1$, $v = d$ in which case it is regular everywhere in $\bar{G}$ except at $s = 1$ where it has a simple pole; moreover $L_u(s)L_v(s)$ has no zeroes in $\bar{G}$ and

$$L_u(s)L_v(s) = O(\log^2|t|) \qquad (s \in G,\ |t| \geqslant 3).$$

To calculate the residue of the function $L_1(s)L_d(s)$ at $s=1$ we notice that by Kronecker's formula

$$L_u(s)L_v(s) = \zeta_d(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})N(\mathfrak{a})^{-s}$$

so that

$$L_1(s)L_d(s) = \zeta_d(s,\chi_0) = \sum N(\mathfrak{a})^{-s}.$$

Consequently $L_1(s)L_d(s)$ has residue $h\varkappa$ at $s=1$, where h is the class number of the field and $\varkappa$ is the usual constant associated to k.

These considerations and formula (7) immediately give the following

LEMMA. *The function $f(s,\chi)$ is regular in G; in fact for $\chi\neq\chi_0$, i.e., for $u\neq 1$, $f(s,\chi)$ is regular in $\bar{G}$.*

Moreover for any χ,

$$f(s,\chi) = O(\log|t|) \qquad (s\in G,\ |t|\geqslant 3). \tag{8}$$

Finally $f(s,\chi_0)\sqrt{s-1}$ is regular in $\bar{G}$ and its expansion at $s=1$ begins with the constant

$$a = \sqrt{h\varkappa}\prod_{l|d}(1-l^{-1})^{-1/2}\prod_{\left(\frac{d}{r}\right)=-1}(1-r^{-2})^{-1/2}. \tag{9}$$

Here $\sqrt{s-1}$ is to take positive values for $s>1$.

Let now $x>\sqrt{3}$. We have

$$\sum_{n\leqslant x} b(n,\chi)\log\frac{x}{n} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^s}{s^2}f(s,\chi)\,ds = \frac{1}{2\pi i}\int_{2-ix^2}^{2+ix^2}\frac{x^s}{s^2}f(s,\chi)\,ds + O(1).$$

Changing the path of integration, we get

$$\sum_{n\leqslant x} b(n,\chi)\log\left(\frac{x}{n}\right) \tag{10}$$
$$= -\frac{1}{2\pi i}\left(\int_B^C+\int_C^D+\int_D^E+\int_E^F+\int_F^E+\int_E^G+\int_G^H+\int_H^A\right)\frac{x^s}{s^2}f(s,\chi)\,ds + O(1).$$

Here we are integrating $\frac{x^s}{s^2}f(s,\chi)$ along the upper edge of the cut in the fourth integral on the right and along the lower edge in the fifth.

Estimating the remaining six integrals on the right with the help of (8), we obtain, from (10), with a suitable $a>0$,

$$\sum_{n\leqslant x} b(n,\chi)\log\frac{x}{n} = -\frac{1}{2\pi i}\int_{EFE}\frac{x^s}{s^2}f(s,\chi)\,ds + O(xe^{-a\sqrt{\log x}}). \tag{11}$$

The integral on the right vanishes for non-principal χ, while for $\chi=\chi_0$, it equals

$$-2\int_{\vartheta}^{1}\frac{x^s}{s^2}f(s,\chi)\,ds,$$

the integration being performed along the lower edge. Thus, (11) gives

$$\sum_{n\leqslant x} b(n,\chi)\log\frac{x}{n} = \frac{\delta_\chi}{\pi i}\int_{\vartheta}^{1}\frac{x^s}{s^2}f(s,\chi)\,ds + O(xe^{-a\sqrt{\log x}}) \tag{12}$$

where

$$\delta_\chi = \begin{cases}1 & \text{if} \quad \chi=\chi_0,\\ 0 & \text{if} \quad \chi\neq\chi_0.\end{cases}$$

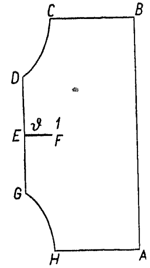

Fig. 2

We now estimate

$$\frac{1}{\pi i}\int_{\vartheta}^{1}\frac{x^s}{s^2}f(s,\chi_0)\,ds.$$

Since the integration is to be taken along the lower edge, we have to take, for s between ϑ and 1, the following expansion:

$$\frac{f(s,\chi_0)}{s^2} = \frac{ai}{\sqrt{1-s}} + a_1 i(1-s)^{1/2} + a_3 i(1-s)^{3/2} + \ldots \tag{13}$$

where $\sqrt{1-s}$ is to take positive values for $\vartheta\leqslant s\leqslant 1$. Retaining the terms up to (and including) $a_{2m-1}(1-s)^{m-1/2}$ and making simple estimates we get:

$$\frac{1}{\pi i}\int_{\vartheta}^{1}\frac{x^s}{s^2}f(s,\chi_0)\,ds$$
$$= \frac{x}{\pi\sqrt{\log x}}\left[a\Gamma\left(\frac{1}{2}\right) + \frac{a_1\Gamma(3/2)}{\log x} + \ldots + \frac{a_{2m-1}\Gamma(m+1/2)}{(\log x)^m}\right] + O\left(\frac{x}{(\log x)^{m+3/2}}\right).$$

This, together with (12), gives

$$\sum_{n\leqslant x} b(n,\chi)\log\frac{x}{n} \tag{14}$$
$$= \frac{\delta_\chi x}{\pi\sqrt{\log x}}\left[a\Gamma\left(\frac{1}{2}\right) + a_1\frac{\Gamma(3/2)}{\log x} + \ldots + \frac{a_{2m-1}\Gamma(m+1/2)}{(\log x)^m}\right] + O\left(\frac{x}{(\log x)^{m+3/2}}\right).$$

We now let

$$b_n = 2^{1-g}\sum_{\chi} b(n,\chi) \tag{15}$$

where, in the summation, χ runs through all the real characters of the group of ideal classes of k. We now notice that the real characters of the ideal class group in the restricted sense of k are exactly the characters of the group of genera in the restricted sense of k. By the theorem of Gauss the number of genera in k is 2^{g-1}. It follows that $b_n = 0$ unless n is the norm of an ideal in the principal genus, i.e., the norm of a totally positive number of k, in which case $b_n = 1$. Thus the number $B_k(x)$ of rational integers $\leqslant x$ which are norms of totally positive numbers of k is given by

$$B_k(x) = \sum_{n\leqslant x} b_n. \tag{16}$$

Summing (14) over all real χ, we get

$$\sum_{n\leqslant x} b_n \log\frac{x}{n} = \frac{2^{1-g}}{\pi}\cdot\frac{x}{\sqrt{\log x}}\times \tag{17}$$

$$\times\left[a\Gamma\left(\frac{1}{2}\right)+a_1\frac{\Gamma(3/2)}{\log x}+\ldots+a_{2m-1}\frac{\Gamma(m+1/2)}{(\log x)^m}\right]+O\left(\frac{x}{(\log x)^{m+3/2}}\right).$$

Taking $m = 4$, we easily obtain the following

THEOREM. *Let k be a quadratic field with discriminant d and let $B_k(x)$ denote the number of rational integers $\leqslant x$ which are norms of totally positive numbers of k. Then*

$$B_k(x) = \frac{2^{1-g}}{\sqrt{\pi}}\cdot\frac{x}{\sqrt{\log x}}\left[a+\frac{a_1-a}{2\log x}\right]+O\left(\frac{x}{\log^{5/2}x}\right).$$

Here g denotes the number of distinct rational primes dividing the discriminant d and the constant a is given by (9).

We remark that for imaginary quadratic fields we can give an explicit expression for a_1 using the first limit formula of Kronecker.

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Estimates of $\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}$

by

A. H. KRUSE (New Mexico)

1. Introduction. Throughout this paper, ω is the set of all nonnegative integers, R is the set of all real numbers, and $R_{\rm irr}$ is the set of all irrational real numbers. For each $x\in R$, $[x]$ is the largest integer not greater than x, and $\operatorname{fr}(x) = x-[x]$ is the fractional part of x. For each $x\in R$, we define

$$\langle x\rangle = \min\bigl(\operatorname{fr}(x), \operatorname{fr}(-x)\bigr),$$

whence $\langle x\rangle$ is the distance between x and the set of integers.

If, in some context, "A" and "B" are expressions and C is a condition (perhaps a conjunction of several conditions) on whatever variables may appear in "A" and "B", then "$A = O(B)$ *under* (or *relative to*, or *for*) C" means as usual that A/B subject to C is bounded (here $0/0 = 0$ and $A/0 = \infty$ if $A \neq 0$), and "$A \asymp B$ *under* (or *relative to*, or *for*) C" will mean that $A = O(B)$ under C and $B = O(A)$ under C. Sometimes instead of "$A = O(B)$ under C" or "$A\asymp B$ under C" we shall write "$A = O(B)$ (C)" or "$A\asymp B$ (C)" respectively.

Throughout this paper, k, m, n, N, and p are restricted without additional mention to be integers.

(Consider the condition C: $x\in R_{\rm irr}$, $N\geqslant 1$, $s\in R$, and $t\in T$.) This paper is concerned with estimating sums of the form

$$\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}\ \Bigl(\asymp \sum_{k=1}^{N} k^{-s}|\sin k\pi x|^{t} \text{ under C if } T \text{ is a bounded subset of } R\Bigr) \tag{1}$$

with s and t nonnegative real numbers. The writer knows of no general treatment of this problem. Hardy and Littlewood ([3], pp. 216-217) showed that if $s\geqslant t\geqslant 1$, then, for each $x\in R_{\rm irr}$ such that

$$1 = O(j^s\langle jx\rangle^t)\quad (j\in\omega\setminus\{0\}), \tag{2}$$

the sum in (1) is $O\bigl((\log N)^2\bigr)$ for $N\geqslant 2$. As remarked in [3], p. 214, (2) is equivalent to

$$q_{n+1}(x) = O\bigl(q_n(x)^{s/t}\bigr)\quad (n\in\omega), \tag{3}$$