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On the difference of consecutive terms of sequences defined by divisibility properties

by

P. ERDÖS (Urbana, Illinois)

Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers satisfying $\sum_i \frac{1}{a_i} < \infty$; denote by $b_1 < b_2 < \ldots$ the sequence of integers not divisible by any a. It is well known that the b's have positive density and hence it follows that $b_{i+1}/b_i \to 1$. It is easy to see that this result is best possible as it stands, in other words, that there is no function $f(x)$ tending to infinity as $x \to \infty$, so that for $x > x_0$ there always is a b in $(x, x+x/f(x))$. To see this let $\sum_k \varepsilon_k < \infty$ and let $n_k \to \infty$ sufficiently fast. Let the sequence a_i consist of the integers in the intervals $(n_k, n_k(1+\varepsilon_k))$, $k = 1, 2, \ldots$ Clearly

$$\sum_k \frac{1}{a_k} < \sum_k \varepsilon_k < \infty$$

and if $n_k \to \infty$ so fast that $\varepsilon_k n_k > n_k/f(n_k)$, then the interval $(n_k, n_k + n_k/f(n_k))$ clearly contains no b's. On the other hand if we assume that the a's are pairwise relatively prime, we can make very much stronger statements about $b_{i+1}-b_i$. In fact, we shall prove the following theorems. (Throughout this paper $c, c_1, \ldots$ will denote positive absolute constants.)

THEOREM 1. *Let*

$$(1) \qquad \sum_i \frac{1}{a_i} < \infty, \qquad (a_i, a_j) = 1.$$

Then there is an absolute constant c (independent of our sequence $a_1 < \ldots$) so that for all sufficiently large x the interval $(x, x+x^{1-c})$ contains b's.

Theorem 1 can probably be improved a great deal and quite possibly $b_{i+1}-b_i = o(b_i^{\varepsilon})$ holds for every $\varepsilon > 0$ if $i > i_0(\varepsilon)$. On the other hand I shall show that there is a definite limit to the improvement of Theorem 1.

Theorem 2. *There is a sequence $a_1 < \ldots$ satisfying* (1) *so that for infinitely many i* ($\exp z = e^z$)

$$b_{i+1} - b_i > \exp\left(\tfrac{1}{4}(\log b_i \log\log b_i)^{1/2}\right).$$

Denote by $B(u, v)$ the number of b's in the interval (u, v), and let a be the density of the sequence $b_1 < \ldots$

Theorem 3. *Let $f(x)/x^{1-\varepsilon} \to \infty$ for every $\varepsilon > 0$. Then*

$$B(x, x+f(x)) = (a+o(1))f(x). \tag{2}$$

Theorem 3 is best possible. Assume that there is a sequence $x_k \to \infty$ so that there is an $\varepsilon > 0$ for which $f(x_k) < x_k^{1-\varepsilon}$. Then there is a sequence $a_1 < \ldots$ satisfying (1) so that (2) does not hold.

Before we prove our theorems we discuss a special case. Let the a's be the squares of primes; then the b's are the squarefree numbers $q_1 < \ldots$ The problem of estimating the maximum possible order of $q_{i+1} - q_i$ is very difficult. On the one hand it is known [2] that for every $\varepsilon > 0$ and infinitely many i

$$q_{i+1} - q_i > (1-\varepsilon)\frac{\pi^2}{12}\log q_i/\log\log q_i \tag{3}$$

and on the other hand [5]

$$q_{i+1} - q_i = o(q_i^{c+\varepsilon}), \quad \text{where} \quad c = \frac{109556}{494419} = 0.22158534.$$

It seems certain that $q_{i+1} - q_i = o(q_i^{\varepsilon})$ for every $\varepsilon > 0$ but this must be very deep.

Let $a_1 < \ldots$ be any sequence satisfying (1). Let

$$a_1 \ldots a_i \leqslant x < a_1 \ldots a_i a_{i+1}. \tag{4}$$

Using the Chinese remainder theorem and an elementary sieve process, it is easy to see that to every $\varepsilon > 0$ there is an $x_0 = x_0(\varepsilon)$ so that for every $x > x_0(\varepsilon)$ there is a $b_j < x$ for which

$$b_{j+1} - b_j > (1-\varepsilon) i \prod_{r=1}^{\infty}\left(1 - \frac{1}{a_r}\right)^{-1}, \tag{5}$$

where i is defined by (4). The proof of (5) which is very similar to that of (3) will not be given in this paper.

In general, (5) is best possible. It is not difficult to construct an infinite sequence $a_1 < \ldots$ satisfying (1) so that for every $\varepsilon > 0$ and $x > x_0(\varepsilon)$ every interval

$$\left(t, t + (1+\varepsilon) i \prod_{r=1}^{\infty}\left(1 - \frac{1}{a_r}\right)^{-1}\right), \quad t < x \tag{6}$$

contains a b_j (i is given by (4)).

All the sequences $a_1 < \ldots$ which I constructed to satisfy (6) increase very fast. I do not know to what an extent this is necessary. I could not prove that there is a sequence satisfying (1) and $a_k < k^2$ say, so that $b_{i+1} - b_i = o(b_i^{\varepsilon})$ for every $\varepsilon > 0$.

Now we prove Theorem 1. We use an idea of Estermann-Roth [4]. We need the following

Lemma. *Let $m < d_1 < \ldots < d_t < m+y$ be a sequence of integers satisfying $(d_i, d_j) = 1$, $1 \leqslant i < j \leqslant t$. Put $\max t = R(m, y)$. Then*

$$R(m, y) < c_1 y/\log y.$$

Clearly for each $p < y$ there can be at most one d_i which is a multiple of p. Hence by a simple argument ($\pi(y)$ denotes the number of primes $\leqslant y$ and $A(m, y)$ denotes the number of integers $m < u < m+y$ all whose prime factors are greater than y) we have

$$R(m, y) \leqslant \pi(y) + A(m, y). \tag{7}$$

Now by a well-known result (easily deduced by Brun's method),

$$A(m, y) < c_2 y/\log y. \tag{8}$$

Lemma 1 immediately follows from (7) and (8).

Now we are ready to prove Theorem 1. Let $c < \min 1/(2+2c_1)$ and $k = k(c)$ a sufficiently large integer. Denote by $I_1(x)$ the number of integers $x < t < x + x^{1-c}$ for which

$$t \equiv 0 \pmod{a_i} \quad \text{for some} \quad 1 \leqslant i \leqslant k.$$

$I_2(x)$ denotes the number of integers $x < t < x + x^{1-c}$ for which

$$t \equiv 0 \pmod{a_i} \quad \text{for some} \quad a_k < a_i \leqslant x^{1-c},$$

and finally $I_3(x)$ denotes the number of integers $x < t < x + x^{1-c}$ satisfying

$$\begin{cases} t \equiv 0 \pmod{a_j} & \text{for some} \quad x^{1-c} < a_j < x + x^{1-c}, \quad \text{but} \\ t \not\equiv 0 \pmod{a_i} & \text{for all} \quad a_i \leqslant x^{1-c}. \end{cases} \tag{9}$$

We evidently have

$$B(x, x+x^{1-c}) \geqslant x^{1-c} - I_1(x) - I_2(x) - I_3(x). \tag{10}$$

A simple sieve argument shows (using $\sum \frac{1}{a_i} < \infty$) that for every $\eta > 0$ if k is sufficiently large ($k > k_0(\eta)$), then

$$I_1(x) < x^{1-c}(1-a+\eta). \tag{11}$$

Again using $\sum \frac{1}{a_i} < \infty$ we obtain that for $k > k_0(\eta)$

$$I_2(x) \leqslant \sum_{a_k < a_i \leqslant x^{1-c}} \left(\left[\frac{x+x^{1-c}}{a_i} \right] - \left[\frac{x}{a_i} \right] \right) < x^{1-c} \sum_{i>k} \frac{1}{a_i} + \sum_{a_i \leqslant x^{1-c}} 1 < \eta x^{1-c}. \tag{12}$$

Since, by $\sum \frac{1}{a_i} < \infty$, $\sum_{a_i < y} 1 = o(y)$. In the estimation of $I_1(x)$ and $I_2(x)$ we did not use $(a_i, a_j) = 1$. This condition will be needed in the estimation of $I_3(x)$.

By assumption, we have $c < \frac{1}{2}$. The integers t satisfying (9) must then be of the form $b_i a_j$ where

$$x < b_i a_j < x + x^{1-c}, \quad a_j > x^{1-c}. \tag{13}$$

To see this observe that if t satisfies (9) we must have $t \equiv 0 \pmod{a_j}$ for some $a_j > x^{1-c}$ and by $c < \frac{1}{2}$, $t/a_j \leqslant x^c$, whence by (9), t/a_j must be a b, say b_i.

For fixed i the number of integers of the form (13) is by Lemma 1 clearly not greater than (now we use $(a_i, a_j) = 1$)

$$\sum_{\frac{x}{b_i} < a_j < \frac{x+x^{1-c}}{b_i}} 1 \leqslant R\left(\frac{x}{b_i}, \frac{x^{1-c}}{b_i}\right) < c_1 \frac{x^{1-c}}{\log x/b_i} < c_1 \frac{x^{1-c}}{(1-2c)\log x - 1}. \tag{14}$$

Hence by (14) the number of integers satisfying (13) is less than

$$\frac{c_1 x^{1-c}}{(1-2c)\log x - 1} \sum_{b_i < x^c + 1} \frac{1}{b_i}. \tag{15}$$

Since the density of the b's is a, we have

$$\sum_{b_i < x^c + 1} \frac{1}{b_i} = ca \log x + o(\log x). \tag{16}$$

From (13), (15), and (16) we finally obtain

$$I_3(x) < \left(1+o(1)\right) \frac{c_1}{1-2c} x^{1-c} ca < \frac{a}{2} x^{1-c} \tag{17}$$

since $c < 1/(2+2c_1)$.

From (10), (11), (12), and (17) we finally obtain that, for sufficiently small η

$$B(x, x+x^{1-c}) > x^{1-c} - x^{1-c}(1-a+\eta) - \eta x^{1-c} - \frac{a}{2} x^{1-c} = \left(\frac{a}{2} - 2\eta\right) x^{1-c} > 0$$

which proves Theorem 1.

Now we pass to Theorem 3. The proof of Theorem 3 is very similar to that of Theorem 1 and we leave it to the reader. On the other hand we prove in some detail that Theorem 3 is best possible. Let $f(x)$ be such that there is a sequence $x_k \to \infty$ for which $f(x_k) < x_k^{1-\varepsilon}$ for a fixed $\varepsilon > 0$. We then show that there is a subsequence of the x_k (denoted for simplicity also by x_k) and a sequence $a_1 < \ldots$ satisfying (1) so that

$$B(x_k, x_k + x_k^{1-\varepsilon}) < (a-\eta) x_k^{1-\varepsilon} \tag{18}$$

for a fixed $\eta > 0$. In other words, (2) cannot hold.

We construct our sequence $a_1 < \ldots$ satisfying (1) as follows: Let the sequence x_k tend to infinity sufficiently fast. The sequence $a_1 < \ldots$ consists of the primes in the intervals

$$\left(\frac{x_k}{t}, \frac{x_k}{t} + \frac{x_k^{1-\varepsilon}}{t}\right), \quad k = 1, 2, \ldots; \ 1 \leqslant t < x_k^{\varepsilon}.$$

A simple computation which we leave to the reader shows that our sequence satisfies (1).

Clearly,

$$B(x_k, x_k + x_k^{1-\varepsilon}) = x_k^{1-\varepsilon} - U_1 - U_2 \tag{19}$$

where U_1 denotes the number of integers in $(x_k, x_k + x_k^{1-\varepsilon})$ which are divisible by an $a_i \leqslant x_k^{1-\varepsilon}$ and U_2 denotes the number of those integers in $(x_k, x_k + x_k^{1-\varepsilon})$ which are divisible by an a_i in $(x_k^{1-\varepsilon}, x_k + x_k^{1-\varepsilon})$ but not divisible by any $a_i \leqslant x_k^{1-\varepsilon}$. Denote the density of the b's by a. By a simple sieve process we obtain

$$U_1 = \left(a + o(1)\right) x^{1-\varepsilon}. \tag{20}$$

As in the proof of Theorem 1 we obtain that the integers of U_2 are of the form

$$b_j a_i, \ b_j < x_k^{\varepsilon} + 1, \quad x_k^{1-\varepsilon} < a_i < x_k + x_k^{1-\varepsilon}$$

and hence by the definition of the a_i's these are the numbers of the form

$$b_j p, \quad 1 \leqslant b_j < x_\varepsilon^k, \quad \frac{x_k}{b_j} < p < \frac{x_k + x_k^{1-\varepsilon}}{b_j}.$$

Now by the theorem of Hoheisel [3] for sufficiently small ε the number of primes in

$$\left(\frac{x_k}{b_j}, \frac{x_k+x_k^{1-\varepsilon}}{b_j}\right)$$

is greater than

$$\frac{1}{2}\cdot\frac{x_k^{1-\varepsilon}}{b_j \log x_k},$$

and hence the number of integers satisfying (21) (or U_2) is greater than

$$U_2 > \frac{x_k^{1-\varepsilon}}{2\log x_k}\sum_{b_j<x_k^{\varepsilon}}\frac{1}{b_j} = \big(\alpha+o(1)\big)\frac{\varepsilon}{2}x_k^{1-\varepsilon} \tag{22}$$

since the density of the b's is α. (18) follows from (19), (20), and (22), and hence we proved that Theorem 3 is best possible.

Now we prove Theorem 2. A theorem of de Bruijn [1] states that if $\psi(x, y)$ denotes the number of integers $\leqslant x$ whose all prime factors are $\leqslant y$ and $y > (\log x)^2$, then

$$\psi(x, y) < \frac{x}{s!} \quad \text{where} \quad y^s \leqslant x < y^{s+1}. \tag{23}$$

Put $u_k = (\log x_k \log\log x_k)^{1/2}$. From (23) we obtain by a simple computation that for every k if x_k is sufficiently large, then

$$\psi\left(x_k, k^2\exp\left(\frac{u_k}{4}\right)\right) < x_k\exp(-u_k). \tag{24}$$

Let

$$I_r^{(k)}, \quad r = 1, 2, \ldots, \left[\frac{x_k}{2}\exp\left(-\frac{u_k}{4}\right)\right],$$

be disjoint intervals of length $[\exp(u_k/4)]$ in $(x_k/2, x_k)$. It immediately follows from (24) that for at least one of these intervals, say $I_{r_0(k)}^{(k)}$, all the integers in $I_{r_0(k)}^{(k)}$ have their greatest prime factor greater than $k^2\exp(u_k/4)$. For every $k = 1, 2, \ldots$ and each integer of $I_{r_0(k)}^{(k)}$, consider the greatest prime factor of this integer. The set of all these primes will be our sequence $a_1 < a_2 < \ldots$ Clearly $\sum_i \frac{1}{a_i} < \infty$ since for a fixed k we obtain $[\exp(u_k/4)]$ primes all greater than $k^2\exp(u_k/4)$; hence

$$\sum_i \frac{1}{a_i} < \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

By our construction, none of the intervals $I_{r_0(k)}^{(k)}$, $k = 1, 2, \ldots,$ contain any b's and hence Theorem 2 is proved.

The proof of Theorem 2 leads to the following questions which seem to be of independent interest.

Put

$$F(u, v) = \min \sum_i \frac{1}{p_i} \tag{25}$$

where in (25) the summation is extended over a set $p_1 < \ldots$ of primes for which every $u \leqslant m < v$ is divisible by at least one p_i. Similarly,

$$f(u, v) = \min \sum \frac{1}{a_i} \tag{26}$$

where in (26) the minimum is taken over all sets of integers $a_1 < \ldots,$ $(a_i; a_j) = 1$ for which every $u \leqslant m < v$ is divisible by at least one a_i. $g(u, v)$ is defined as $f(u, v)$ but the condition $(a_i, a_j) = 1$ is omitted. Clearly

$$g(u, v) \leqslant f(u, v) \leqslant F(u, v)$$

and

$$g(2, v) = f(2, v) = F(2, v) = \sum_{p<v}\frac{1}{p}.$$

It seems difficult to obtain good estimations for $f(u, v)$ and $F(u, v)$. I proved that then

$$\overline{\lim_{u=\infty}}\, F(u, u+t) = \tfrac{1}{2} \quad \text{and} \quad \lim_{u=\infty} f(u, u+t) = 0,$$

$g(u, v)$ is easier to handle. It is easy to see that for $v \leqslant 2u$

$$g(u, v) = \sum_{l=u}^{v-1}\frac{1}{l},$$

but if v is large compared to u, then $g(u, v)$ may also be hard to determine.

The proof of Theorem 1 gives that there is an $\varepsilon > 0$ and a $c > 0$ so that for every u

$$f(u, u+u^{1-c}) > \varepsilon \tag{27}$$

but (27) is probably very far from being best possible.

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UNIVERSITY OF ILLINOIS
Urbana, Illinois

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On two problems of Erdös, Szüsz and Turán concerning diophantine approximations

by

H. Kesten* (Ithaca, N. Y.) and V. T. Sós (Budapest)

1. Introduction. The present paper concerns itself with the following pair of problems posed by Erdös, Szüsz and Turán [2]:

Problem 1. *For $A > 0$, $c \geqslant 1$, let*

$$S(N, A, c) = \text{set of } \xi \in [0,1] \text{ which satisfy } |b\xi - a| \leqslant Ab^{-1} \text{ for some integers } a, b \text{ with } N \leqslant b \leqslant cN, (a, b) = 1.$$

Does

$$\lim_{N\to\infty} |S(N, A, c)| \tag{1.1}$$

exist, and if so, what is its value? (If C is a set, $|C|$ denotes its Lebesgue measure.)

If $|b\xi - a| \leqslant (2b)^{-1}$, then a/b must be a continued fraction convergent of ξ. ([5], Chapter 10.) The next problem is therefore closely related to problem 1.

Problem 2. *For $c \geqslant 1$, let*

$$T(N, c) = \text{set of } \xi \in [0,1] \text{ which have at least one continued fraction convergent } p_n/q_n \text{ with } N \leqslant q_n \leqslant cN.$$

Does

$$\lim_{N\to\infty} |T(N, c)| \tag{1.2}$$

exist, and if so, what is its value?

Originally, these problems were treated by means of the methods of the article immediately following this one [7]. It was noticed, however, by the second author that a much simpler, almost self contained treatment of these problems is possible and it is our aim to present this treatment here.

* Alfred P. Sloan Fellow.