Возможен, как всегда, и другой путь доказательства этих метрических теорем, элементарным, но громоздким подсчетом, но оценки тригонометрических сумм привлекли нас своей простотой.

Цитированная литература

[1] А. О. Гельфонд, *Трансцендентные и алгебраические числа,* Москва 1952, p. 224.

[2] M. Lazard, *Les zéros d'une fonction analytique d'une variable sur un corps valué complet,* Institut des hautes études scientifiques, Publ. Mathem. 14 (1962), pp. 223-251.

[3] A. Rényi, *Representations for real numbers and their ergodic properties,* Acta Math. Acad. Scient. Hungar 8 (3-4) (1900), pp. 477-493.

[4] А. О. Гельфонд, *Об одном общем свойстве систем счисления,* И. А. Н. 23 (6) (1959), pp. 809-814.

[5] H. Davenport, P. Erdös, W. I. Le Veque, *On Weils criterion for uniform distributions,* Michigan Math. Journ. 10 (1963), pp. 311-314.

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Further developments in the comparative prime-number theory III

by

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1. As well-known, Chebyshev (see Chebyshev [1]) asserted without a proof that (p standing for primes)

$$\lim_{x\to+\infty}\sum_{p>2}(-1)^{(p-1)/2}e^{-p/x} = -\infty, \tag{1.1}$$

i.e. "there are more primes $\equiv 3$ (4) than $\equiv 1$ (4) in Abel's sense". This is undecided until now; but as well-known (see Hardy-Littlewood [1], Landau [1], [2]) it is equivalent to the fact that (with $s = \sigma+it$)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^s}\neq 0,\qquad \sigma>\frac{1}{2}.$$

The same could have been proved for the sum

$$\sum_{p>2}(-1)^{(p-1)/2}\log p\cdot e^{-p/x} \tag{1.2}$$

and analogously for

$$\sum_{p\equiv 1(3)}\log p\cdot e^{-p/x}-\sum_{p\equiv 2(3)}\log p\cdot e^{-p/x}. \tag{1.3}$$

By these are essentially all moduli k with $\varphi(k) = 2$ settled. As to the next difficult question $\varphi(k) = 4$, the simplest is the case $k = 8$. It turned out (see Knapowski-Turán [1]) that for the functions

$$\sum_{p\equiv 1(8)}\log p\cdot e^{-p/x}-\sum_{p\equiv l(8)}\log p\cdot e^{-p/x} \tag{1.4}$$

we have an analogous situation as before; but as a new phenomenon, we proved that for $0<\delta<c_1$ for each $l_1\neq l_2$ among 3, 5, 7 we have

(1.5)

$$\max_{\delta^{-1/3}\leqslant x\leqslant\delta^{-1}}\Big\{\sum_{p\equiv l_1(8)}\log p\cdot e^{-p/x}-\sum_{p\equiv l_2(8)}\log p\cdot e^{-p/x}\Big\}>\frac{1}{\sqrt{\delta}}\,e^{-41\frac{\log(1/\delta)\log_3(1/\delta)}{\log_2(1/\delta)}}$$

and hence also

(1.6)
$$\min_{\delta^{-1/3}\leqslant x\leqslant\delta^{-1}}\Big\{\sum_{p\equiv l_1(8)}\log p\cdot e^{-p/x}-\sum_{p\equiv l_2(8)}\log p\cdot e^{-p/x}\Big\}<-\frac{1}{\sqrt{\delta}}e^{-41\frac{\log(1/\delta)\log_3(1/\delta)}{\log_2(1/\delta)}}.$$

These results would suggest for the general k that if l_1 resp. l_2 are quadratic residue resp. non-residue $\bmod k$ (i.e. l_1 and l_2 are of opposite quadratic character) then for

$$\sum_{p\equiv l_1(k)}\log p\cdot e^{-p/x}-\sum_{p\equiv l_2(k)}\log p\cdot e^{-p/x} \tag{1.7}$$

we have a situation, analogous to that of (1.1) or (1.3) or (1.4), whereas if l_1 and l_2 have the *same* quadratic character $\bmod k$, then for the function (1.7) we have a situation analogous to (1.5) and (1.6). By other words if l_1 and l_2 have opposite quadratic character $\bmod k$ then "definitive preponderance in Abel's sense" holds if and only if the generalized Riemann-conjecture holds for all $L(s,\chi)$-functions $\bmod k$, whereas in the case when l_1 and l_2 are of the same quadratic character, there is no definitive preponderance even in Abel's sense. A closer analysis however revealed (see Knapowski-Turán [2]) that owing to the "small" zeros of the L-functions a proof of any of these assertions for large k's would be difficult in particular the first. In the same paper we made the observation that if we replace the Abel-means

$$\sum_{p\equiv l_1(k)}\log p\cdot e^{-p/x}-\sum_{p\equiv l_2(k)}\log p\cdot e^{-p/x}$$

by

$$\sum_{p\equiv l_1(k)}\log p\cdot e^{-\frac{1}{r}\log^2\frac{p}{x}}-\sum_{p\equiv l_2(k)}\log p\cdot e^{-\frac{1}{r}\log^2\frac{p}{x}}$$

or with the notation

$$\varepsilon_k(n,l_1,l_2)=\begin{cases}+1 & \text{for}\quad n\equiv l_1(k),\\ -1 & \text{for}\quad n\equiv l_2(k),\\ 0 & \text{otherwise}\end{cases} \tag{1.8}$$

by

$$F_k(x,l_1,l_2)\overset{\text{def}}{=}\sum_{p}\varepsilon_k(p,l_1,l_2)e^{-\frac{1}{r}\log^2\frac{p}{x}} \tag{1.9}$$

with suitable $r=r(x)$, the situation changes to a certain extent. In particular it is so for "good" k-values, i.e. those for which there is an $E=E(k)$ such that

$$\prod_{\chi \bmod k}L(s,\chi)\neq 0 \tag{1.10}$$

for

$$\sigma\geqslant\tfrac{1}{2},\qquad |t|\leqslant E(k)\qquad (0<E(k)\leqslant\tfrac{1}{2})$$

(Haselgrove-condition). For such k-values (whose number is probably infinite) we showed at least that for $l_1=1$ and $l_2=$ quadratic non-residue $\bmod k$, the relation

$$\lim_{x\to+\infty}F_k(x,1,l_2)=-\infty \tag{1.11}$$

for all $a_1(k)\leqslant r=r(x)\leqslant\log x$ holds if and only if the generalized Riemann-conjecture holds for k.

2. In the present note we shall deal with the case

$$l_1=1,\qquad l_2=l=\text{quadratic residue}\bmod k. \tag{2.1}$$

For this case we shall prove in correspondance with (1.5) and (1.6) the

THEOREM I. *For "good" k's in the case* (2.1) *and for*

$$T>\max(c,e^{4e^{3k}},e^{(20\pi)^6/E(k)^6}) \tag{2.2}$$

there exist x_1, x_2 in the interval

$$(Te^{-(\log T)^{5/6}},\ Te^{(\log T)^{11/15}}) \tag{2.3}$$

such that for suitable

$$(2\log T)^{2/3}\leqslant\nu_j\leqslant(2\log T)^{2/3}+(2\log T)^{2/5} \tag{2.4}$$

the inequalities

$$\begin{gathered}\sum_{p}\varepsilon_k(p,l,1)\log p\cdot e^{-\frac{1}{\nu_1}\log^2\frac{p}{x_1}}>\sqrt{T}e^{-c'\log^{5/6}T},\\ \sum_{p}\varepsilon_k(p,l,1)\log p\cdot e^{-\frac{1}{\nu_2}\log^2\frac{p}{x_2}}<-\sqrt{T}e^{-c'\log^{5/6}T}\end{gathered} \tag{2.5}$$

hold.

This is a special case of

THEOREM II. *In the case* (2.1) *for "good" k's if $\varrho_0=\beta_0+i\gamma_0$ is a zero of an $L(s,\chi')$-function* $\bmod k$ *with*

$$\beta_0\geqslant\tfrac{1}{2},\qquad \gamma_0>0,\qquad \chi'(l)\neq 1,$$

there exist for

$$T>\max(c,e^{4e^{3k}},e^{(20\pi)^6/E(k)^6},e^{e^{10|\varrho_0|}})$$

x_1, x_2-numbers in the interval

$$(Te^{-(\log T)^{5/6}},\ Te^{(\log T)^{11/15}})$$

such that the inequalities (2.5) *hold with $\sqrt{T}$ replaced by T^{β_0}.*

However we shall confine ourselves to the proof of Theorem I; that of Theorem II follows mutatis mutandis.

As in paper II of this series we can conclude directly as to the discrepancy of primes $\equiv 1\ (k)$ and $\equiv l\ (k)$ if l is a quadratic residue mod k. So we assert the

THEOREM III. *For "good" k's in the case* (2.1) *for T's satisfying* (2.2) *there are U_ν-numbers with*

$$Te^{-\log^{6/7}T} \leqslant U_1 < U_2 \leqslant Te^{\log^{6/7}T},$$

resp.

$$Te^{-\log^{6/7}T} \leqslant U_3 < U_4 \leqslant Te^{\log^{6/7}T}$$

so that

$$\sum_{U_1 \leqslant p \leqslant U_2} \varepsilon_k(p, 1, l) > \sqrt{T}e^{-c'\log^{5/6}T}$$

and

$$\sum_{U_3 \leqslant p \leqslant U_4} \varepsilon_k(p, 1, l) < -\sqrt{T}e^{-c'\log^{5/6}T}.$$

Since the proof runs exactly like that in our paper II, we omit the details.

3. For the proof we shall need some lemmata.

LEMMA I. *Let for a positive m and $n \leqslant N$ the z_j's with*

(3.1) $$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_{h_1}| \geqslant \ldots \geqslant |z_n|$$

are such that with a $0 < \varkappa \leqslant \pi/2$

(3.2) $$\varkappa \leqslant |\operatorname{arc} z_j| \leqslant \pi \quad (j = 1, \ldots, n);$$

furhter, let h resp. h_1 be defined by

(3.3) $$|z_h| > \frac{4N}{m+N(3+\pi/\varkappa)}$$

resp. by

(3.4) $$\begin{cases} |z_{h_1}| < |z_h| - \dfrac{2N}{m+N(3+\pi/\varkappa)}, & \textit{if there is such an } h_1, \\ h_1 = n & \textit{otherwise} \end{cases}$$

and finally

(3.5) $$B \overset{\text{def}}{=} \min_{h \leqslant j < h_1} \operatorname{Re}\Big(\sum_{v=1}^{j} b_v\Big).$$

Then there are integer ν_1 and ν_2 with

(3.6) $$m \leqslant \nu_1, \nu_2 \leqslant m+N(3+\pi/\varkappa)$$

such that

$$\operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu_1} \geqslant \frac{B}{2N+1}\left(\frac{N}{24(m+N(3+\pi/\varkappa))}\right)^{2N}\left(\frac{|z_h|}{2}\right)^{m+N(3+\pi/\varkappa)}$$

and

$$\operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu_2} \leqslant -\frac{B}{2N+1}\left(\frac{N}{24(m+N(3+\pi/\varkappa))}\right)^{2N}\left(\frac{|z_h|}{2}\right)^{m+N(3+\pi/\varkappa)}.$$

The proof of this lemma one can find in Knapowski-Turán [3] as Theorem 4.1.

LEMMA II. *If $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are real with*

$$|\alpha_\nu| \geqslant U\ (>0),$$

further

$$\Delta > 1/U$$

and with a $\gamma > 1$

$$\sum_\nu \frac{1}{1+|\alpha_\nu|^\gamma} \leqslant V\ (<\infty),$$

then every real interval of length Δ contains a ξ-value such that for all ν-indices the inequality

$$\{\alpha_\nu \xi + \beta_\nu\} \geqslant \frac{1}{24V} \cdot \frac{1}{1+|\alpha_\nu|^\gamma}$$

holds ($\{x\}$ *denoting the distance of x from the next integer*).

For the proof of this lemma, see Knapowski-Turán [7].

LEMMA III. *For any given k modulus there exists a broken line W in the vertical strip $\frac{1}{200} \leqslant \sigma \leqslant \frac{1}{100}$, symmetrical to the real axis, consisting alternately of vertical resp. horizontal segments, each horizontal strip of width* 1 *containing at most one of the horizontal segments and on which for all χ's* mod k *the inequalities*

$$\left|\frac{L'}{L}(s, \chi)\right| \leqslant c_1 k \log^2 k (2+|t|),$$

$$\left|\frac{L'}{L}(2s, \chi)\right| \leqslant c_1 k \log^2 k (2+|t|)$$

hold.

The proof of this lemma is contained mutatis mutandis in the book of the second of us (see Turán [1]).

4. Now we can turn to the proof of our theorem. If l (with $(l, k) = 1$) is a quadratic residue mod k, then the solutions of $x^2 \equiv l\ (k)$ form obviously a coset according to the subgroup formed by the solutions of $x^2 \equiv 1\ (k)$

in the multiplicative group of reduced residue-classes $\bmod k$. Let the solutions of $x^2 \equiv l\ (k)$ resp. $x^2 \equiv 1\ (k)$ be

$$\alpha_1, \alpha_2, \ldots, \alpha_\mu \quad \text{resp.} \quad \beta_1, \beta_2, \ldots, \beta_\mu. \tag{4.1}$$

Then

$$-\frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \frac{L'}{L}(s, \chi) = \sum_{p \equiv l(k)} \frac{\log p}{p^s} + \sum_{p^2 \equiv l(k)} \frac{\log p}{p^{2s}} + f_1(s) \tag{4.2}$$

$$= \sum_{p \equiv l(k)} \frac{\log p}{p^s} + \sum_{\nu=1}^{\mu} \sum_{p \equiv \alpha_\nu(k)} \frac{\log p}{p^{2s}} + f_1(s)$$

$$= \sum_{p \equiv l(k)} \frac{\log p}{p^s} - \sum_{\nu=1}^{\mu} \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(\alpha_\nu) \frac{L'}{L}(2s, \chi) + f_2(s),$$

where generally $f_\nu(s)$ stand for functions regular for $\sigma \geqslant 0{,}34$ and satisfying here the inequality

$$|f_\nu(s)| \leqslant c_2 \tag{4.3}$$

c, c', c_1, c_2 and later $c_3, \ldots$ denoting positive numerical constants.

(4.2) gives the identity

$$\frac{1}{\varphi(k)} \sum_{\chi} (1 - \bar{\chi}(l)) \frac{L'}{L}(s, \chi) + \frac{1}{\varphi(k)} \sum_{\nu=1}^{\mu} \sum_{\chi} (\bar{\chi}(\alpha_\nu) - \bar{\chi}(\beta_\nu)) \frac{L'}{L}(2s, \chi) \tag{4.4}$$

$$\stackrel{\text{def}}{=} \Phi_k(s, l) = \sum_{p} \varepsilon_k(p, l, 1) \frac{\log p}{p^s} + f_3(s)$$

for $\sigma \geqslant 0{,}34$. Now with a T in (2.2) let

$$D \stackrel{\text{def}}{=} (2 \log T)^{1/3}. \tag{4.5}$$

5. Next we consider all $\varrho = \sigma_\varrho + it_\varrho$ zeros of all $L(s, \chi)$ functions $\bmod k$ satisfying

$$|t_\varrho| \leqslant 2\sqrt{D} \tag{5.1}$$

and apply Lemma II with $\gamma = \frac{11}{10}$, $U = \frac{1}{8\pi} E(k)$ to the numbers

$$\alpha_\nu = \frac{t_\varrho}{4\pi} \quad \text{and} \quad \frac{t_\varrho}{8\pi},$$

$$\beta_\nu = \frac{1}{8\pi} \operatorname{Im}(\varrho^2) \quad \text{and} \quad \frac{1}{32\pi} \operatorname{Im}(\varrho^2).$$

Then one can choose evidently [1]

$$V = c_4 k \log k$$

and thus Lemma II assures the existence of a b_0 in

$$\left(3 \leqslant \frac{D}{2} \leqslant\right) D - \frac{10\pi}{E(k)} \leqslant b_0 \leqslant D \tag{5.2}$$

so that for all ϱ's in (5.1)

$$\left\{\frac{1}{2\pi} \cdot \frac{t_\varrho}{2} b_0 + \frac{1}{2\pi} \operatorname{Im} \frac{\varrho^2}{4}\right\} \geqslant \frac{c_5}{1 + |t_\varrho|^{11/10}} \cdot \frac{1}{k \log k}$$

and

$$\left\{\frac{1}{2\pi} \cdot \frac{t_\varrho}{4} b_0 + \frac{1}{2\pi} \operatorname{Im} \frac{\varrho^2}{16}\right\} \geqslant \frac{c_5}{1 + |t_\varrho|^{11/10}} \cdot \frac{1}{k \log k},$$

i.e.

$$\frac{c_6}{(1 + |t_\varrho|^{11/10}) k \log k} \leqslant |\operatorname{arc}(e^{\frac{1}{4}(\varrho^2 + 2b_0\varrho)})| \leqslant \pi$$

and

$$\frac{c_6}{(1 + |t_\varrho|^{11/10}) k \log k} \leqslant |\operatorname{arc}(e^{\frac{1}{4}(\varrho^2/4 + b_0\varrho)})| \leqslant \pi.$$

Since from (2.2) and (4.5) we have

$$e^k \leqslant (\log T)^{1/3} < D, \tag{5.3}$$

we get the inequalities

$$\begin{aligned} \frac{c_6}{(1 + |t_\varrho|^{11/10}) \log^2 D} &\leqslant |\operatorname{arc}(e^{\frac{1}{4}(\varrho^2 + 2b_0\varrho)})| \leqslant \pi, \\ \frac{c_6}{(1 + |t_\varrho|^{11/10}) \log^2 D} &\leqslant |\operatorname{arc}(e^{\frac{1}{4}(\varrho^2/4 + b_0\varrho)})| \leqslant \pi \end{aligned} \tag{5.4}$$

for all ϱ's in (5.1).

6. Fixing b_0 that way, let r be an integer to be determined later so that

$$D^2 \leqslant r \leqslant D^2 + D^{6/5} \tag{6.1}$$

and consider the integral

$$I_l(r) = \frac{1}{2\pi i} \int_{(2)} e^{\frac{r}{4}(s+b_0)^2} \Phi_k(s, l)\, ds. \tag{6.2}$$

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[1] Here we used the fact that the number of zeros of any $L(s, \chi)$ in the half-strip $\lambda \leqslant t \leqslant \lambda + 1$, $\sigma > 0$ is at most $c_3 \log k (2 + |\lambda|)$.

Using the integral-formula (see e.g. Knapowski-Turán [2])

$$\frac{1}{2\pi i}\int_{(2)} e^{\frac{r}{4}(s+b_0)^2-\lambda s}\,ds = \frac{1}{\sqrt{\pi r}}\, e^{\frac{r}{4}b_0^2-\frac{1}{r}\left(\lambda-\frac{rb_0}{2}\right)^2}$$

we get from (4.4) and (6.2)

$$I_l(r) = \frac{e^{\frac{r}{4}b_0^2}}{\sqrt{\pi r}}\sum_p \varepsilon_k(p,l,1)\, e^{-\frac{1}{r}\left(\log p-\frac{rb_0}{2}\right)^2} + \frac{1}{2\pi i}\int_{(0,34)} e^{\frac{r}{4}(s+b_0)^2} f_3(s)\,ds.$$

Using (4.3) we get for the absolute-value of this last integral—shifting it to the vertical line $\sigma = 0{,}34$—the upper bound

$$\frac{c_2}{2\pi}\int_{-\infty}^{\infty} e^{\frac{r}{4}\{(b_0+0{,}34)^2-v^2\}}\,dv < c_7 e^{\frac{r}{4}(b_0+0{,}34)^2},$$

i.e. owing to (4.5), (5.2) and (6.1)

(6.4) $$< c_7 e^{\frac{r}{4}b_0^2} e^{\frac{r}{2}b_0 0{,}34} e^{\frac{r}{16}} < \frac{T^{\frac{2}{5}}}{\sqrt{\pi r}}\, e^{\frac{r}{4}b_0^2},$$

if only c in (2.2) is sufficiently large. On the other hand, inserting in (6.2) the left-side for $\Phi_k(s,l)$ and shifting the line of integration to W we get

(6.5) $$I_l(r) = \frac{1}{\varphi(k)}\sum_\chi \left(1-\bar\chi(l)\right)\sum_{\varrho(\chi)}{}' e^{\frac{r}{4}(\varrho+b_0)^2} + \frac{1}{2\varphi(k)}\sum_{\nu=1}^{\mu}\sum_\chi \left(\bar\chi(\alpha_\nu)-\bar\chi(\beta_\nu)\right)\sum_{\varrho(\chi)}{}'' e^{\frac{r}{4}\left(\frac{\varrho}{2}+b_0\right)^2} + \frac{1}{2\pi i}\int_{(W)} e^{\frac{r}{4}(s+b_0)^2}\Phi_k(s,l)\,ds,$$

where Σ' resp. Σ'' means that the respective summation must be extended only to those ϱ's for which ϱ resp. $\varrho/2$ is right from W. For the absolute value of the last integral in (6.5) Lemma III gives the upper bound

$$c_7 k^2 \int_{-\infty}^{\infty} \log^2 k(2+|t|)\, e^{\frac{r}{4}\left\{\left(b_0+\frac{1}{100}\right)^2-t^2\right\}}\,dt < c_8 k^2 \log^2 k\, e^{\frac{r}{4}\left(b_0+\frac{1}{100}\right)^2}$$

which in turn is owing to (2.2), (5.2) and (6.1)

(6.6) $$< e^{\frac{r}{4}b_0^2}\frac{T^{1/50}}{\sqrt{\pi r}}.$$

(6.5), (6.6), (6.4) and (6.3) give, taking real parts the inequality

(6.7) $$\Bigg|\sum_p \varepsilon_k(p,l,1)\log p\cdot e^{-\frac{1}{r}\left(\log p-\frac{rb_0}{2}\right)^2} - \sqrt{\pi r}\,\mathrm{Re}\Bigg\{\sum_\chi \frac{1-\bar\chi(l)}{\varphi(k)}\sum_{\varrho(\chi)}{}' e^{\frac{r}{4}(2\varrho b_0+\varrho^2)} + \sum_{\nu=1}^{\mu}\sum_\chi \frac{\bar\chi(\alpha_\nu)-\bar\chi(\beta_\nu)}{2\varphi(k)}\sum_{\varrho(\chi)}{}'' e^{\frac{r}{4}\left(b_0\varrho+\frac{\varrho^2}{4}\right)}\Bigg\}\Bigg| < c_9 T^{\frac{2}{5}}.$$

7. Now we estimate (trivially) the contribution of ϱ's satisfying

(7.1) $$|t_\varrho| > 2\sqrt{D}.$$

Using the footnote on p. 121 this contribution is absolutely

(7.2) $$< c_{10}\Bigg[e^{\frac{r}{4}(1+2b_0)}\sum_{n\geqslant 2\sqrt{D}-1} e^{-\frac{r}{4}n^2}\log kn + e^{\frac{r}{4}\left(\frac{1}{4}+b_0\right)}\sum_{n\geqslant 2\sqrt{D}-1} e^{-\frac{r}{4}\cdot\frac{n^2}{4}}\log kn\Bigg] < c_{11} e^{\frac{r}{2}(\sqrt{D}-D)} < c_{12}.$$

Let G be the domain right from W satisfying $|t| \leqslant 2\sqrt{D}$ and

(7.3) $$\max_{\substack{\varrho(\chi)\,\mathrm{in}\,G\\ \chi(l)\neq 1}} |e^{\frac{1}{4}(\varrho^2+2b_0\varrho)}| \overset{\mathrm{def}}{=} |e^{\frac{1}{4}(\varrho_1^2+2\varrho_1 b_0)}| \overset{\mathrm{def}}{=} M.$$

Hence from (6.7), (7.2) and (7.3) we get the inequality

(7.4) $$\Bigg|\sum_p \varepsilon_k(p,l,1)\log p\cdot e^{-\frac{1}{r}\left(\log p-\frac{rb_0}{2}\right)^2} - \sqrt{\pi r}\,|e^{\frac{1}{4}(\varrho_1^2+2b_0\varrho_1)}|^r\,\mathrm{Re}\Bigg\{\sum_\chi \frac{1-\bar\chi(l)}{\varphi(k)}\sum_{\varrho(\chi)\in G}\left(e^{\frac{1}{4}\{\varrho^2+2b_0\varrho-\mathrm{Re}(\varrho_1^2+2\varrho_1 b_0)\}}\right)^r + \sum_{\nu=1}^{\mu}\sum_\chi \frac{\bar\chi(\alpha_\nu)-\bar\chi(\beta_\nu)}{2\varphi(k)}\sum_{\varrho(\chi)\in G}\left(e^{\frac{1}{4}\left\{\frac{\varrho^2}{4}+\varrho b_0-\mathrm{Re}(\varrho_1^2+2\varrho_1 b_0)\right\}}\right)^r\Bigg\}\Bigg| < c_{13}T^{2/5}.$$

8. Until now the integer r was subjected only to the restriction (6.1); now we shall determine it using our lemmata. Let us denote the expression

$$\mathrm{Re}\Big\{\sum_\chi \ldots\Big\}$$

in (7.4) by $Z(r)$; we shall try to use Lemma I with

(8.1) $$e^{\frac{1}{4}(\varrho^2+2b_0\varrho)-\frac{1}{4}\mathrm{Re}(\varrho_1^2+2b_0\varrho_1)}$$

resp.

(8.2) $$e^{\frac{1}{4}\left(\frac{\varrho^2}{4}+b_0\varrho\right)-\frac{1}{4}\mathrm{Re}(\varrho_1^2+2b_0\varrho_1)}$$

as z_j-vectors, calling them z_j's of first (resp. of second) category. Correspond ingly we shall choose as b_j-coefficients the numbers

(8.3) $$\frac{1-\overline{\chi}(l)}{\varphi(k)}$$

resp.

(8.4) $$\frac{\overline{\chi}(a_\nu)-\overline{\chi}(\beta_\nu)}{2\varphi(k)}$$

and call them b_j's of first resp. of second category. First we have to verify

(8.5) $$\max_j |z_j| = 1.$$

For the z_j's of first category this is evident from the definition of ϱ_1. To verify it also for the z_j's of second category we remark first that owing to a theorem of Siegel (see Siegel [1]) there is a c_{14} such that each $L(s,\chi)$ has a zero in the domain

(8.6) $$\sigma \geqslant \tfrac{1}{2}, \quad |t| \leqslant c_{14};$$

this holds especially for the L-functions belonging to χ's with $\chi(l)\neq 1$. Denoting by $\varrho_0=\sigma_0+it_0$ any of *such* zeros we have (M in (7.3))

(8.7) $$M \geqslant |e^{\frac{1}{4}(\varrho_0^2+2b_0\varrho_0)}| = e^{\frac{1}{4}(\sigma_0^2-t_0^2+2b_0\sigma_0)} \geqslant e^{\frac{1}{4}(b_0-c_{14}^2)}.$$

In order to show that for the z_j's of second category even the sharper inequality

(8.8) $$|z_j| \leqslant e^{-2}$$

holds, it suffices owing to (8.2) to show

$$e^{\frac{1}{4}\left(\frac{\sigma_\varrho^2-t_\varrho^2}{4}+b_0\sigma_\varrho\right)} < e^{\frac{1}{4}(b_0-c_{14}^2)-2}$$

or *a fortiori*

(8.9) $$\tfrac{1}{4}+b_0\sigma_\varrho < b_0-c_{14}^2-8.$$

But owing to the classical theorem we have

$$\sigma_\varrho < \max\left\{1-\frac{c_{15}}{\log k(2+|t_\varrho|)},\ 1-\frac{c_{15}}{k}\right\},$$

i.e. in G, using also (2.2),

$$\sigma_\varrho < \max\left\{1-\frac{c_{15}}{\log k(2+2\sqrt{D})},\ 1-\frac{c_{15}}{k}\right\} < 1-\frac{c_{16}}{\log D};$$

hence if c in (2.2) is sufficiently large, using (5.2) we get

$$c_{16}\frac{b_0}{\log D} \geqslant \frac{c_{16}}{2}\cdot\frac{D}{\log D} > 9+c_{14}^2,$$

i.e.

$$\frac{1}{4}+b_0\sigma_\varrho \leqslant \frac{1}{4}+b_0-c_{16}\frac{b_0}{\log D} < b_0-c_{14}^2-8$$

and (8.9) — whence (8.8) and (8.5) — holds indeed.

9. The number of terms in $Z(r)$ is owing to the footnote on p. 121

(9.1) $$\leqslant c_{17}\varphi(k)\sqrt{D}\log kD < \sqrt{D}\log^3 D \overset{\text{def}}{=} N.$$

What will play the role of $\varkappa$? From (5.4) and $|t_\varrho| \leqslant 2\sqrt{D}$ we could choose as $\varkappa$

$$\frac{c_6}{\{1+(2\sqrt{D})^{11/10}\}\log^2 D} > c_{18}D^{-3/5}\log D.$$

Hence

(9.2) $$\varkappa = D^{-3/5}.$$

For m we choose

(9.3) $$m = D^2.$$

As to h, we choose

(9.4) $$h = 1;$$

then (3.3) is obviously satisfied if c in (2.2) is sufficiently large. As to z_{h_1} we shall choose it so that no b_j of second category should contribute to B. This choice is fulfilled if z_{h_1} is the absolutely greatest among the z_j's of second category. Then we have owing to (8.8)

$$|z_{h_1}| \leqslant e^{-2} < 1-\frac{6}{7} < 1-\frac{2\sqrt{D}\log^3 D}{D^2} < |z_h| - \frac{2N}{m+N(3+\pi/\varkappa)},$$

i.e. (3.4) is fulfilled too. Now in B we have only b_j's with nonnegative real part, i. e.

$$B \geqslant \min_{\substack{\chi \\ \chi(l)\neq 1}} \mathrm{Re}\,\frac{1-\bar{\chi}(l)}{\varphi(k)} \geqslant \frac{8}{k^3} > \frac{8}{\log^3 D}. \tag{9.5}$$

With the above choices the interval $(m, m+N(3+\pi/\varkappa))$ is certainly contained in the interval (6.1), i.e. r can be chosen according to Lemma I. Hence $r=\nu_1$ and ν_2 can be determined so that

$$(2\log T)^{2/3} \leqslant \nu_1, \qquad \nu_2 \leqslant (2\log T)^{2/3}+(2\log T)^{2/5} \tag{9.6}$$

and

$$\begin{aligned} Z(\nu_1) &> \left(\frac{\sqrt{D}\log^3 D}{24(D^2+4D^{11/10}\log^3 D)}\right)^{2\sqrt{D}\log^3 D}\frac{8}{\log^3 D}. \\ &\frac{1}{3\sqrt{D}\log^3 D}\left(\frac{1}{2}\right)^{D^2+4D^{11/10}\log^3 D} > e^{-D^2} = e^{-(2\log T)^{2/3}} \end{aligned} \tag{9.7}$$

and analogously

$$Z(\nu_2) < -e^{-(2\log T)^{2/3}}. \tag{9.8}$$

10. To complete the proof we have to give a lower bound to

$$M_j = \sqrt{\pi\nu_j}\,|e^{\frac{1}{4}(\varrho_1^2+2b_0\varrho_1)}|^{\nu_j}, \qquad j=1,2. \tag{10.1}$$

Owing to the maximal-definition of ϱ_1 and that of ϱ_0 we get for $j=1,2$

$$M_j \geqslant 2^{\frac{1}{3}}\sqrt{\pi}\log^{\frac{1}{3}}T\,|e^{\frac{1}{4}(\varrho_0^2+2b_0\varrho_0)}|^{\nu_j}$$

and the second factor, using (9.6), (8.6), (5.2), (4.5) and (2.2),

$$\begin{aligned} &= e^{\frac{\nu_j}{4}(\sigma_0^2-t_0^2+2\sigma_0 b_0)} \geqslant e^{-\frac{c_{14}^2}{4}\nu_j}(e^{\frac{1}{2}b_0\nu_j})^{\frac{1}{2}} \\ &\geqslant e^{-\frac{c_{14}^2}{2}\log^{2/3}T}\, e^{\frac{1}{4}D^2\left(D-\frac{10\pi}{E(k)}\right)} > \sqrt{T}e^{-c_{19}(\log T)^{5/6}}; \end{aligned}$$

hence

$$M_j > \sqrt{T}e^{-c_{20}(\log T)^{5/6}}.$$

Putting in (7.4) for $j=1,2$

$$\frac{b_0 r}{2} = \frac{b_0\nu_j}{2} = \log x_j,$$

the proof is finished. (2.3) presents no difficulties.

References

P. L. Chebyshev

[1] *Lettre de M. le professeur Tchébychev à M. Fuss, sur un nouveau théorème rélatif aux nombres premiers contenus dans les formes 4n+1 et 4n+3*, Bull. de la Classe phys.-math. de l'Acad. Imp. des Sciences St. Petersburg 11 (1853), p. 208.

G. H. Hardy and J. E. Littlewood

[1] *Contributions to the theory of Riemann zeta-function and the theory of distribution of primes*, Acta Math. 41 (1918), pp. 119-196.

S. Knapowski and P. Turán

[1] *Comparative prime number theory VIII*, Acta Math. Hung. 14 (1963), pp. 251-268.

[2] *Further developments in the comparative prime-number theory II*, Acta Arithm. 10 (1964), pp. 293-313.

[3] *Comparative prime-number theory III*, Acta Math. Hung. 13 (3-4) (1962), pp. 343-364.

E. Landau

[1], [2] *Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie, I*, resp. *II*, Math. Zeitschr. 1 (1918), pp. 1-24, resp. 213-219.

C. L. Siegel

[1] *On the zeros of Dirichlet L-functions*, Annals of Math. (1945), pp. 409-422.

P. Turán

[1] *Eine neue Methode in der Analysis und deren Anwendungen*, Acad. Kiadó 1953. A completely rewritten English edition will appear in the Interscience Tracts series.

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