ACTA ARITHMETICA
XI (1966)

Sum-free sets of integers

by

H. L. ABBOTT and L. MOSER (Edmonton)

A set S of integers is said to be *sum-free* if $a, b \in S$ then $a+b \notin S$. The case where $a = b$ is not excluded, that is, $a \in S$ implies $2a \notin S$.

A well known theorem of I. Schur ([4]) states that if the integers $1, 2, \ldots, [n!e]$ are split in an arbitrary manner into n sets, at least one of the sets fails to be sum-free. This leads us to define $f(n)$ as the largest positive integer m for which there exists some way of splitting the integers $1, 2, \ldots, m$ into n sum-free sets.

It is easy to verify that $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. In 1961, L. D. Baumert ([1]), with the aid of a high speed computer, showed that $f(4) = 44$. Since Baumert's work has not been published we exhibit one of the ways he found of splitting the integers $1, 2, \ldots, 44$ into four sum-free sets.

A	B	C	D
1	2	4	9
3	7	6	10
5	8	13	11
15	18	20	12
17	21	22	14
19	24	23	16
26	27	25	29
28	33	30	31
40	37	32	34
42	38	39	35
44	43	41	36

The value of $f(n)$ is not known for $n \geqslant 5$ and it seems to be quite difficult to determine $f(n)$, even for $n = 5$.

From Schur's theorem we get

$$f(n) \leqslant [n!e]-1 \tag{1}$$

and no general improvement on this upper bound for $f(n)$ has been obtained up to the present time, although the known values of $f(n)$ indicate that (1) is not best possible. On the other hand, Schur proved that

$$f(n+1) \geqslant 3f(n)+1 \tag{2}$$

and from (2) and the fact that $f(4) = 44$ we get, for $n \geqslant 4$,

$$f(n) \geqslant \frac{89(3)^{n-4}-1}{2}. \tag{3}$$

The main result that we wish to establish in this paper is that

$$f(n) > 89^{\frac{n}{4}-c\log n} \tag{4}$$

for some absolute constant c and all sufficiently large n. (4) is clearly better than (3).

We find it convenient to define a function g as follows: If $f(n-1) < l \leqslant f(n)$, then $g(l) = n$. $g(l)$ is thus the smallest number of sum-free sets into which the integers $1, 2, \ldots, l$ can be partitioned. It follows from (3) that for l sufficiently large

$$g(l) < \log l. \tag{5}$$

In order to prove (4) we shall need the following

THEOREM 1. *For all positive integers m and k,*

$$f\big(km+g(kf(m))\big) \geqslant \big(2f(m)+1\big)^k-1. \tag{6}$$

If we set $m = 4$ in (6) and use the fact that $f(4) = 44$ we get

$$f\big(4k+g(44k)\big) \geqslant 89^k-1 \tag{7}$$

and it is not difficult to see that (5) and (7) imply (4).

Proof of Theorem 1. Let $X = 2f(m)+1$ and write the numbers $1, 2, \ldots, X^k-1$ in base X. Call a number *good* if each of its digits does not exceed $f(m)$ and call a number *bad* if at least one of its digits exceeds $f(m)$. We shall show that the good numbers can be partitioned into $g\big(kf(m)\big)$ sum-free sets and the bad numbers into km sum-free sets. The theorem will then follow.

Let $A_1, A_2, \ldots, A_{g(kf(m))}$ be disjoint sum-free sets containing the numbers $1, 2, \ldots, kf(m)$. Divide the good numbers into sets $B_1, B_2, \ldots, B_{g(kf(m))}$ by placing a number in class B_j if the sum of its digits belongs to A_j. This can be done since the sum of the digits of a good number does not exceed $kf(m)$. It is not difficult to see that the sets $B_1, B_2, \ldots, B_{g(kf(m))}$ are sum-free.

Divide the bad numbers into k classes $C_1, C_2, \ldots, C_k$ by placing $a = a_1+a_2X+a_3X^2+\ldots+a_jX^{j-1}+\ldots+a_kX^{k-1}$ in class C_j if $a_i \leqslant f(m)$ for $i = 1, 2, \ldots, j-1$ and $a_j \geqslant f(m)+1$. Next divide each of $C_1, C_2, \ldots, C_k$ into m sets as follows: Let $D_1, D_2, \ldots, D_m$ be disjoint sum-free sets containing the numbers $1, 2, \ldots, f(m)$, and split the numbers in C_j into m sets $D_{j1}, D_{j2}, \ldots, D_{jm}$ by placing $a = a_1+a_2X+a_3X^2+\ldots+a_jX^{j-1}+\ldots+a_kX^{k-1}$ in D_{jl} if $a_j \equiv -u \pmod X$ for some $u \in D_l$. Since a_j is one of the numbers $f(n)+1, f(n)+2, \ldots, 2f(n)$ exactly one such u can be found. It remains to be shown that D_{jl} is sum-free. Suppose that we can find $a, b, c \in D_{jl}$ such that $a+b = c$. We have

$$a = \sum_{i=1}^{k} a_i X^{i-1}, \quad b = \sum_{i=1}^{k} b_i X^{i-1}, \quad c = \sum_{i=1}^{k} c_i X^{i-1}$$

where $a_i, b_i, c_i \leqslant f(m)$ for $i = 1, 2, .., j-1$, a_j, $b_j, c_j \geqslant f(m)+1$ and $a_j \equiv -u \pmod X$, $b_j \equiv -v \pmod X$, and $c_j \equiv -w \pmod X$ where $u, v, w \in D_l$. Since $a_j+b_j = X+c_j$, it follows that $u+v \equiv w \pmod X$, and since $u, v, w \leqslant f(m)$ we must have $u+v = w$. However, this contradicts the fact that D_l is sum-free. The bad numbers have therefore been partitioned into km sum-free sets and the proof of the theorem is complete.

It seems likely that (4) could be improved even further if one knew the value of $f(n)$ for some value of $n \geqslant 5$.

While the upper and lower bounds for $f(n)$ are quite far apart, we can still gain a little more insight into the behavior of $f(n)$. We show, using (6), that $\lim\limits_{n\to\infty} f(n)^{1/n}$ exists, although we cannot decide whether the limit is finite or infinite. Let

$$\alpha = \liminf_{n\to\infty} f(n)^{1/n} \leqslant \limsup_{n\to\infty} f(n)^{1/n} = \beta.$$

Suppose first that $\beta < \infty$. Let $\varepsilon > 0$ be given, and let m be the smallest integer for which

$$f(m)^{1/m} > \beta-\varepsilon. \tag{8}$$

If $k \geqslant k_0(\varepsilon)$,

$$km+g\big(kf(m)\big) < [km(1+\varepsilon)]. \tag{9}$$

Let

$$[km(1+\varepsilon)] \leqslant n \leqslant [(k+1)m(1+\varepsilon)]. \tag{10}$$

Then

$$f(n) \geqslant f\big([km(1+\varepsilon)]\big) > f\Big(km+g\big(kf(m)\big)\Big) \geqslant \big(2f(m)+1\big)^k-1 > f(m)^k$$

where we have used (10), (9) and (6). Finally, using (10) and (8) we get

$$\liminf_{n\to\infty} f(n)^{1/n} \geqslant (\beta-\varepsilon)^{1/(1+\varepsilon)}.$$

It follows that $\alpha = \beta$. The case $\beta = \infty$ can be disposed of in a similar manner.

In conclusion, we mention an application to a problem in graph coloring. Let $g(n)$ be the largest positive integer for which there exists some way of coloring the edges of a complete graph on $g(n)$ vertices in n colors without forcing the appearance of a monochromatic triangle. That $g(n)$ exists follows from a well known theorem of F. P. Ramsay ([3]), and in fact in [2] it is proved that

$$g(n) \leqslant [n!e]. \tag{11}$$

However, it seems that no lower bound for $g(n)$ appears in the literature. Here we prove that

$$g(n) \geqslant f(n)+1 \tag{12}$$

and hence, in view of (4), that

$$g(n) > 89^{\frac{1}{4}n - c\log n}. \tag{13}$$

In order to prove (12), let $A_1, A_2, \ldots, A_n$ be disjoint sum-free sets containing the integers $1, 2, \ldots, f(n)$. Let G be a complete graph with vertices $P_0, P_1, \ldots, P_{f(n)}$. Color the edges of G in the n colors $C_1, C_2, \ldots, C_n$ by coloring the edge joining P_s and P_t color C_j if $|s-t| \in A_j$. Suppose there results a triangle with vertices P_s, P_t and P_r all of whose edges are colored C_j. We may assume $s > t > r$. Then $s-t,\ s-r,\ t-r \in A_j$. But $(s-t)+(t-r) = (s-r)$ and this contradicts the fact that A_j is sum--free.

It is interesting to observe that (11) and (12) afford an independent proof of (1).

References

[1] L. D. Baumert, *Sum-free sets* (unpublished).

[2] R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Can. Jour. Math. 7 (1955), pp. 1-7.

[3] F. P. Ramsay, *On a problem in formal logic*, Proc. Lond. Math. Soc. 30 (1930), pp. 264-286.

[4] I. Schur, *Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$*, Jahresber. Deutsch. Math. Verein. 25 (1916), pp. 114-117.

Reçu par la Rédaction le 5. 5. 1965

ACTA ARITHMETICA
XI (1966)

On the difference $\pi(x) - \mathrm{li}(x)$

by

R. Sherman Lehman (Berkeley, Cal.)

1. Introduction. The prime number theorem states that $\pi(x)$, the number of primes less than or equal to x, is asymptotically equal to $\mathrm{li}(x)$ as $x \to \infty$ where

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0+} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right\}.$$

It is a remarkable fact that the difference $\pi(x) - \mathrm{li}(x)$ is negative for all values of x at which $\pi(x)$ has been calculated exactly. In particular, Rosser ([11], p. 72) has shown that the difference is negative for all $x \leqslant 10^8$. Nevertheless, Littlewood ([9]) has proved that there is a positive number K such that

$$\frac{\log x \{\pi(x) - \mathrm{li}(x)\}}{x^{1/2} \log\log\log x}$$

is greater than K for arbitrarily large values of x and less than $-K$ for arbitrarily large values of x. Thus the situation represented by the calculations does not continue indefinitely. Skewes ([12]) has obtained a very large upper bound for the first x for which the difference is positive, namely $\exp\exp\exp\exp(7.705)$.

In this paper we first derive an explicit formula for a certain average of the difference $\pi(e^u) - \mathrm{li}(e^u)$. We then describe how this explicit formula can be combined with numerical computations performed by a computer to show that between 1.53×10^{1165} and 1.65×10^{1165} there are more than 10^{500} successive integers x for which $\pi(x) > \mathrm{li}(x)$.

2. Explicit formulas. For background information we refer to Ingham ([4]), [5]).

Throughout this paper $\varrho = \beta + i\gamma$ will denote a zero of the Riemann zeta function $\zeta(s)$ for which $0 < \beta < 1$. We denote by ϑ a number satisfying $|\vartheta| \leqslant 1$. The number denoted will, in general, be different for different occurrences and may depend on variables.