Sum-free sets of integers

by

H. L. Abbott and L. Moser (Edmonton)

A set $S$ of integers is said to be sum-free if $a, b \in S$ then $a + b \notin S$. The case where $a = b$ is not excluded, that is, $a \in S$ implies $2a \notin S$.

A well known theorem of I. Schur (1) states that if the integers $1, 2, \ldots, [n/e]$ are split in an arbitrary manner into $n$ sets, at least one of the sets fails to be sum-free. This leads us to define $f(n)$ as the largest positive integer $m$ for which there exists some way of splitting the integers $1, 2, \ldots, m$ into a sum-free sets.

It is easy to verify that $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. In 1961, L. D. Baumert (11), with the aid of a high speed computer, showed that $f(4) = 44$. Since Baumert's work has not been published we exhibit one of the ways he found of splitting the integers $1, 2, \ldots, 44$ into four sum-free sets.

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The value of $f(n)$ is not known for $n \geq 5$ and it seems to be quite difficult to determine $f(n)$, even for $n = 5$.

From Schur's theorem we get

$$(1) \quad f(n) \leq [n/e] - 1$$

and no general improvement on this upper bound for $f(n)$ has been ob-
tained up to the present time, although the known values of f(n) indicate that (1) is not best possible. On the other hand, Schar proved that

\[ f(n+1) > 3f(n)+1 \]

and from (2) and the fact that f(4) = 44 we get, for \( n \geq 4 \),

\[ f(n) > \frac{89(3)^{n-1}-1}{2}. \]

The main result that we wish to establish in this paper is that

\[ f(n) > 89^n \]

for some absolute constant \( c \) and all sufficiently large \( n \). (4) is clearly better than (3).

We find it convenient to define a function \( g \) as follows: If \( f(n-1) < i < f(n) \), then \( g(i) = n \). \( g(i) \) is thus the smallest number of sum-free sets into which the integers 1, 2, ..., \( i \) can be partitioned. It follows from (3) that for \( n \) sufficiently large

\[ g(n) < \log n. \]

In order to prove (4) we shall need the following

**Theorem 1.** For all positive integers \( m \) and \( b \),

\[ f(km +gf(m))] > (2f(m)+1)^b - 1. \]

If we set \( m = 4 \) in (6) and use the fact that \( f(4) = 44 \) we get

\[ f(4k + g(4k)) > 89^k - 1 \]

and it is not difficult to see that (5) and (7) imply (4).

**Proof of Theorem 1.** Let \( X = 2f(m)+1 \) and write the numbers 1, 2, ..., \( X^n - 1 \) in base \( X \). Call a number good if each of its digits does not exceed \( f(m) \) and call a number bad if at least one of its digits exceeds \( f(m) \). We shall show that the good numbers can be partitioned into \( g(f(m)) \) sum-free sets and the bad numbers into \( km \) sum-free sets. The theorem will then follow.

Let \( A_1, A_2, ..., A_{g(f(m))} \) be disjoint sum-free sets containing the numbers 1, 2, ..., \( f(m) \). Divide the good numbers into sets \( B_1, B_2, ..., B_{g(f(m))} \) by placing a number in class \( B_i \) if the sum of its digits belongs to \( A_i \). This can be done since the sum of the digits of a good number does not exceed \( f(m) \). It is not difficult to see that the sets \( B_1, B_2, ..., B_{g(f(m))} \) are sum-free.

Divide the bad numbers into \( k \) classes \( C_1, C_2, ..., C_k \) by placing

\[ a = a_i + a_i X + a_i X^2 + ... + a_i X^{i-1} + a_i X^i \]

in class \( C_i \) if \( a_i < f(m)+1 \) for \( i = 1, 2, ..., j-1 \) and \( a_j \geq f(m)+1 \). Next divide each of \( C_1, C_2, ..., C_k \) into \( m \) sets as follows: Let \( D_1, D_2, ..., D_m \) be disjoint sum-free sets containing the numbers 1, 2, ..., \( f(m) \), and split the numbers in \( C_i \) into \( m \) sets \( D_{i1}, D_{i2}, ..., D_{im} \) by placing

\[ a = a_i + a_i X + a_i X^2 + ... + a_i X^{j-1} + a_i X^j \]

in \( D_{ij} \) if \( a_i = u \pmod{X} \) for some \( u \in \mathbb{D_i} \). Since \( a_i \) is one of the numbers \( f(m)+1, f(n)+2, ..., 2f(m) \) exactly one such \( u \) can be found. It remains to be shown that \( D_c \) is sum-free. Suppose that we can find \( a, b, c \in D_c \) such that \( a + b = c \). We have

\[ a = \sum_{i=1}^{k} a_i X^{i-1}, \quad b = \sum_{i=1}^{k} b_i X^{i-1}, \quad c = \sum_{i=1}^{k} a_i X^{i-1} \]

where \( a_i, b_i, c_i \leq f(m) \) for \( i = 1, 2, ..., j-1 \), \( a_j, b_j, c_j \geq f(m)+1 \) and \( a_j = -u \pmod{X} \), \( b_j = -v \pmod{X} \), and \( c_j = -w \pmod{X} \) where \( u, v, w \in D_c \). Since \( a_j + b_j = X + a_j \), it follows that \( u + v = w \pmod{X} \), and since \( u, v, w \leq f(m) \) we must have \( u + v = w \). However, this contradicts the fact that \( D_c \) is sum-free. The bad numbers have therefore been partitioned into \( km \) sum-free sets and the proof of the theorem is complete.

It seems likely that (4) could be improved even further if one knew the value of \( f(n) \) for some value of \( n \).

While the upper and lower bounds for \( f(n) \) are quite far apart, we can still gain a little more insight into the behavior of \( f(n) \). We show, using (6), that \( \lim f(n)^{1/n} \) exists, although we cannot decide whether the limit is finite or infinite. Let

\[ a = \lim_{n \to \infty} f(n)^{1/n} \leq \lim_{n \to \infty} f(n)^{1/n} = \beta. \]

Suppose first that \( \beta < \infty \). Let \( \varepsilon > 0 \) be given, and let \( m \) be the smallest integer for which

\[ f(m)^{1/m} > \beta - \varepsilon. \]

If \( k \geq k(\varepsilon) \),

\[ km + g(f(m))] < [km(1 + \varepsilon)]. \]

Let

\[ \lfloor km(1 + \varepsilon) \rfloor \leq n \leq \lfloor (k+1)m(1 + \varepsilon) \rfloor. \]

Then

\[ f(n) > f([km(1 + \varepsilon)]) > f(km + g(f(m))) > (2f(m)+1)^k - 1 > f(n)^k \]

where we have used (10), (9) and (6). Finally, using (10) and (8) we get

\[ \lim f(n)^{1/n} = (\beta - \varepsilon)^{(1+\varepsilon)} \]

It follows that \( a = \beta \). The case \( \beta = \infty \) can be disposed of in a similar manner.
In conclusion, we mention an application to a problem in graph coloring. Let \( g(n) \) be the largest positive integer for which there exists some way of coloring the edges of a complete graph on \( g(n) \) vertices in \( n \) colors without forcing the appearance of a monochromatic triangle. That \( g(n) \) exists follows from a well known theorem of F. P. Ramsey [3]), and in fact in [2] it is proved that

\[
(11) \quad g(n) \leq \lceil n/4 \rceil.
\]

However, it seems that no lower bound for \( g(n) \) appears in the literature. Here we prove that

\[
(12) \quad g(n) \geq f(n) + 1
\]

and hence, in view of (4), that

\[
(13) \quad g(n) \geq 89^{\frac{n}{105}}.
\]

In order to prove (12), let \( A_1, A_2, \ldots, A_n \) be disjoint sum-free sets containing the integers 1, 2, \ldots, \( f(n) \). Let \( G \) be a complete graph with vertices \( P_1, P_2, \ldots, P_{100} \). Color the edges of \( G \) in the \( n \) colors \( C_1, C_2, \ldots, C_n \) by coloring the edge joining \( P_s \) and \( P_t \) color \( C_s \) if \( s \neq t \neq C_s \). Suppose there result a triangle with vertices \( P_s, P_t, P_r \) all of whose edges are colored \( C_s \). We may assume \( s > t > r \). Then \( s-t, s-r, t-r \in A_1 \). But \( s-t, s-r, t-r \in A_1 \) and this contradicts the fact that \( A_1 \) is sum-free.

It is interesting to observe that (11) and (12) afford an independent proof of (1).

References


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On the difference \( \pi(x) - \text{li}(x) \)

by

R. Sherman Lehman (Berkeley, Cal.)

1. Introduction. The prime number theorem states that \( \pi(x) \), the number of primes less than or equal to \( x \), is asymptotically equal to \( \text{li}(x) \) as \( x \to \infty \) where

\[
\text{li}(x) = \int_2^x \frac{dt}{\log t} + \int_1^2 \frac{dt}{\log t}.
\]

It is a remarkable fact that the difference \( \pi(x) - \text{li}(x) \) is negative for all values of \( x \) at which \( \pi(x) \) has been calculated exactly. In particular, Rosser [11], p. 72 has shown that the difference is negative for all \( x \leq 10^4 \). Nevertheless, Littlewood [19] has proved that there is a positive number \( K \) such that

\[
\log \frac{\pi(x) - \text{li}(x)}{x^{\beta}} \log \log x
\]

is greater than \( K \) for arbitrarily large values of \( x \) and less than \( -K \) for arbitrarily large values of \( x \). Thus the situation represented by the calculations does not continue indefinitely. Skewes [12] has obtained a very large upper bound for the first \( x \) for which the difference is positive, namely \( \exp(\exp(7.75)) \).

In this paper we first derive an explicit formula for a certain average of the difference \( \pi(x) - \text{li}(x) \). We then describe how this explicit formula can be combined with numerical computations performed by a computer to show that between \( 1.38 \times 10^{37} \) and \( 1.65 \times 10^{37} \) there are more than \( 10^{37} \) successive integers \( x \) for which \( \pi(x) > \text{li}(x) \).

2. Explicit formulas. For background information we refer to Ingham [4], [5].

Throughout this paper \( \rho = \beta + i\gamma \) will denote a zero of the Riemann zeta function \( \zeta(s) \) for which \( 0 < \beta < 1 \). We denote by \( \vartheta \) a number satisfying \( |\vartheta| < 1 \). The number denoted will, in general, be different for different occurrences and may depend on variables.