to c modulo H which are less than x for which $G(t)$ has all of its prime factors greater than $x^{1/R}$, i.e., $G(t)$ has at most $R\cdot m$ prime factors, m being the degree of the polynomial. Since the polynomial $F(n)$ of Theorem 1 is equal to $F(c)G(t)$, the theorem follows if we set

$$A = Rm+A_1,$$

where A_1 is the number of prime factors of $F(c)$.

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On Mordell's theorem

by

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1. Suppose that $R(\sqrt{d})$ is a real quadratic field with fundamental discriminant d, main unit $E_1 = T_1+U_1\sqrt{d}$ and class number $h(d)$. Following Berger and Leopoldt ([6], [21]), we introduce the generalized Bernoulli numbers B_χ^k belonging to a primitive residue character χ modulo $f \geqslant 1$ by the relation(1)

$$\sum_{r=1}^{f}\frac{\chi(r)te^{rt}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_\chi^k\frac{t^k}{k!}, \qquad |t| < \frac{2\pi}{f}.$$

Then the results we find in some Mordell's articles ([24]-[27]), and in the article by Ankeny and Chowla ([4]) demonstrate the equivalence of two facts

$$U_1 \equiv 0 (\operatorname{mod} p),$$

$$B_\chi^{(p-1)/2} \equiv 0 \ (\operatorname{mod} p),$$

where $f=1$ for $d=p\equiv 1(\operatorname{mod} 4)$, and $f=4$ for $d=4p$, $p\equiv 3(\operatorname{mod} 4)$.

This fact was first stated by Kiselev ([15], [16]) and later independently by Ankeny, Artin and Chowla ([1], [2], [4]), but Mordell succeeded without Dirichlet's formulae which have not up to now been proved with the help of elementary methods.

In this note, by extending Mordell's method of p-adic logarithm, there is demonstrated the

THEOREM. *Let $R(\sqrt{d})$ be a real quadratic field with fundamental discriminant $d=np$, $p>3$, an odd prime number and $1\leqslant n<p$. The congruence*

$$U_1 \equiv 0 (\operatorname{mod} p^l) \tag{1}$$

(1) As for arithmetical properties of B_χ^k, see articles [9], [18], [19], [21], [29]. We remark also that for $f=1$ and $f=4$, generalized Bernoulli numbers correspond to usual Bernoulli and Euler numbers.

holds if and only if

(2) $$\frac{1}{m}B_\chi^m \equiv 0 \pmod{p^l},$$

where l is any positive integer, $m = ((p-1)/2)p^{l-1}$, $\varepsilon = (-1)^{(p-1)/2}$ *and* $\chi(x) = \left(\frac{\varepsilon n}{x}\right)$ *is Kronecker's symbol.*

The demonstration is obtained with the help of an estimate for the class number $h(d)$.

2. It is known that $h(d) = O(\sqrt{d})$ and a more exact result follows from Hardy-Littlewood's hypothesis ([3])

$$h(d) = O\left(\sqrt{d}\,\frac{\ln\ln d}{\ln d}\right).$$

Also for any $\varepsilon > 0$ there exists infinitely many real quadratic fields such that $h(d) > d^{1/2-\varepsilon}$ (see [5]).

For the purpose of our note, for example, the following estimate is sufficient (²)

(3) $$h(d) < \sqrt{d}.$$

This specification of the result of Polya ([18]), Schur ([32]), Landau ([20]) and Hua Loo Ken ([12]) can be easily derived if we consider Dirichlet's formulae for class number of real quadratic field in the form

(4) $$h(d) = \frac{\sqrt{d}}{2\ln E_1}\mathscr{L}(1|\chi),$$

where $\mathscr{L}(1|\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n}$ and $\chi(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol.

Indeed for $E_1 = \frac{1}{2}(t+u\sqrt{d})$ provided $t^2-u^2d = \pm 4$ the inequality $t \geq u\sqrt{d-4}$ holds, so that

$$E_1 > \tfrac{1}{2}u(\sqrt{d}+\sqrt{d-4}) > \sqrt{d-3},$$

and since $\mathscr{L}(1|\chi) < \frac{1}{2}\ln d + 1$ (see [12], lemma 5), then

$$h(d) < \sqrt{d}\,\frac{\frac{1}{2}\ln d+1}{\ln(d-3)}.$$

(²) Note that many recent articles ([3], [8], [11], [22]) contain much weaker results given sometimes for special cases. As for $d < -4$ we can get $h(d) < \frac{1}{3}\sqrt{|d|}\ln|d|$.

Inequality (3) follows since $(\frac{1}{2}\ln d+1)/\ln(d-3) < 1$ for $d \geq 17$ and for $d = 5, 8, 12, 13$, as known, $h(d) = 1$. Therefore (3) is true for all d.

3. Let $k = R(\varrho)$ with $\varrho = \exp(2\pi i/p)$, where p is an odd prime rational integer and R is rational number field. Then provided $n \geq 1$ and $(n, p) = 1$, we consider a field $K = k(\zeta) = R(\xi)$, where $\zeta = \exp(2\pi i/n)$ and $\xi = \exp(2\pi i/np)$. From the theorem of class field theory (for example, see [35], Ch. 3, par. 12), the odd prime ideal $1-\varrho$ from field k decomposes in K into the product of different prime ideals so that if $\mathfrak{p}$, a prime ideal of K and $\mathfrak{p}\,|\,(1-\varrho)$, then $\mathfrak{p}\,\|\,(1-\varrho)$ and therefore together with $(1-\varrho)^{p-1}\,\|\,p$, we have $\mathfrak{p}^{p-1}\,\|\,p$. Selecting an integer $\pi \in K$ with $\mathfrak{p}\,\|\,\pi$, we consider the π-adic algebraic integers $a, b \in K$ such that $a \equiv b \equiv 1 \pmod{\mathfrak{p}}$. For these numbers, we define the quotient

$$f(a) = \frac{a^{p^l}-1}{p^l}.$$

Since the formulae

$$a \equiv b \pmod{p^l\mathfrak{p}}, \quad \text{and} \quad (a+p^l\pi\beta)^{p^l} \equiv a^{p^l}+p^{2l}\pi\gamma \pmod{p^{2l}\mathfrak{p}},$$

where β, γ are π-adic integers of K, imply that $f(a) \equiv f(b) \pmod{p^l\mathfrak{p}}$, so the quotient $f(a)$ is characterized by residues $\bmod\, p^l\mathfrak{p}$. This quotient has the properties of a logarithm (³), namely, if $a \equiv b \equiv 1 \pmod{\mathfrak{p}}$, then

1. $f(ab) \equiv f(a)+f(b) \pmod{p^l\mathfrak{p}}$, since

$$\frac{(ab)^{p^l}-1}{p^l} = \frac{a^{p^l}b^{p^l}-a^{p^l}+a^{p^l}-1}{p^l} = a^{p^l}\frac{b^{p^l}-1}{p^l}+\frac{a^{p^l}-1}{p^l}, \quad a^{p^l} \equiv 1 \pmod{p^l\mathfrak{p}},$$

2. $f\left(\frac{a}{b}\right) \equiv f(a)-f(b) \pmod{p^l\mathfrak{p}}$.

Using these two properties, we conclude that

3. if $a_i \equiv b_i \equiv 1 \pmod{\mathfrak{p}}$, $1 \leq i \leq s$, and

(5) $$\prod_i a_i - \prod_i b_i \equiv 0 \pmod{p^l\mathfrak{p}},$$

then we have

(6) $$\sum_{i=1}^{s}\frac{a_i^{p^l}-b_i^{p^l}}{p^l} \equiv 0 \pmod{p^l\mathfrak{p}}.$$

(³) For details concerning relation between $f(a)$ and p-adic logarithm for case $l = 1$, see [2], and for any $l > 1$ [23], [29].

Indeed, from $\prod_i a_i - \prod_i b_i \equiv 0 \pmod{p^l \mathfrak{p}}$, it follows that

$$\prod_i \frac{a_i}{b_i} \equiv 1 \pmod{p^l \mathfrak{p}}$$

or

$$\Big(\prod_i \frac{a_i}{b_i}\Big)^{p^l} = 1 + p^{2l}\pi\gamma$$

with some π-adic integer $\gamma \in K$, so that

$$\frac{1}{p^l}\Big\{\Big(\prod_i \frac{a_i}{b_i}\Big)^{p^l} - 1\Big\} \equiv 0 \pmod{p^l \mathfrak{p}}$$

and, together with properties 1 and 2 for $f(a)$, we obtain property 3.

Mordell himself uses his p-adic logarithm method only in the cases $n = 1$ and $n = 4$.

4. Supposing $d = np$ is a fundamental discriminant of the real quadratic field $R(\sqrt{d})$ with $1 \leqslant n < p$, we conclude that $h(d) < p$ (see (3)). Firstly we assume that $n > 1$. Then Dirichlet's formula for the class-number $h(d)$ of field $R(\sqrt{d})$ may be written as

$$E_1^{2h} = \prod_{\left(\frac{d}{b}\right) = -1, 0<b<d} (1-\xi^b)^2,$$

or

$$E_1^{-2h} = \prod_{\left(\frac{d}{a}\right) = +1, 0<a<d} (1-\xi^a)^2,$$

where $\left(\frac{d}{r}\right)$ is Kronecker's symbol. Then since

$$E_1^{-2} = (T_1 - U_1\sqrt{d})^2,$$

and

$$\frac{E_1^{2h} - E_1^{-2h}}{2} = U_1(2hT_1^{2h-1} + C_{2h}^3 T_1^{2h-3} U_1^2 d + \ldots)\sqrt{d},$$

the conditions

(7) $$U_1 \equiv 0 \pmod{p^l}$$

and

(8) $$\frac{\prod_b (1-\xi^b)^2 - \prod_a (1-\xi^a)^2}{2} \equiv 0 \pmod{p^l \mathfrak{p}}$$

follow one from the other. Here the prime ideal $\mathfrak{p}$ of $R(\xi)$ divides $(1-\varrho)$, $\varrho = \exp(2\pi i/p)$.

Writing $\zeta = \exp(2\pi i/n)$, $\varepsilon = (-1)^{(p-1)/2}$ and selecting for every b with condition $\left(\frac{d}{b}\right) = -1$ two positive integers μ and ν by

$$b \equiv \begin{cases} n\mu \pmod{p}, & 0 < \mu < p, \\ p\nu \pmod{n}, & 0 < \nu < n, \end{cases}$$

so that

$$\xi^b = \zeta^\nu \varrho^\mu \quad \text{and} \quad -1 = \left(\frac{d}{b}\right) = \left(\frac{\varepsilon n}{b}\right)\left(\frac{\varepsilon p}{b}\right) = \varepsilon\left(\frac{\varepsilon n}{\nu}\right)\left(\frac{\varepsilon p}{\mu}\right),$$

we obtain

$$1-\xi^b = 1-\zeta^\nu\varrho^\mu = \varrho^\mu - \zeta^\nu\varrho^\mu + 1 - \varrho^\mu = \varrho^\mu(1-\zeta^\nu)[1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}].$$

Since $(\varrho^\mu, \mathfrak{p}) = (1-\zeta^\nu, \mathfrak{p}) = 1$ acting by analogy with $1-\xi^a$, where $\left(\frac{d}{a}\right) = +1$, we rewrite (8) in the form

$$\prod_{0<\mu<p, 0<\nu<n, \left(\frac{\varepsilon p}{\mu}\right) = -\varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^2 - \prod_{0<\mu<p, 0<\nu<n, \left(\frac{\varepsilon p}{\mu}\right) = \varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^2 \equiv 0 \pmod{p^l \mathfrak{p}}.$$

Then using (5) and (6) we suppose that every product consists of $\varphi(d)$ numbers and cancelling by 2 we state

$$\frac{1}{p^l}\Big\{\sum_{\left(\frac{\varepsilon p}{\mu}\right) = -\varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^{p^l} - \sum_{\left(\frac{\varepsilon p}{\mu}\right) = \varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^{p^l}\Big\} \equiv 0 \pmod{p^l \mathfrak{p}}.$$

Further

$$\frac{1}{p^l}\sum_{\substack{0<\nu<n \\ (\nu,n)=1}}\Big\{\sum_{\left(\frac{\varepsilon p}{\mu}\right) = -\varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^{p^l} - \sum_{\left(\frac{\varepsilon p}{\mu}\right) = \varepsilon\left(\frac{\varepsilon n}{\nu}\right)} [1+(\varrho^{-\mu}-1)(1-\zeta^\nu)^{-1}]^{p^l}\Big\} \equiv 0 \pmod{p^l \mathfrak{p}}$$

or after raising to power and changing the order of summation,

$$(9)\quad \sum_{k=1}^{p^l}\frac{1}{p^l}C_{p^l}^k\Big\{\sum_\nu(1-\zeta^\nu)^{-k}\Big[\sum_{\left(\frac{\varepsilon p}{\mu}\right)=-\varepsilon\left(\frac{\varepsilon n}{\nu}\right)}(\varrho^{-\mu}-1)^k-\sum_{\left(\frac{\varepsilon p}{\mu}\right)=\varepsilon\left(\frac{\varepsilon n}{\nu}\right)}(\varrho^{\mu}-1)^k\Big]\Big\}\equiv 0\,(\mathrm{mod}\,p^l\mathfrak{p}).$$

Note that

$$\sum_{\left(\frac{\varepsilon p}{\mu}\right)=-\varepsilon\left(\frac{\varepsilon n}{\nu}\right)}\varrho^{-\mu i}=\begin{cases}\dfrac{p-1}{2}, & \text{if}\quad p\,|\,i,\\ \dfrac{1}{2}\Big[-1+\varepsilon\left(\dfrac{\varepsilon p}{\mu}\right)\left(\dfrac{\varepsilon p}{i}\right)\sqrt{\varepsilon p}\Big], & \text{if}\quad p\nmid i,\end{cases}$$

and hence

$$\sum_{\left(\frac{\varepsilon p}{\mu}\right)=-\varepsilon\left(\frac{\varepsilon n}{\nu}\right)}(\varrho^{-\mu}-1)^k-\sum_{\left(\frac{\varepsilon p}{\mu}\right)=\varepsilon\left(\frac{\varepsilon n}{\nu}\right)}(\varrho^{-\mu}-1)^k=\frac{1}{2}\left(\frac{\varepsilon n}{\nu}\right)\sqrt{\varepsilon p}\sum_{i=0}^{k}(-1)^{k-i}C_k^i\left(\frac{\varepsilon p}{i}\right).$$

It is easy to see that

$$\frac{1}{p^l}C_{p^l}^kC_k^i\left(\frac{\varepsilon p}{i}\right)\equiv(-1)^k\frac{1}{k}C_k^i\left(\frac{\varepsilon p}{i}\right)(\mathrm{mod}\,p^l),$$

since $\left(\frac{\varepsilon p}{i}\right)=0$, if $p\,|\,i$, and, if $p\nmid i$, then $\frac{1}{k}C_k^i$ is a p-adic integer. Then from (9)

$$\sqrt{\varepsilon p}\sum_{k=1}^{p^l}\frac{(-1)^{k-1}}{k}\Big\{\sum_\nu\left(\frac{\varepsilon n}{\nu}\right)(1-\zeta^\nu)^{-k}\sum_{i=0}^{k}(-1)^{k-i}C_k^i\left(\frac{\varepsilon p}{i}\right)\Big\}\equiv 0\,(\mathrm{mod}\,p^l\mathfrak{p}).$$

Since from all i with $m=\frac{1}{2}(p-1)p^{l-1}$,

$$\left(\frac{\varepsilon p}{i}\right)\equiv i^m(\mathrm{mod}\,p^l),$$

so $\sum_{i=0}^{k}(-1)^{k-i}C_k^i\left(\frac{\varepsilon p}{i}\right)\equiv\Delta^k0^m(\mathrm{mod}\,p^l)$, where Δ^k0^m is a k-any finite difference.

Therefore

$$(10)\qquad -\sqrt{\varepsilon p}\sum_{k=1}^{p^l}\sum_\nu\frac{\left(\frac{\varepsilon n}{\nu}\right)\Delta^k0^m}{k(\zeta^\nu-1)^k}\equiv 0\,(\mathrm{mod}\,p^l\mathfrak{p}).$$

We notice that $\Delta^k0^m=0$ for $k>m$.

With the help of Kiselev's identity ([16], [17])

$$\frac{1}{r}\sum_{\tau=1}^{r-1}B_s\left(\frac{\tau}{r}\right)\tau^{vt}=\frac{(-1)^{s-1}s}{r^s}\sum_{k=1}^{s}\frac{\Delta^k0^s}{k(\tau^t-1)^k},$$

where $r>1$, v — integers, $0<v\leqslant r-1$, $\tau=\exp(2\pi i/r)$ and $B_s(x)$ is a Bernoulli polynomial, we conclude from (10) that

$$\frac{1}{m}\sum_\nu\left(\frac{\varepsilon n}{\nu}\right)\sum_{\tau=0}^{n-1}B_m\left(\frac{\tau}{n}\right)\zeta^{\tau\nu}\equiv 0\,(\mathrm{mod}\,\mathfrak{p}^{(p-1)(l-\frac{1}{2})+1}).$$

Noticing that

$$\sum_\nu\left(\frac{\varepsilon n}{\nu}\right)\zeta^{v\nu}=\left(\frac{\varepsilon n}{v}\right)\sqrt{\varepsilon n},$$

we obtain at last

$$\frac{1}{m}B_\chi^m\equiv 0\,(\mathrm{mod}\,\mathfrak{p}^{(p-1)(l-\frac{1}{2})+1}),$$

where $\chi(v)=\left(\frac{\varepsilon n}{v}\right)$ is Kronecker's symbol and B_χ^m rational numbers, so that

$$\frac{1}{m}B_\chi^m\equiv 0\,(\mathrm{mod}\,p^l).$$

In the case $d=p\equiv 1\,(\mathrm{mod}\,4)$, provided $\prod_{1\leqslant r\leqslant p-1}(1-\varrho)^r=p$ with $\varrho=\exp(2\pi i/p)$ we see that $U_1\equiv 0\,(\mathrm{mod}\,p^l)$ is equivalent to

$$\frac{\prod_b(1-\varrho^b)^2-\prod_a(1-\varrho^a)^2}{2}\equiv 0\,(\mathrm{mod}\,p^{l+1}\mathfrak{p}).$$

Using the given above method for $l'=l+1$ and writing $m'=((p-1)/2)p^l$, finally from Kummer's congruence,

$$\frac{B_\chi^{m'}}{m'}\equiv\frac{B_\chi^m}{m}(\mathrm{mod}\,p^l)$$

we get

$$\frac{1}{m}B_\chi^m\equiv 0\,(\mathrm{mod}\,p^l),$$

where in this case $B_\chi^m=B_m$ are usual Bernoulli numbers.

5. In connection with all said above, there is a hypothesis first suggested in the article [2], completed in [17] and repeatedly discussed in [33], [34], [19], [31]. Namely a far advanced calculation gives a right to suppose that in case when the discriminant of field $R(\sqrt{d})$ is $d = p$ or $d = 4p$ ($p \equiv 1$, $p \equiv 3 \pmod 4$ respectively and p an odd prime integer), we have

$$U_1 \not\equiv 0 \pmod p$$

or (just the same)

$$B_\chi^{(p-1)/2} \equiv 0 \pmod p.$$

Here χ is an character mod f, $f = 1$ or 4.

It is also interesting to observe that if $d < -4$, so for $d = -p$, $p \equiv 3 \pmod 4$ and $d = -4p$, $p \equiv 1 \pmod 4$ (in both cases p is an odd prime integer), we have[4]

$$(11) \qquad h(d) \equiv -2B_\chi^{(p+1)/2} \pmod p,$$

where χ is a character mod$f$, $f = 1$ or 4, $\chi(-1) = -1$ for $f = 4$.

Therefore since $h(d) < \frac{1}{3}\sqrt{|d|}\ln|d|$ for $d < -4$, in above cases we have $h(d) < p$ and thus

$$B_\chi^{(p+1)/2} \not\equiv 0 \pmod p.$$

Lastly we note that when the square free kernel of a composite discriminant contains more then one prime integer, this hypothesis is false. For example, for $d = 184 = 8.23$, fundamental unit is $E_1 = 24335 + 897\sqrt{184}$ and $U_1 = 897 = 29.23$.

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On the zeros of L-functions

by

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Introduction

1. Let $L(s,\chi)$ be any L-function of Dirichlet with a character χ to modulus $D>2$. Using an unproved hypothesis in 1945, Linnik proved (see [10], §17) that for any $\lambda\epsilon[0,\log D]$ and $t_0\epsilon[-\log^3 D,\log^3 D]$ the number of zeros of $L(s,\chi)$ lying in the rectangle $(1-\lambda/\log D\leqslant\sigma\leqslant 1,$ $t_0\leqslant t\leqslant t_0+1)$ in the plane of the complex variable $s=\sigma+it$ does not exceed $e^{c_0\lambda}$, where c_0 (and later on $c, c', c_1, c_2, \ldots$) stands for an appropriate absolute constant >0 (¹). In 1944 Linnik [9] proved by a very complicated method that the number of functions $L(s,\chi)$ having at least one zero in the rectangle $\{1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t|\leqslant\min(\lambda^{100},\log^3 D)\}$ does not exceed $e^{c_2\lambda}$. Ten years later Rodosskiĭ ([12], pp. 333-341) gave a simpler proof, but merely for the rectangles $(1-\lambda/\log D\leqslant\sigma\leqslant 1,$ $|t|\leqslant e^{\lambda}/\log D)$. In 1961 Turán [13] proved by his new method a slightly more general result: The number of zeros of the function $Z(s)=\prod_{\chi}L(s,\chi)$ in the rectangle $(1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t-t_0|\leqslant e^{\lambda}/\log D)$ with $|t_0|<D^{1/2}$ does not exceed $e^{c_3\lambda}$.

The height of the rectangle considered by Turán or Rodosskiĭ for a large D and $\lambda<\log\log\log D$ (for example) is very small. In order to eliminate this restriction I have combined Turán's method with some ideas taken from Linnik's paper [10]. By these means I have succeeded in proving the following

THEOREM. (i) *For any $T\geqslant D$ and $\lambda\epsilon[0,\log T]$ the number of zeros of the function $L(s,\chi)$ in the rectangle*

$$(1)\qquad (1-\lambda/\log T\leqslant\sigma\leqslant 1,\ |t|\leqslant T)$$

does not exceed $e^{c\lambda}$

(ii) *The same is true for the function $Z(s)=\prod_{\chi}L(s,\chi)$.*

(¹) Linnik's proof is based on the following hypothesis: Any circle of radius $1/\log D$ with the centre in the rectangle $(1-\log\log D/\log D<\sigma<1,\ |t|<\log^3 D)$ contains no more than c_1 zeros of $L(s,\chi)$. He promised (see [10], pp. 111 and 118) to publish another proof for the case in which this hypothesis does not hold. Twenty years have elapsed since, but no proof of this kind has been published yet.