the conclusion follows from (22), (23) and the multiplicative property of the norm.

Remark. In connection with Theorem 5 let us remark that the theorem of Bauer gives an answer to a question of D. H. Lehmer ([6], p. 436) concerning possible types of homogeneous polynomials $F(x, y)$ of degree $\frac{1}{2}\varphi(n)$ such that when $(x, y) = 1$, the prime factors of $F(x, y)$ either divide n or are of the form $nk\pm 1$. (If $f(x) = x^3+x^2-2x-1$, then $y^3f(x/y)$ is an example of such polynomial for $n = 7$.) The answer is that all such polynomials must be of the form $A\prod_{i=1}^{\frac{1}{2}\varphi(n)}(x-a_iy)$, where a_i runs through all conjugates of a primitive element of the field $Q\left(2\cos\frac{2}{n}\pi\right)$ and A is a rational integer.

Note added in proof. In connection with Theorem 2 a question arises whether solvable fields of degree p^2 (p prime) are Bauerian. J. L. Alperin has proved that the answer is positive if the field is primitive and $p > 3$. P. Roquette has found a proof for the case where the Galois group of the normal closure is a p-group (oral communication).

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An extension of the theorem of Bauer and polynomials of certain special types

by

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1. For a given algebraic number field K let us denote by $P(K)$ the set of those rational primes which have a prime ideal factor of the first degree in K. M. Bauer [1] proved in 1916 the following theorem:

If K is normal, then $P(\Omega)\subset P(K)$ implies $\Omega\supset K$. (The converse implication is immediate).

In this theorem, inclusion $P(\Omega)\subset P(K)$ can be replaced by a weaker assumption that the set of primes $P(\Omega)-P(K)$ is finite, which following Hasse we shall denote by $P(\Omega)\leqslant P(K)$.

In the preceding paper [8], one of us has characterized all the fields K for which $P(\Omega)\leqslant P(K)$ implies that Ω contains one of the conjugates of K and has called such fields *Bauerian*. The characterization is in terms of the Galois group of the normal closure $\bar{K}$ of K and is not quite explicit. Examples of non-normal Bauerian fields given in that paper are the following: fields K such that $\bar{K}$ is solvable and $\left(\frac{|\bar{K}|}{|K|}, |K|\right) = 1$ (1), fields of degree 4. The aim of the present paper is to exhibit a class of Bauerian fields that contains all normal and some non-normal fields. We say that a field K has property (N) if there exists a normal field L of degree relatively prime to the degree of K such that the composition KL is the normal closure of K. We have

Theorem 1. *If K and Ω are algebraic number fields and K has property* (N) *then $P(\Omega)\leqslant P(K)$ implies that Ω contains one of the conjugates of K.*

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(1) We let | | denote both the degree of the field over Q and the order of the group.

Not all fields K such that $\bar{K}$ is solvable and $\left(\frac{|\bar{K}|}{|K|}, |K|\right) = 1$ possess property (N). We have however

THEOREM 2. *If K is a number field such that $\left(\frac{|\bar{K}|}{|K|}, |K|\right) = 1$ and the Galois group of $\bar{K}$ is supersolvable, then K has property* (N).

In particular K can be any field of prime degree such that $\bar{K}$ is solvable or any field generated by $\sqrt[n]{a}$, where a, n are rational integers and $(n, \varphi(n)) = 1$. The field $Q(\sqrt[6]{2})$ does not possess property (N), it is however Bauerian. (It follows from a theorem of Flanders (cf. [7], Th. 167) and results of the preceding paper that $Q(\sqrt[n]{a})$ is Bauerian if $n \not\equiv 0 \bmod 8$.) We have no example of non-normal field K with property (N), such that $\bar{K}$ is non-solvable however one could construct such a field provided there are fields corresponding to every Galois group.

The original Bauer's theorem has been applied in [2] to characterize polynomials $f(x)$ with the property that in every arithmetical progression there is an integer x such that $f(x)$ is a norm of an element of a given normal field K. The method used in [2] can be modified in order to obtain

THEOREM 3. *Let K be a field having property* (N) *and let $N_{K/Q}(\omega)$ denote the norm from K to the rational field Q. Let $f(x)$ be a polynomial over Q such that the multiplicity of each zero of $f(x)$ is relatively prime to $|K|$. If in every arithmetical progression there is an integer x such that*

$$f(x) = N_{K/Q}(\omega) \quad \text{for some } \omega \in K,$$

then

$$f(x) = N_{K/Q}(\omega(x)) \quad \text{for some } \omega(x) \in K[x].$$

The proofs of Theorems 1-3 given in § 3 are independent of the preceding paper [8] and assume only the original Bauer's theorem. They are preceded in § 2 by some lemmata of seemingly independent interest. Theorems 1 and 3 could be proved by the methods and results of [8]. We retain the present proofs since they use, as do the statements of the theorems, only the language of field theory. We refer to [8] for examples showing that an extension of the theorems to an arbitrary field K is impossible.

2. LEMMA 1. *Let fields K and L have the following properties: L is normal, (degree K, degree L) $= 1$, KL is normal. Then for any field Ω the inclusion*

$$\Omega L \supset KL \tag{1}$$

implies that Ω contains one of the conjugates of K.

Proof. It follows from (1) that

$$\Omega KL = \Omega L. \tag{2}$$

Since KL is normal and L is normal, we have

$$|\Omega KL| = \frac{|\Omega|\,|KL|}{|\Omega \cap KL|}, \tag{3}$$

$$|\Omega L| = \frac{|\Omega|\,|L|}{|\Omega \cap L|}. \tag{4}$$

(Cf. [6], § 19.5, Satz 1).

Since clearly $|KL| = |K|\,|L|$, we get from (2), (3) and (4)

$$|\Omega \cap KL| = |K|\,|\Omega \cap L|. \tag{5}$$

Let $\mathfrak{G}$ be the Galois group of KL. And let $\mathfrak{H}$, $\mathfrak{J}$, $\mathfrak{N}$ be subgroups of $\mathfrak{G}$ corresponding to K, $\Omega \cap KL$ and L, respectively.

In view of (5)

$$[\mathfrak{G}:\mathfrak{H}] \,|\, [\mathfrak{G}:\mathfrak{J}], \quad \text{thus} \quad |\mathfrak{J}|\,|\,|\mathfrak{H}| \quad \text{and} \quad (|\mathfrak{J}|, |\mathfrak{N}|) = 1.$$

On the other hand, since $\mathfrak{H}\mathfrak{N} = \mathfrak{G}$, and $\mathfrak{N}$ is normal, it can be easily shown that

$$\mathfrak{J}\mathfrak{N} = (\mathfrak{J}\mathfrak{N} \cap \mathfrak{H})\mathfrak{N}.$$

Thus $\mathfrak{J}$ and $\mathfrak{J}\mathfrak{N} \cap \mathfrak{H}$ are two representative subgroups of $\mathfrak{J}\mathfrak{N}$ over $\mathfrak{N}$ and by Theorem 27 ([9], Chapter IV) they are conjugate. The theorem in question had been deduced from the conjecture now proven [3] that all groups of odd orders are solvable. It follows that $\mathfrak{J}$ is contained in a certain conjugate of $\mathfrak{H}$, thus $\Omega \cap KL$ contains a suitable conjugate of K and the same applies to Ω, q. e. d.

The first two assumptions of Lemma 1 are necessary as shown by the following examples

1. $K = Q(e^{2\pi i/3})$, $L = Q(\sqrt[3]{2})$, $\Omega = Q(e^{2\pi i/3}\sqrt[3]{2})$ (²),
2. $K = Q(i)$, $L = Q(\sqrt{2})$, $\Omega = Q(\sqrt{-2})$.

As to the third assumption, namely that KL is normal, we can show that it is necessary provided that there exists a field with Galois group $\mathfrak{G}$, where $\mathfrak{G}$ is the wreath product of $\mathfrak{S}_4$ acting on 4 isomorphic copies of the simple group $\mathfrak{G}_{168}$. Then in the counterexample, K is a field of degree 7^4 corresponding to the wreath product of $\mathfrak{S}_4$ acting on 4 isomorphic copies of a subgroup $\mathfrak{H}$ of $\mathfrak{G}_{168}$ of index 7, L is a normal field of degree 24 corresponding to the product of 4 copies of $\mathfrak{G}_{168}$. The construction of Ω

(²) We owe this example to Mr. Surinder Sehgal.

and the proof that it furnishes a counterexample is complicated and will be omitted.

LEMMA 2. *In any supersolvable group $\mathfrak{G}$ for each set Π of primes either there is a normal Π-subgroup $\neq 1$ or there is a normal Hall*[3] *$\hat{\Pi}$-group $\neq 1$ ($\hat{\Pi}$ is the set of all prime divisors of $|\mathfrak{G}|$ not contained in Π).*

Proof. If this lemma would be false, then there would be a supersolvable group $\mathfrak{G} \neq 1$ of minimal order for which it would be false.

If Π or $\hat{\Pi}$ are empty then the statement is trivial. Let Π and $\hat{\Pi}$ be non-empty. Since $\mathfrak{G} \neq 1$, there is a maximal normal subgroup $\mathfrak{M} \neq \mathfrak{G}$. Since $\mathfrak{G}$ is solvable $[\mathfrak{G}:\mathfrak{M}]$ is a prime p. If $\mathfrak{M}$ contains a normal Π-subgroup $\mathfrak{N} \neq 1$, then $\langle \mathfrak{N}^{\mathfrak{G}} \rangle$ is a normal Π-subgroup $\neq 1$ of $\mathfrak{G}$, a contradiction. Hence $\mathfrak{M}$ contains no normal Π-subgroup. Since $\mathfrak{M}$, a subgroup of a supersolvable group, itself is supersolvable, it follows from the minimal property of $\mathfrak{G}$ that $\mathfrak{M}$ contains a normal Hall $\hat{\Pi}$-group $\mathfrak{J}$. A normal Hall subgroup of a solvable group is the unique subgroup of its order (cf. [4], Th. 9.3.1). Therefore $\mathfrak{J}$ must be a characteristic subgroup of $\mathfrak{M}$ and hence a normal subgroup of $\mathfrak{G}$. If $p \in \Pi$ then $\mathfrak{J}$ is normal Hall $\hat{\Pi}$-group of $\mathfrak{G}$, a contradiction. Hence $p \in \hat{\Pi}$. It follows that

(6) *the index of every maximal normal subgroup of $\mathfrak{G}$ is a prime number belonging to $\hat{\Pi}$.*

Now let $\mathfrak{N} \neq 1$ be a minimal normal subgroup of $\mathfrak{G}$. Since $\mathfrak{G}$ is supersolvable, it follows that $\mathfrak{N}$ is of prime order, say q. Since we have assumed $\mathfrak{G}$ does not have a normal Π-subgroup, $q \in \hat{\Pi}$. Suppose $\mathfrak{G}/\mathfrak{N}$ contains a normal Π-subgroup $\mathfrak{H}/\mathfrak{N} \neq 1$. Since $\mathfrak{H}$ is solvable it contains a q-complement $\mathfrak{J} \neq 1$. The group $\mathfrak{J}$ is a Hall Π-subgroup of $\mathfrak{H}$. If $\mathfrak{J}$ is normal in $\mathfrak{H}$, it follows (cf. [4], Th. 9.3.1) that $\mathfrak{J}$ is a characteristic subgroup of $\mathfrak{H}$ and hence $\mathfrak{J} \neq 1$ would be a normal Π-subgroup of $\mathfrak{G}$ contrary to hypothesis. It follows that $\mathfrak{J}$ is not normal in $\mathfrak{H}$. In particular $\mathfrak{J}$ does not commute elementwise with $\mathfrak{N}$. Thus $\mathfrak{J}$ is not contained in $\mathfrak{Z}_{\mathfrak{N}}$ the centralizer of $\mathfrak{N}$.

The group $\mathfrak{Z}_{\mathfrak{N}}$ is normal in $\mathfrak{G}$. It follows that the index $[\mathfrak{G}:\mathfrak{Z}_{\mathfrak{N}}]$ is divisible by a prime $r \in \Pi$.

On the other hand, the factor group of the normalizer over the centralizer satisfies

$$\mathfrak{N}_{\mathfrak{N}}/\mathfrak{Z}_{\mathfrak{N}} \cong \mathfrak{G}/\mathfrak{Z}_{\mathfrak{N}}$$

so that it is isomorphic to a subgroup of the automorphism group of the cyclic group $\mathfrak{N}$. Hence $\mathfrak{G}/\mathfrak{Z}_{\mathfrak{N}}$ is abelian and therefore contains a normal subgroup $\mathfrak{M}_1/\mathfrak{Z}_{\mathfrak{N}}$ of index r. Hence $\mathfrak{G}$ contains a maximal normal subgroup

[3] A Hall subgroup is a subgroup whose order and index are relatively prime.

$\mathfrak{M}_1$, of prime index r, where $r \in \Pi$, contrary to (6). It follows that $\mathfrak{G}/\mathfrak{N}$ does not contain a nontrivial normal Π-subgroup.

Since $\mathfrak{G}/\mathfrak{N}$ is also supersolvable, it follows from the minimal property of $\mathfrak{G}$ that $\mathfrak{G}/\mathfrak{N}$ contains a normal Hall $\hat{\Pi}$-subgroup, say $\mathfrak{H}/\mathfrak{N}$. But then $\mathfrak{H}$ is a normal Hall $\hat{\Pi}$-subgroup of $\mathfrak{G}$, contrary to hypothesis.

Not all solvable groups possess the property enunciated in the lemma, e.g. $\mathfrak{S}_4$. On the other hand groups possessing this property need not be solvable, e.g. the direct product of $\mathfrak{A}_5$ and $\boldsymbol{Z}_{30}$. We have not found another well known class of finite groups which possess the property besides supersolvable groups.

LEMMA 3. *Let $G(x)$ be a polynomial with integral coefficients, irreducible over Q and let $G(\theta) = 0$. Let J be any subfield of $Q(\theta)$. Then*

$$G(x) = aN_{J/Q}\big(H(x)\big)$$

identically, where $H(x)$ is a polynomial over J.

Proof: See [2], Lemma 2.

3. Proof of Theorem 1. Let L be a normal field such that $(|K|, |L|) = 1$ and $KL = \bar{K}$. Assume that $P(\Omega) \leqslant P(K)$. We have

(7) $$P(\Omega L) \subset P(\Omega) \frown P(L) \leqslant P(K) \frown P(L).$$

Let q be a large prime, $q \in P(K) \frown P(L)$ and let

$$q = \mathfrak{q}_1 \mathfrak{q}_2 \ldots \mathfrak{q}_g$$

be its factorization in $\bar{K}$. Since $\bar{K}$ is normal we have

$$N_{\bar{K}/Q}(\mathfrak{q}_i) = q^{|\bar{K}|/g}.$$

Now, let $\mathfrak{p}$ be the prime ideal factor of q of degree 1 in L. We have

(8) $$N_{\bar{K}/Q}\mathfrak{p} = N_{L/Q}N_{KL/L}\mathfrak{p} = q^{|K|}.$$

On the other hand,

$$\mathfrak{p} = \mathfrak{q}_{i_1}\mathfrak{q}_{i_2} \ldots \mathfrak{q}_{i_s},$$

whence

(9) $$N_{\bar{K}/Q}\mathfrak{p} = \prod_{j=1}^{s} N_{\bar{K}/Q}\mathfrak{q}_{i_j} = q^{|\bar{K}|s/g}.$$

It follows from (8) and (9) that

$$|K| = \frac{|\bar{K}|}{g}s = \frac{|K|\,|L|}{g}s;$$

hence

(10) $$|L|\,|\,g.$$

In this proof that fact that L is normal has not been used, thus by symmetry

$$|K|\,|g.$$

Since $(|K|,|L|)=1$, $|K|\,|L|\,|g$, thus $g=|KL|=|\bar{K}|$ and $q\in P(KL)$. This shows that $P(K)\cap P(L)\leqslant P(KL)$ and we get from (7)

$$P(\Omega L)\leqslant P(KL).$$

By the theorem of Bauer it follows that $\Omega L\supset KL$ and by Lemma 1, Ω contains a conjugate of K, q. e. d.

Proof of Theorem 2. Let $\mathfrak{G}$ be the Galois group of $\bar{K}$, $\mathfrak{H}$ the subgroup of $\mathfrak{G}$ belonging to K, Π the set of primes dividing the order of $\mathfrak{H}$. Since $|\mathfrak{G}|=nm$, with $(n,m)=1$, $\mathfrak{H}$ is a Hall Π-subgroup of $\mathfrak{G}$ and hence (cf. [4], Th. 9.3.1) any normal Π-subgroup of $\mathfrak{G}$ is a subgroup of $\mathfrak{H}$. By Lemma 2 either there is in $\mathfrak{G}$ a normal Π-subgroup $\neq 1$ or there is a normal Hall $\hat{\Pi}$-subgroup. The first case is impossible since then $\mathfrak{H}$ would contain a non-trivial normal subgroup of $\mathfrak{G}$, thus there would be a normal field between K and $\bar{K}$. Therefore, there is in $\mathfrak{G}$ a normal subgroup $\mathfrak{N}$ such that $|\mathfrak{N}|\,|\mathfrak{H}|=|\mathfrak{G}|$. Let L be the field belonging to $\mathfrak{N}$. Clearly L is normal, $(|K|,|L|)=1$, $KL=\bar{K}$ and therefore the field K has property (N), q. e. d.

Proof of Theorem 3. Let

$$f(x)=cf_1(x)^{e_1}f_2(x)^{e_2}\dots f_r(x)^{e_r}, \tag{11}$$

where $c\neq 0$ is a rational number and $f_1(x),f_2(x),\dots,f_r(x)$ are coprime polynomials with integral coefficients, each irreducible over Q and where $e_1,e_2,\dots,e_r$ are non-zero integers. Put

$$F(x)=f_1(x)f_2(x)\dots f_r(x).$$

Since the discriminant of $F(x)$ is not zero, there exist polynomials $A(x)$, $B(x)$ with integral coefficients such that

$$F(x)A(x)+F'(x)B(x)=D, \tag{12}$$

identically, where D is a non-zero integer.

Let θ be a zero of some $f_j(x)$ and set $\Omega=Q(\theta)$. Let L be a normal field postulated by the assumption that K has property (N) and let $q\in P(\Omega L)$ be a large prime. Clearly $q\in P(\Omega)$ and by the theorem of Dedekind, the congruence

$$f_j(x)\equiv 0\ (\mathrm{mod}\ q)$$

is soluble. Let x_0 be a solution. By (12) we have $F'(x_0)\not\equiv 0\ (\mathrm{mod}\ q)$, whence

$$F(x_0+q)\not\equiv F(x_0)\ (\mathrm{mod}\ q^2).$$

By choosing x_1 to be either x_0 or x_0+q, we can ensure that

$$f_j(x_1)\equiv 0\ (\mathrm{mod}\ q),\qquad F(x_1)\not\equiv 0\ (\mathrm{mod}\ q^2),$$

whence $f_j(x_1)\not\equiv 0\ (\mathrm{mod}\ q^2)$ and $f_i(x_1)\not\equiv 0\ (\mathrm{mod}\ q)$ for $i\neq j$. By the hypothesis of the theorem there exists $x_2\equiv x_1\ (\mathrm{mod}\ q^2)$ such that

$$f(x_2)\equiv N_{K/Q}(\omega)\quad \text{for some } \omega\in K. \tag{13}$$

From the preceding congruences we have

$$f(x_2)\equiv 0\ (\mathrm{mod}\ q^{e_j}),\qquad f(x_2)\not\equiv 0\ (\mathrm{mod}\ q^{e_j+1}). \tag{14}$$

Let the prime ideal factorization of q in $\bar{K}=KL$ be

$$q=\mathfrak{q}_1\mathfrak{q}_2\dots\mathfrak{q}_g.$$

Since $\bar{K}$ is normal, we have

$$N_{\bar{K}/Q}\mathfrak{q}_i=q^{|\bar{K}|/g}.$$

Write the prime ideal factorization of ω in $\bar{K}$ in the form

$$(\omega)=\mathfrak{q}_1^{a_1}\mathfrak{q}_2^{a_2}\dots\mathfrak{q}_g^{a_g}\mathfrak{A}\mathfrak{B}^{-1},$$

where $\mathfrak{A},\mathfrak{B}$ are ideals in K relatively prime to q. Then

$$N_{K/Q}(\omega)=q^{|K|(a_1+a_2+\dots+a_g)/g}N_{K/Q}(\mathfrak{A})N_{K/Q}(\mathfrak{B})^{-1} \tag{15}$$

and $N_{K/Q}(\mathfrak{A})$, $N_{K/Q}(\mathfrak{B})$ are relatively prime to q.

It follows from (13), (14) and (15) that

$$|K|(a_1+a_2+\dots+a_g)/g=e_j,\qquad \text{thus}\qquad |K|\,|e_jg.$$

However, we assumed $(|K|,e_j)=1$, whence $|K|\,|g$. On the other hand $q\in P(L)$ and so by the argument in the paragraph culminating with (10), $|L|\,|g$. Since $(|K|,|L|)=1$, $|K|\,|L|\,|g$, thus $g=|KL|$ and $q\in P(KL)$. This shows that $P(\Omega L)\leqslant P(KL)$. By the theorem of Bauer it follows that $\Omega L\supset KL$ and by Lemma 1, Ω contains a conjugate of K, say K'. Applying Lemma 3 with $G(x)=f_j(x)$, $J=K'$ we conclude that

$$f_j(x)=a_jN_{K'/Q}\big(H_j(x)\big),$$

where $H_j(x)$ is a polynomial over K'. Clearly

$$f_j(x)=a_jN_{K/Q}\big(H_j'(x)\big),$$

where $H_j'(x)$ is a conjugate of H_j with coefficients in K.

By (11) and the multiplicative property of the norm, we get

$$f(x)=aN_{K/Q}\big(h(x)\big),$$

where $h(x)$ is a polynomial over K. By the hypothesis of the theorem, taking x to be a suitable integer, we infer that a is the norm of an element a of K. Putting $\omega(x) = ah(x)$, we obtain $f(x) = N_{K/Q}(\omega(x))$, identically, q. e. d.

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Quadratic Diophantine equations with a parameter

by

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We have proved in [2] the following result: Let $f(t)$ be a polynomial with integral coefficients and suppose that every arithmetical progression contains an integer t such that $F(x, y, t) = x^2+y^2-f(t) = 0$. Then $F(x(t), y(t), t) = 0$ identically, where $x(t)$ and $y(t)$ are polynomials with integral coefficients. This can be extended to $F(x, y, t) = x^2+\Delta y^2-f(t)$ provided $x(t), y(t)$ are allowed to have rational coefficients. An example is given in [5] showing that an analogous theorem does not hold for a general polynomial $F(x, y, t)$ even if we assume solubility for all integers t, and the question is raised there of the connection between the solubility of $F(x, y, t) = 0$ in rationals x, y for a suitable t from every arithmetical progression and the solubility in rational functions $x(t), y(t)$ (cf. also [4], Problems 5 and 6). In this paper we prove (Theorem 2) that such a connection does exist if $F(x, y, t)$ is of degree at most two in x and y. Whether, under the last assumption, the solubility in integers implies the solubility in polynomials with rational coefficients we do not know even in the simple case

$$F(x, y, t) = a(t)xy+b(t)x+c(t)$$

(a solution in polynomials with integral coefficients need not exist as is shown by the example $a(t) = 0$, $b(t) = 2$, $c(t) = t(t+1)$). On the other hand, it is easy to deduce from our Theorem 2 the result on sums of two squares mentioned at the beginning.

We start with a theorem on quadratic forms over $Q(t)$, where Q denotes the rational field.

Theorem 1. *Let $a(t)$, $b(t)$ be polynomials with integral coefficients. Suppose that every arithmetical progression contains some integer t such that the equation*

$$a(t)x^2+b(t)y^2 = z \tag{1}$$