A theorem on generalized Dedekind sums

by

L. CARLETTI (Durham, North Carolina)

1. Put

\[(\omega) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{k} & (x \neq \text{integer}), \\ 0 & (x \text{ integer}). \end{cases}\]

The Dedekind sum \(s(h, k)\) is defined by

\[(1.1) \quad s(h, k) = \sum_{\nu \equiv 0 \pmod{k}} \left( \frac{hu}{k} \right) \left( \frac{\nu}{k} \right),\]

where the summation is extended over a complete residue system \((\mod{k})\). It is well known that \(s(h, k)\) satisfies

\[(1.2) \quad 12hk(s(h, k) + s(k, h)) = h^2 - 3hk + k^2 + 1,\]

where \(h\) and \(k\) are relatively prime.

Bademacher, at the 1963 Number Theory Institute in Boulder, proved the following generalization of (1.2). Define

\[(1.3) \quad s(h, k; x, y) = \sum_{\nu \equiv 0 \pmod{k}} \left( \frac{hu + y}{k} + \frac{x}{k} \right) \left( \frac{\nu + y}{k} \right),\]

where \(x, y\) are arbitrary real numbers. Then

\[(1.4) \quad s(h, k; x, y) + s(h, k; y, x) = \frac{1}{4} \delta(x) \delta(y) + ((\omega))((y)) + \frac{1}{2} \left\{ \frac{h}{k} B_2(y) + \frac{1}{hk} B_3(ky + \nu) + \frac{k}{h} B_2(\nu) \right\},\]

where \((h, k) = 1,\)

\[\delta(x) = \begin{cases} 1 & (x \text{ integral}), \\ 0 & (\text{otherwise}) \end{cases}\]

* Supported in part by NSF grant GP-1693.
and 
$$ B_n(x) = B_0(x - (x)), \quad B_0(x) = x^0 = x + \frac{1}{2}. $$

When $x = y = 0$, (1.4) reduces to (1.2). Rademacher’s proof of (1.4) appeared in [6].

Let $B_n(x)$ be the Bernoulli polynomial of degree $n$ defined by

$$ \frac{t^n}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} $$

and let $B_n(x)$ be the Bernoulli function defined by means of

$$ B_n(x) = B_n(x - [x]). $$

Apostol ([1], [2]) introduced the sum

$$ s_p(k, x) = \sum_{n=0}^{\infty} B_n \left( \frac{\mu + \nu + x}{k} \right) \left( \frac{x}{k} \right) $$

and proved the reciprocity formula

$$ (p+1)(B_k + kB) + k \frac{p}{2} = B_k + kB $$

where $(k, k) = 1$, $p$ odd, $p > 1$ and $B_p = B_p(0)$. For another proof of (1.6) see [5].

Rademacher’s definition of $s_p(k, x, y)$ suggests that we define

$$ s_p(k, x) = \sum_{n=0}^{\infty} B_n \left( \frac{\mu + \nu + x}{k} \right) \left( \frac{x}{k} \right) $$

which reduces to $s_p(x)$ when $x = y = 0$. It is evident from (1.7) that

$$ s_p(k, x, y) = s_p(k, x, y + 1) = s_p(k, h, x, y), $$

so that there is no loss in generality in assuming that

$$ 0 \leq x < 1, \quad 0 \leq y < 1. $$

The writer ([3]) has proved the following

**Theorem.** Let $(k, k) = 1$ and assume that $x, y$ satisfy (1.9). Then

$$ s_p(k, x, y) = \sum_{n=0}^{\infty} B_n \left( \frac{\mu + \nu + x}{k} \right) \left( \frac{x}{k} \right) $$

for all $p > 0$.

It is understood that to evaluate $(B_k + kB + ky + ky)^{p+1}$ we expand by the multinomial theorem and then replace $B^n$ by $B_n$; alternatively we have

$$ (B_k + kB + ky + ky)^{p+1} = \sum_{r=1}^{p+1} \left( \begin{array}{c} p+1 \\ r \\ \end{array} \right) k^r \left( \frac{\mu + \nu + x}{k} \right)^{p+1} B_r(y) B_{p-r+1}(x). $$

This suggests the following equivalent formulation of (1.10) in which (1.9) is no longer assumed:

$$ (p+1)(h^2 + h + y)^{p+1} = \sum_{r=1}^{p+1} \left( \begin{array}{c} p+1 \\ r \\ \end{array} \right) k^r \left( \frac{\mu + \nu + x}{k} \right)^{p+1} B_r(y) B_{p-r+1}(x) $$

The object of the present paper is to give a new and simpler proof of (1.10).

2. We recall that $B_n(x)$ satisfies the multiplication theorem:

$$ B_n(kx) = \frac{k^n}{x^n} \sum_{n=0}^{\infty} B_n \left( \frac{x}{k} \right). $$

Applying (2.1) to (1.7) we get

$$ s_p(k, x, y) = \frac{k^n}{x^n} \sum_{n=0}^{\infty} B_n \left( \frac{\mu + \nu + x}{k} \right) \left( \frac{x}{k} \right), $$

so that

$$ s_p(k, x, y) = \frac{k^n}{x^n} \sum_{n=0}^{\infty} B_n \left( \frac{\mu + \nu + x}{k} \right) \left( \frac{x}{k} \right) $$

We shall now assume that $\mu, \nu$ satisfy

$$ 0 \leq \mu < 1, \quad 0 \leq \nu < 1. $$

As we have seen above there is no loss in generality in making this assumption. Put

$$ x = \frac{\mu}{k} + \frac{\nu}{k} + \frac{\mu + \nu}{k}, $$

so that

$$ 0 \leq x < 2 \quad 0 \leq \mu + \nu < k. $$

Since

$$ B_1(x) = x - \frac{1}{2} \quad (0 \leq x < 1), $$

it follows that, for $\mu, \nu$ satisfying (2.3),

$$ B_1(x) + B_1(y) = B_1(x + y) + \frac{1}{2} f(x + y), $$

where

$$ f(x) = \begin{cases} -1 & (0 \leq x < 1), \\ +1 & (1 \leq x < 2). \end{cases} $$
For brevity put
\[(2.8) \quad S_p = \frac{\hbar^p s_p(h, k, z, y) + \hbar^p s_p(k, h, y, x)}{2} - \frac{\hbar^p s_p(h, z, y, x)}{2}.
\]
Then (2.2) becomes
\[(2.9) \quad (\hbar^p s_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} (B_1(c) + \frac{1}{2} f(c)) B_p(c).
\]
In view of (2.7) this becomes
\[(2.10) \quad (\hbar^p s_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} (B_1(c) + \frac{1}{2} f(c)) B_p(c) - \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p(c).
\]
Now by (2.1) we have
\[(2.11) \quad \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p(c) = (\hbar^p s_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} (B_1(c) + \frac{1}{2} f(c)) B_p(c) = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p(c).
\]
so that
\[(2.12) \quad (\hbar^p s_p = T_p - U_p + \frac{1}{2} (\hbar^p s_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_1(c) B_p(c),
\]
where
\[(2.13) \quad U_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p(c) = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p(c).
\]

3. It will be convenient to put
\[(3.1) \quad \tau = \frac{\hbar^p s_p(h, y, k).}{2}
\]
Since \(\hbar\) and \(k\) are relatively prime, it is evident from (2.11) that
\[(3.2) \quad T_p = \sum_{s=1}^{k-1} B_1\left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right) B_p\left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right).
\]
If we put
\[(3.3) \quad \tau = \left[\frac{s}{\hbar}\right] + \zeta, \quad 0 \leq \zeta < 1,
\]
so that \(\zeta\) is the fractional part of \(s\), then (3.2) reduces to
\[(3.4) \quad T_p = \sum_{s=1}^{k-1} B_1\left(\frac{s}{\hbar k} + \frac{\zeta}{\hbar k}\right) B_p\left(\frac{s}{\hbar k} + \frac{\zeta}{\hbar k}\right).
\]
Now we have, for \(p \geq 1,
\[(3.5) \quad B_1(\tau) B_p(\tau) = B_{p+1}(\tau) + \frac{1}{p+1} \sum_{t=1}^{p+1} \frac{1}{2} B_p B_{p-t+1}(\tau) + \frac{1}{p+1} B_{p+1}.
\]
Indeed (3.5) is a special case of a formula for \(B_n(u)(u)\) (see e.g. [4]).
Put \(\mu = \left(\mu + \zeta\right)/k\) in (3.5) and sum over \(\mu\). We get
\[(3.6) \quad T_p = \sum_{s=1}^{k-1} B_1\left(\frac{\mu + \zeta}{k}\right) B_p\left(\frac{\mu + \zeta}{k}\right).
\]
Therefore by (3.4)
\[(3.7) \quad T_p = \frac{p}{p+1} \left(\hbar^p s_p(\zeta) + \frac{1}{p+1} (\hbar^p s_p(B(k+\zeta) p + \frac{1}{p+1} (\hbar^p s_p(\zeta).\right)
\]
As for \(U_p\), we have
\[(3.8) \quad U_p = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} B_p\left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right).
\]
Now
\[(3.9) \quad \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} \left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right) = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} \left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right),
\]
where
\[(3.10) \quad \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} \left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right) = \sum_{s=1}^{k-1} \sum_{c=s}^{k-1} \left(\frac{s}{\hbar k} + \frac{\hbar}{\hbar k}\right).
\]
We may assume without loss of generality that \(k < h\), so that
\[
\frac{s}{h} = y + \frac{kz}{h} < 2.
\]
Hence in the outer summation on the right of (3.8) we have

\[ \mu < k - 1 - \left[ \frac{z}{k} \right] \]

(3.10)

Let us assume first, in evaluating the inner sum on the right of (3.8), that \( h \mu + s \) is not an integral multiple of \( k \). Then the sum is equal to

\[ \frac{h^{-r}}{r+1} \left[ B_{r+1} \left( h + \frac{h \mu}{k} + \frac{s}{k} \right) \right] \]

(3.7)

Thus by (3.7)

\[ U_p = \frac{h^{-p}}{p+1} \sum_{\mu} \sum_{p+1} \frac{\mu + s}{k} \left( \frac{h^{-r} \mu + \frac{s}{k}}{r+1} \right) B_{r+1} \left( \frac{h \mu}{k} + \frac{s}{k} \right) \]

(3.11)

But

\[ \sum_{\mu} \sum_{p+1} \frac{\mu + s}{k} \left( \frac{h^{-r} \mu + \frac{s}{k}}{r+1} \right) B_{r+1} \left( \frac{h \mu}{k} + \frac{s}{k} \right) \]

(3.12)

provided \( s < h \). When \( s \geq h \) we must delete the terms corresponding to \( \mu = k - 1 \) in the right member of (3.11). Since

\[ \left[ \frac{h(k-1)}{k} + \frac{s}{k} \right] = h, \]

it is evident that (3.12) holds for all \( s \).

We have

\[ (h(kB + hB + \zeta)^{p+1} = (hk(B + 1) + (B + \zeta)^{p+1} \]

(3.13)

Since \( (B + 1)^{p+1} = B^p (p \neq 1) \), the above sum is equal to

\[ \sum_{\mu} \sum_{p+1} \frac{\mu + s}{k} \left( \frac{(h(kB + hB + \zeta)^{p+1} + (p+1)hk(B + \zeta)^p \]

Therefore (3.12) becomes

\[ U_p = \frac{(hk)^{-p}}{p+1} \left( h(kB + hB + \zeta)^{p+1} \right) \]

(3.14)

When \( h \mu + s \) is an integral multiple of \( k \), the above proof requires modification. We get in this case

\[ \sum_{\text{all } c} \left( \frac{w + \frac{s}{kh}}{k} \right) = \frac{k^{-r}}{r+1} \left( B_{r+1} \left( \frac{w + \frac{s}{kh}}{k} \right) \right) \]

(3.15)
The remainder of the proof goes through without change. Therefore (3.13) holds without exception.

We now substitute from (3.6) and (3.12) in (2.10) to get
\[
(\lambda)\gamma^{-p} s_p = \frac{p}{p+1} \left( \lambda(\lambda)^{-p} B_{p+1}(\ell) + \frac{(\lambda h)\gamma}{p+1} (\lambda B + \ell)^{p+1} + \frac{1}{2}(\lambda h\lambda)^{-p} B_p(\ell) \right)
- \frac{(\lambda h)\gamma}{p-1} ((\lambda h B + h + \ell)^p + (\lambda h + B + y)^p) + \frac{1}{2}(\lambda h\lambda)^{-p} B_p(\ell)
\]
This evidently completes the proof of (1.10) when \( p \geq 1 \).

When \( p = 0 \), we have
\[
S_0(h, k; x, y) = \sum_{\mu(\text{mod} k)} B_1 \left( \frac{\mu + y}{h} \right) B_1(y),
so that
\[
S_0 = h s_0(h, k; x, y) + h s_0(k, h; y, x) = h B_1(y) + k B_1(x).
On the other hand,
\[
(Bh + Bk + hy + kx) = h B_1(y) + k B_1(x).
Since \( 0 < \pi < 1, 0 < y < 1 \), it is clear that (1.10) holds when \( p = 0 \).

References


Reçu par la Rédaction le 23. 11. 1964