Then,

$$(4.1)\qquad Q \leqslant \frac{\log(\omega/\sqrt{2})+\omega}{\omega}\left\{1+\frac{1}{2\omega}\right\} \quad \text{for} \quad \omega \geqslant \log 10.$$

For $\omega \geqslant \sqrt{2e}$ ($>\log 10$) this function is decreasing, and for $\omega = \sqrt{2e}$ it is $<\frac{3}{2}$. This proves Theorem 2.

It remains to prove Theorem 1 for $1 < u < e^{10}$. Obviously, for $u \leqslant e^8$, Theorem 1 is a consequence of Theorem 2. For $u > e^8$ however, the corresponding ω being > 6.4, (4.1) gives for $u < e^{10}$

$$Q < 1.4 < 1+\frac{4}{\log u}.$$

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An improvement of Selberg's sieve method I

by

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The sieve method is concerned with counting in a finite set M of positive integers those elements which are not divisible by any prime $p < z$ or rather with estimating the number $A(M; z)$ of these elements from above and from below. If z is sufficiently large compared with the largest element in M the numbers counted in $A(M; z)$ will have a restricted number of prime divisors, and it is this fact that makes estimates of $A(M; z)$ interesting. The estimates, however, depend essentially upon only the number of elements in M, say $|M|$.

For relatively small values of z the classical sieve of Eratosthenes-Legendre is quite satisfactory (§ 1). However, the interest lies in larger values of z which were first treated successfully by Viggo Brun. Later, Buchstab was able to give improved estimates, starting from initial estimates, by using identities of Meissel's type (cf. (2.2)). For an upper estimate the sieve method was formulated generally and this estimate minimized by A. Selberg. He also gave a lower estimate which can be derived from his upper estimate by using Buchstab's method (cf. Ankeny-Onishi [1]). In applications a combinatorial argument of Kuhn led to further improvements. In a series of papers Y. Wang very successfully combined all methods mentioned above.

We propose in this paper a method (based on Theorems 1 and 4) which leads to a new two-sided estimate (Theorem 5) by employing only the simplest upper estimate of Selberg for $z^2 > |M|$ (cf. Corollary to Theorem 2) and extensions of the classical estimate (Theorem 3).

The new estimate improves Selberg's upper and lower estimate for $z^2 < |M|$, and cannot be further improved upon by Buchstab's method. In particular, the new lower estimate is positive already for $z^{2+\varepsilon} < |M|$ (with arbitrary $\varepsilon > 0$ and large z), which decides a question left open by A. Selberg ([15], p. 292). Furthermore, our estimate is completely uniform for all sets of a certain regular behaviour (condition $\mathrm{H}_k(M)$). This condition restricts, however, our present presentation to what might be called the linear sieve. Exceptional primes are allowed which we combine to a product k. Thereby, we may apply our results to numbers in a short interval belonging to a residue class mod k and having at most r prime factors (Theorems 6, 7, and 8). Here, we make use of Kuhn's method. Thus, in particular, we obtain the following theorems.

THEOREM 9. *For $\varepsilon > 0$ and $x \geqslant x_0(\varepsilon)$ there exist at least two integers n in the interval*

$$x - x^{\frac{14}{25}+\varepsilon} < n \leqslant x$$

having at most two prime factors.

THEOREM 10. *For $\varepsilon > 0$, $k \geqslant k_0(\varepsilon)$, and $(k, l) = 1$, there exists an integer*

$$n \equiv l \bmod k, \quad 1 \leqslant n \leqslant k^{\frac{25}{11}+\varepsilon}$$

having at most two prime factors.

These problems respectively, were first discussed by Viggo Brun ([6]) and W. Fluch ([8]). The best exponents, previously found are $\frac{10}{17}$ (Y. Wang [18]) and $\frac{7}{2}$ (S. Uchiyama [16]).

1. The sieve of Eratosthenes. Throughout the paper let z denote a real number $\geqslant 2$, k a positive integer, and(1)

$$P_k(z) = \prod_{\substack{p<z \\ p\nmid k}} p, \quad R_k(z) = \prod_{\substack{p<z \\ p\nmid k}} \left(1 - \frac{1}{p}\right).$$

Generalizing the terminology of the introduction we set

$$A_k(M; z) = |\{n \in M;\ n \not\equiv 0 \bmod p\ \forall p | P_k(z)\}|,$$

and, we denote by M_d, for any positive integer d, the set of all integers m such that $md \in M$. (The discussion of the more general number $|\{n \in M;\ n \not\equiv a_p \bmod p\ \forall p|P_k(z)\}|$ with arbitrary integers a_p can be reduced to the case $a_p = 0$ by a suitable translation of M in view of the Chinese Remainder Theorem.) Let y denote a real number > 1. For given M and k we shall impose upon y (> 1) the condition

$$\mathrm{H}_k(M) \qquad \left| |M_d| - \frac{y}{d} \right| \leqslant 1, \quad \forall (d, k) = 1.$$

Clearly, the existence of y presupposes a certain regular distribution of the numbers in M.

Under this assumption Legendre's formula or the sieve of Eratosthenes in the form of

$$A_k(M; z) = \sum_{n \in M} \sum_{\substack{d|n \\ d|P_k(z)}} \mu(d) = \sum_{d|P_k(z)} \mu(d) \sum_{m \in M_d} 1$$

(1) An empty product shall be one, similarly an empty sum shall be zero.

can be applied in a well-known manner to obtain

$$(1.1) \qquad A_k(M; z) = y \sum_{d|P_k(z)} \frac{\mu(d)}{d} + O(2^{\pi(z)}) = yR_k(z) + O(2^z).$$

This estimate is satisfactory for sufficiently small values of z, and indicates the nature of the leading term in $A_k(M; z)$. The O-constant in (1.1) is independent of M, k, z and y.

Throughout the paper O-estimates shall be understood as uniform with respect to all admissible parameters. The corresponding positive absolute O-constants shall be called $c_1, c_2, \ldots$ etc.

2. A basic identity.

THEOREM 1. *Let $2 \leqslant z_1 \leqslant z \leqslant \sqrt{y}$ and set*

$$y_j = \frac{y}{p_1 \cdots p_j}, \quad j = 1, 2, \ldots$$

Then, for all positive integers r,

$$(2.1) \qquad A_k(M; z) = A_k(M; z_1) + \sum_{1 \leqslant i \leqslant r-1} (-1)^i \sum_{\substack{z_1 \leqslant p_i < \ldots < p_1 < z \\ p_j < \sqrt{y_j}, p_j \nmid k, j=1,\ldots,i}} A_k(M_{p_1 \ldots p_i}; z_1) +$$
$$+ (-1)^r \sum_{\substack{z_1 \leqslant p_r < \ldots < p_1 < z \\ p_j < \sqrt{y_j}, p_j \nmid k, j=1,\ldots,r}} A_k(M_{p_1 \ldots p_r}; p_r) +$$
$$+ \sum_{1 \leqslant i \leqslant r} (-1)^i \sum_{\substack{z_1 \leqslant p_i < \ldots < p_1 < z \\ p_j < \sqrt{y_j}, p_j \nmid k, j=1,\ldots,i-1 \\ \sqrt{y_i} \leqslant p_i < y_i, p_i \nmid k}} A_k(M_{p_1 \ldots p_i}; p_i).$$

This holds independently of condition $\mathrm{H}_k(M)$.

Proof. The proof depends upon the identity

$$(2.2) \qquad A_k(M; z) = |M| - \sum_{\substack{p<z \\ p\nmid k}} A_k(M_p; p),$$

which is essentially due to Buchstab ([7], p. 1241, see also Meissel [11]). There is a similar identity

$$(2.3) \qquad R_k(z) = 1 - \sum_{\substack{p<z \\ p\nmid k}} \frac{R_k(p)}{p},$$

which will be used later (cf. de Bruijn [3], p. 807, Ankeny-Onishi [1], p. 54).

We now proceed by induction with respect to r. Applying (2.2) for both z and z_1 we get by subtraction

(2.4) $$A_k(M;z)=A_k(M;z_1)-\sum_{\substack{z_1\leqslant p<z\\ p<\sqrt{y/p},\,p\nmid k}} A_k(M_p;p)-\sum_{\substack{z_1\leqslant p<z\\ \sqrt{y/p}\leqslant p<y/p,\,p\nmid k}} A_k(M_p;p),$$

i.e. (2.1) for $r=1$.

Suppose (2.1) had been proved for r. Then, in the first sum of (2.4) we apply (2.1) taking M_p, y/p, p for M, y, z, respectively. We also replace p_j by p_{j+1} and p by p_1 which changes y_j/p into y_{j+1} and y/p becomes y_1. Thus,

$$\sum_{\substack{z_1\leqslant p<z\\ p<\sqrt{y/p},\,p\nmid k}} A_k(M_p;p) = \sum_{\substack{z_1\leqslant p_1<z\\ p_1<\sqrt{y_1},\,p_1\nmid k}} A_k(M_{p_1};z_1)+$$
$$+\sum_{1\leqslant i\leqslant r-1}(-1)^i \sum_{\substack{z_1\leqslant p_{i+1}<\ldots<p_1<z\\ p_j<\sqrt{y_j},\,p_j\nmid k,\,j=1,\ldots,i+1}} A_k(M_{p_1\ldots p_{i+1}};z_1)+$$
$$+(-1)^r \sum_{\substack{z_1\leqslant p_{r+1}<\ldots<p_1<z\\ p_j<\sqrt{y_j},\,p_j\nmid k,\,j=1,\ldots,r+1}} A_k(M_{p_1\ldots p_{r+1}};p_{r+1})+$$
$$+\sum_{1\leqslant i\leqslant r}(-1)^i \sum_{\substack{z_1\leqslant p_{i+1}<\ldots<p_1<z\\ p_j<\sqrt{y_j},\,p_j\nmid k,\,j=1,\ldots,i\\ \sqrt{y_{i+1}}\leqslant p_{i+1}<y_{i+1},\,p_{i+1}\nmid k}} A_k(M_{p_1\ldots p_{i+1}};p_{i+1}).$$

Using this in (2.4) we obtain (2.1) for $r+1$.

For later use we collect some estimates for $R_k(z)$ starting with the well-known result of Mertens

$$R_1(z)=\frac{e^{-\gamma}}{\log z}+O\left(\frac{1}{\log^2 z}\right),$$

where γ denotes Euler's constant. This implies

(2.5) $$\frac{R_1(w)}{R_1(z)}=\frac{\log z}{\log w}+O\left(\frac{\log z}{\log^2 w}\right),\qquad z\geqslant w\ (\geqslant 2).$$

LEMMA 2.1. *For* $z\geqslant w\geqslant 2$

(2.6) $$\frac{R_k(w)}{R_k(z)}=O\left(\frac{\log z}{\log w}\right),$$

(2.7) $$\frac{1}{R_k(z)}=O(\log z),$$

(2.8) $$\frac{R_k(w)}{R_k(z)}=\frac{\log z}{\log w}+O\left(\frac{\log z\log\log 3k}{\log^2 w}\right).$$

Proof. In view of

(2.9) $$\frac{R_k(w)}{R_k(z)}=\frac{R_1(w)}{R_1(z)}\prod_{\substack{w\leqslant p<z\\ p\mid k}}\left(1-\frac{1}{p}\right),$$

(2.5) implies (2.6). Taking $w=2$, (2.7) follows from (2.6). Because of (2.6) a proof of (2.8) is required only in case that

(2.10) $$\log\log 3k\leqslant \tfrac{1}{2}\log w$$

holds true. Let $\nu(k)$ denote the number of different prime factors of k. Then

$$1\geqslant\prod_{\substack{w\leqslant p<z\\ p\mid k}}\left(1-\frac{1}{p}\right)\geqslant\left(1-\frac{1}{w}\right)^{\nu(k)}\geqslant 1-\frac{\nu(k)}{w}.$$

Hence, by $\nu(k)=O(\log k)$ and (2.10)

$$\prod_{\substack{w\leqslant p<z\\ p\mid k}}\left(1-\frac{1}{p}\right)=1+O\left(\frac{\log k}{w}\right)=1+O\left(\frac{1}{\log w}\right).$$

This combined with (2.9) and (2.5) yields (2.8).

3. Selberg's upper estimate. For positive real x we define

$$S_k(x,z)=\sum_{\substack{1\leqslant n\leqslant x\\ n\mid P_k(z)}}\frac{1}{\varphi(n)},\qquad \Psi(x,z)=\sum_{\substack{1\leqslant n\leqslant x\\ p(n)<z}}1,\qquad T(x,z)=\sum_{\substack{1\leqslant n\leqslant x\\ p(n)<z}}\frac{1}{n},$$

where $p(n)$ denotes the greatest prime divisor of n, $p(1)=1$.

LEMMA 3.1.

(3.1) $$S_k(x,z)\geqslant\frac{d}{\varphi(d)}S_{kd}\left(\frac{x}{d},z\right),\qquad \textit{if}\quad d\mid P_k(z),$$

3.2) $$R_k(z)S_k(x,z)\geqslant R_1(z)S_1(x,z),$$

(3.3) $$S_1(x,z)\geqslant T(x,z).$$

Proof. We have

(3.4) $$S_k(x,z)=\sum_{t\mid d}\sum_{\substack{1\leqslant n\leqslant x\\ n\mid P_k(z)\\ (n,d)=t}}\frac{1}{\varphi(n)}=\sum_{\substack{t\mid d\\ t\mid P_k(z)}}\frac{1}{\varphi(t)}\sum_{\substack{1\leqslant m\leqslant \frac{x}{t}\\ m\mid P_k(z)\\ (m,t)=1\\ \left(m,\frac{d}{t}\right)=1}}\frac{1}{\varphi(m)}$$
$$=\sum_{\substack{t\mid d\\ t\mid P_k(z)}}\frac{1}{\varphi(t)}S_{kd}\left(\frac{x}{t},z\right).$$

If $d|P_k(z)$ it follows

$$S_k(x,z) \geqslant \sum_{t|d} \frac{1}{\varphi(t)} S_{kd}\left(\frac{x}{d}, z\right) = \frac{d}{\varphi(d)} S_{kd}\left(\frac{x}{d}, z\right)$$

which proves (3.1).

On the other hand we derive from (3.4)

$$S_1(x,z) \leqslant S_d(x,z) \prod_{\substack{p|d\\ p<z}} \left(1+\frac{1}{p-1}\right) = S_d(x,z)\frac{R_d(z)}{R_1(z)}.$$

Let $q(n)$ denote the largest squarefree divisor of n. Then,

$$S_1(x,z) = \sum_{\substack{1\leqslant n\leqslant x\\ p(n)<z}} \frac{\mu^2(n)}{n} \prod_{p|n} \frac{1}{1-1/p} = \sum_{\substack{q(m)\leqslant x\\ p(m)<z}} \frac{1}{m} \geqslant \sum_{\substack{1\leqslant m\leqslant x\\ p(m)<z}} \frac{1}{m} = T(x,z).$$

THEOREM 2. *If y satisfies* $\mathrm{H}_k(M)$, *we have*

$$A_k(M;z) \leqslant \frac{yR_k(z)}{R_1(z)T(\xi,z)} + \Psi^2(\xi,z) \tag{3.5}$$

for arbitrary values of $\xi > 1$.

This is Selberg's well-known upper estimate in a form convenient for our purposes. Note that the k occurs in $R_k(z)$ only, and that for T and Ψ one can obtain asymptotic expansions.

Proof. Let $\xi > 1$. For positive integers d we define

$$\lambda_d = \mu(d)\frac{d}{\varphi(d)}\cdot\frac{S_{kd}(\xi/d,z)}{S_k(\xi,z)}.$$

Because of $\xi>1$ we have $S_k(\xi,z)\geqslant 1$, and

$$\lambda_d = 0 \quad \text{for} \quad d>\xi. \tag{3.6}$$

If $t|P_k = P_k(z)$, $1\leqslant t\leqslant \xi$, then

$$\sum_{\substack{d|P_k\\ d\equiv 0 \bmod t}} \frac{\lambda_d}{d} = \sum_{\substack{m|P_k\\ (m,t)=1}} \frac{\mu(tm)}{\varphi(tm)}\cdot\frac{S_{ktm}(\xi/tm,z)}{S_k(\xi,z)} \tag{3.7}$$

$$= \frac{\mu(t)}{\varphi(t)S_k(\xi,z)} \sum_{\substack{m|P_k\\ (m,t)=1}} \frac{\mu(m)}{\varphi(m)} \sum_{\substack{1\leqslant \nu\leqslant \xi/tm\\ \nu|P_{ktm}(z)}} \frac{1}{\varphi(\nu)}$$

$$= \frac{\mu(t)}{\varphi(t)S_k(\xi,z)} \sum_{\substack{1\leqslant \nu m\leqslant \xi/t\\ \nu m|P_{kt}(z)}} \frac{1}{\varphi(\nu m)} \sum_{m|\nu m}\mu(m) = \frac{\mu(t)}{\varphi(t)S_k(\xi,z)}.$$

By (3.1)

$$|\lambda_d| \leqslant \mu^2(d), \quad \text{if} \quad d|P_k. \tag{3.8}$$

Now, since $\lambda_1 = 1$

$$A_k(M;z) \leqslant \sum_{n\in M}\Big(\sum_{\substack{d|n\\ d|P_k}} \lambda_d\Big)^2 = \sum_{\substack{d_\nu|P_k\\ \nu=1,2}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{n\in M\\ n\equiv 0 \bmod [d_1,d_2]}} 1,$$

where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2. Here, by $\mathrm{H}_k(M)$,

$$\sum_{n\in M_{[d_1,d_2]}} 1 = \frac{y}{[d_1,d_2]} + \theta = \frac{y}{d_1 d_2}\sum_{\substack{t|d_1\\ t|d_2}} \varphi(t) + \theta, \qquad |\theta|\leqslant 1.$$

Hence, by (3.6), (3.8) and (3.7)

$$A_k(M;z) \leqslant y \sum_{\substack{d_\nu|P_k\\ 1\leqslant d_\nu\leqslant\xi\\ \nu=1,2}} \frac{\lambda_{d_1}\lambda_{d_2}}{d_1 d_2} \sum_{\substack{t|d_1\\ t|d_2}} \varphi(t) + \sum_{\substack{d_\nu|P_k\\ 1\leqslant d_\nu\leqslant\xi\\ \nu=1,2}} |\lambda_{d_1}\lambda_{d_2}|$$

$$\leqslant y \sum_{\substack{1\leqslant t\leqslant\xi\\ t|P_k}} \varphi(t)\Big(\sum_{\substack{d|P_k\\ d\equiv 0 \bmod t}} \frac{\lambda_d}{d}\Big)^2 + \Big(\sum_{\substack{d|P_k\\ 1\leqslant d\leqslant\xi}} \mu^2(d)\Big)^2 \leqslant \frac{y}{S_k(\xi,z)} + \Psi^2(\xi,z).$$

Using herein (3.2) and (3.3) we obtain (3.5).

The simplest case of Theorem 2 takes the following form.

COROLLARY. *If y satisfies* $\mathrm{H}_k(M)$ *we have*

$$A_k(M;z) \leqslant yR_k(z)\left\{\frac{2e^\gamma \log z}{\log y} + c_1\frac{\log z \log\log 3y}{\log^2 y}\right\}, \tag{3.9}$$

provided that $z \geqslant \sqrt{y}$.

Proof. Taking

$$\xi^2 = \frac{y}{1+\log^2 y} \quad (>1)$$

Theorem 2 can be applied. Since $z>\xi$,

$$T(\xi,z) = \sum_{1\leqslant n\leqslant\xi} \frac{1}{n} \geqslant \frac{e^{-\gamma}}{R_1(\xi)}$$

by an inequality of Rosser and Schoenfeld ([12], p. 71).

Hence, from (3.5) estimating $\Psi(\xi, z)$ from above trivially by ξ and using (2.5) and (2.7) we obtain

$$A_k(M;z) \leqslant yR_k(z)\left\{\frac{e^\gamma R_1(\xi)}{R_1(z)}+\frac{\xi^2}{yR_k(z)}\right\} \leqslant yR_k(z)\left\{\frac{2e^\gamma \log z}{\log y}+c_1\frac{\log z\log\log 3y}{\log^2 y}\right\}.$$

4. Extensions of the classical estimate.

THEOREM 3. *If y satisfies $\mathrm{H}_k(M)$, we have*

$$A_k(M;z) \leqslant yR_k(z)\{1+O(e^{-\frac{\log y}{\log z}})\} \quad \textit{for} \quad 1 \leqslant \log z \leqslant \log y, \tag{4.1}$$

and

$$A_k(M;z) = yR_k(z)\left\{1+O\left(\frac{1}{\log y}\right)\right\} \quad \textit{for} \quad \log z \leqslant \frac{\log y}{2\log\log 3y}. \tag{4.2}$$

A similar estimate which is not sufficient for our purpose is due to Barban ([2]).

Proof. Using (2.7) we see from (1.1) that Theorem 3 holds true if z is bounded, so that we may assume

$$z > z_0,$$

where z_0 denotes a suitable absolute constant. Then, from A. I. Vinogradov's result ([17]) we infer

$$\Psi(x,z) = O(xe^{-2\frac{\log x}{\log z}}). \tag{4.3}$$

From the identity

$$T(x,z) = \frac{\Psi(x,z)}{x}+\int_1^x \Psi(w,z)\frac{dw}{w^2},$$

since

$$\lim_{x\to\infty} T(x,z) = \sum_{\substack{n=1\\ p(n)<z}}^{\infty}\frac{1}{n} = \prod_{p<z}\frac{1}{1-1/p} = \frac{1}{R_1(z)}, \tag{4.4}$$

it follows

$$\frac{1}{R_1(z)} = \int_1^\infty \Psi(w,z)\frac{dw}{w^2}.$$

Since $T(x,z)$ is non-decreasing in x and so, by (4.4), is $\leqslant 1/R_1(z)$, we find

$$\frac{1}{R_1(z)}-T(x,z) \leqslant \int_x^\infty \Psi(w,z)\frac{dw}{w^2} = O\left(\int_x^\infty e^{-2\frac{\log w}{\log z}}\frac{dw}{w}\right) = O(\log z\, e^{-2\frac{\log x}{\log z}}),$$

in view of (4.3). Taking

$$\xi^2 = \frac{y}{\log z} \quad (\geqslant \sqrt{z} > 1),$$

we observe

$$T(\xi,z) \geqslant \sum_{1\leqslant n\leqslant z^{1/4}}\frac{1}{n},$$

and therefore

$$\frac{1}{R_1(z)T(\xi,z)} \leqslant 1+O(e^{-\frac{\log \xi^2}{\log z}}).$$

Now, from Theorem 2 we obtain by (4.3) and (2.7) the first part of Theorem 3.

Next, since $z > z_0$ the upper estimate of (4.2) follows from (4.1). Therefore, it is sufficient to prove the corresponding estimate from below.

By (2.2) and (2.3)

$$A_k(M;z)-yR_k(z) = A_k(M;z_0)-yR_k(z_0)-\sum_{\substack{z_0\leqslant p<z\\ p\nmid k}}\left\{A_k(M_p;p)-\frac{y}{p}R_k(p)\right\}.$$

Notice that the condition $\mathrm{H}_k(M)$ for y implies $\mathrm{H}_k(M_p)$ for y/p, $p\nmid k$, since $(M_p)_d = M_{pd}$. Hence, using (1.1) at $z = z_0$ and (4.1) we have

$$A_k(M;z)-yR_k(z) \geqslant O(1)-y\sum_{\substack{z_0\leqslant p<z\\ p\nmid k}}\frac{R_k(p)}{p}O(e^{-\frac{\log(y/p)}{\log p}}) \geqslant O(1)+O(ye^{-\frac{\log y}{\log z}}).$$

Because of (2.7) now the proof of (4.2) has been completed.

5. Basic functions and approximate identities. Let([2])

$$\omega(u) = \frac{1}{u}, \quad \varrho(u) = 1, \quad 0<u\leqslant 2, \tag{5.1}$$

$$(u\omega(u))' = \omega(u-1), \quad (u-1)\varrho'(u) = -\varrho(u-1), \quad u\geqslant 2. \tag{5.2}$$

Both functions have been investigated by several authors. De Bruijn ([4], [5]) proved([3])

$$\omega(u) = e^{-\gamma}+O(e^{-u}), \quad \varrho(u) = O(e^{-u}), \quad u\geqslant 1, \tag{5.3}$$

$$\varrho(u)>0, \quad u>0. \tag{5.4}$$

([2]) At the point $u=2$ the right-hand derivative has to be taken.

([3]) For our convenience we have shifted the argument of $\varrho(u)$ by 1.

With these functions we set

$$F(u) = e^{\gamma}\{\omega(u)+\varrho(u)/u\}, \qquad u>0,$$
$$f(u) = e^{\gamma}\{\omega(u)-\varrho(u)/u\}, \qquad u>0.$$

By (5.1) and (5.2) we have

(5.5) $$F(u) = 2e^{\gamma}/u, \quad f(u)=0, \quad 0<u\leqslant 2,$$

(5.6) $$(uF(u))' = f(u-1), \quad (uf(u))' = F(u-1), \quad u\geqslant 2,$$

and hence

(5.7) $$\int_v^u f(t-1)\,dt = uF(u)-vF(v),$$
$$\int_v^u F(t-1)\,dt = uf(u)-vf(v), \qquad 2\leqslant v\leqslant u.$$

Taking $v=2$, because of (5.5) we obtain

$$uF(u) = 2F(2)+\int_2^u f(t-1)\,dt = 2e^{\gamma}, \qquad 2\leqslant u\leqslant 3,$$

and hence

(5.8) $$F(u) = 2e^{\gamma}/u, \qquad 0<u\leqslant 3.$$

Thus,

(5.9) $$uf(u) = 2f(2)+\int_2^u F(t-1)\,dt = 2e^{\gamma}\log(u-1), \qquad 2\leqslant u\leqslant 4.$$

Also, by (5.3)

(5.10) $$F(u) = 1+O(e^{-u}), \quad f(u) = 1+O(e^{-u}), \quad u\geqslant 1,$$

and, by (5.4),

(5.11) $$F(u)-f(u) = \frac{2e^{\gamma}}{u}\varrho(u)>0, \qquad u>0.$$

If $F'(u)=0$ were possible, take the smallest value u_0 of this kind. Then, because of (5.8),

(5.12) $$F'(u_0)=0, \quad u_0>3, \quad F'(u)<0 \text{ for } 0<u<u_0.$$

However, by (5.6) and (5.11) with suitable numbers u_1 and u_2 satisfying $u_0-1<u_1<u_0$, $u_1-1<u_2<u_1$, we derive

$$0 = u_0F'(u_0) = f(u_0-1)-F(u_0) < f(u_0-1)-f(u_0) = -f'(u_1)$$
$$= \frac{f(u_1)-F(u_1-1)}{u_1} < \frac{F(u_1)-F(u_1-1)}{u_1} = \frac{F'(u_2)}{u_1},$$

which contradicts (5.12). Hence $F'(u)<0$ for $u>0$, and therefore with (5.6) and (5.11) again

$$uf'(u) = F(u-1)-f(u) > F(u-1)-F(u) > 0, \qquad u>2.$$

Combining these results with (5.10) we get

(5.13) $$\begin{cases} F(u) \text{ is monotonically decreasing towards } 1, \\ f(u) \text{ is monotonically increasing towards } 1. \end{cases}$$

The following notation will be convenient

(5.14) $$g_\nu(u) = \begin{cases} F(u), & \text{if } \nu\equiv 0 \bmod 2, \\ f(u), & \text{if } \nu\equiv 1 \bmod 2, \end{cases} \qquad u>0.$$

LEMMA 5.1. *For $2\leqslant z_1\leqslant z\leqslant \sqrt{y}$ and both values of ν we have*

$$R_k(z)g_\nu\left(\frac{\log y}{\log z}\right)$$
$$= R_k(z_1)g_\nu\left(\frac{\log y}{\log z_1}\right) - \sum_{\substack{z_1\leqslant p<z \\ p\nmid k}} \frac{R_k(p)}{p}g_{\nu+1}\left(\frac{\log(y/p)}{\log p}\right) + O\left(\frac{R_k(z)\log z\log\log 3k}{\log^2 z_1}\right).$$

Proof. By (2.3) and (2.8), if $z_1\leqslant w\leqslant z$,

(5.15) $$\sigma(w) = \frac{1}{R_k(z)}\sum_{\substack{z_1\leqslant p<w \\ p\nmid k}}\frac{R_k(p)}{p} = \frac{R_k(z_1)}{R_k(z)} - \frac{R_k(w)}{R_k(z)}$$
$$= \log z\left(\frac{1}{\log z_1}-\frac{1}{\log w}\right) + O\left(\frac{\log z\log\log 3k}{\log^2 z_1}\right).$$

We set

$$g(w) = g_{\nu+1}\left(\frac{\log(y/w)}{\log w}\right), \qquad z_1\leqslant w\leqslant z.$$

Because of $w \leqslant \sqrt{y}$ the argument of $g_{\nu+1}$ is $\geqslant 1$. Therefore, using (5.13), $g(w)$ is monotonic, uniformly bounded and continuous in $z_1 \leqslant w \leqslant z$. Hence, by (5.15) and (5.7),

$$\frac{1}{R_k(z)} \sum_{\substack{z_1 \leqslant p < z \\ p \nmid k}} \frac{R_k(p)}{p} g_{\nu+1}\left(\frac{\log(y/p)}{\log p}\right)$$

$$= \sigma(z) g(z) - \int_{z_1}^{z} \sigma(w)\, dg(w)$$

$$= \left(\frac{\log z}{\log z_1} - 1\right) g(z) + O\left(\frac{\log z \log\log 3k}{\log^2 z_1}\right) - \log z \int_{z_1}^{z} \left(\frac{1}{\log z_1} - \frac{1}{\log w}\right) dg(w)$$

$$= \log z \int_{z_1}^{z} \frac{g(w)}{\log^2 w} \cdot \frac{dw}{w} + O\left(\frac{\log z \log\log 3k}{\log^2 z_1}\right)$$

$$= \frac{\log z}{\log y} \int_{\frac{\log y}{\log z}}^{\frac{\log y}{\log z_1}} g_{\nu+1}(t-1)\, dt + O\left(\frac{\log z \log\log 3k}{\log^2 z_1}\right)$$

$$= \frac{\log z}{\log y} \left\{\frac{\log y}{\log z_1} g_\nu\left(\frac{\log y}{\log z_1}\right) - \frac{\log y}{\log z} g_\nu\left(\frac{\log y}{\log z}\right)\right\} + O\left(\frac{\log z \log\log 3k}{\log^2 z_1}\right).$$

Using (2.8) our lemma follows.

With a similar proof we get

LEMMA 5.2. *For $2 \leqslant z_1 \leqslant z \leqslant y$ we have*

$$\sum_{\substack{z_1 \leqslant p < m \\ p \nmid k}} \frac{R_k(p)}{p} e^{-\frac{\log(y/p)}{\log p}} \leqslant R_k(z) e^{-\frac{\log y}{\log z}} \frac{e}{3} \left(1 + c_2 \frac{\log y \log\log 3k}{\log^2 z_1}\right),$$

where $m = \min(z, y^{1/3})$.

Proof. Using (5.15) as before and setting $g(w) = e^{-\frac{\log(y/w)}{\log w}}$, which also is monotonic, continuous, but bounded by $g(m)$ in $z_1 \leqslant w \leqslant m$, we obtain

(5.16)

$$\frac{1}{R_k(z)} \sum_{\substack{z_1 \leqslant p < m \\ p \nmid k}} \frac{R_k(p)}{p} e^{-\frac{\log(y/p)}{\log p}} = \frac{\log z}{\log y} \int_{\frac{\log y}{\log m}}^{\frac{\log y}{\log z_1}} e^{1-t} dt + O\left(\frac{\log z \log\log 3k}{\log^2 z_1} g(m)\right).$$

Noting that

$$\frac{\log z}{\log y} e^{1-\frac{\log y}{\log m}} \leqslant e^{-\frac{\log y}{\log z}} \frac{e}{3},$$

we find that the left-hand side of (5.16) is

$$\leqslant e^{-\frac{\log y}{\log z}} \frac{e}{3} \left(1 + c_2 \frac{\log y \log\log 3k}{\log^2 z_1}\right).$$

THEOREM 4. *Let $2 \leqslant z_1 \leqslant z \leqslant \sqrt{y}$ and set*

$$y_j = \frac{y}{p_1 \ldots p_j}, \qquad j = 1, 2, \ldots$$

Then, for all positive integers r and both values of ν, we have

$$R_k(z) g_\nu\left(\frac{\log y}{\log z}\right)$$

$$= R_k(z_1) g_\nu\left(\frac{\log y}{\log z_1}\right) + \sum_{1 \leqslant i \leqslant r-1} (-1)^i \sum_{\substack{z_1 \leqslant p_i < \ldots < p_1 < z \\ p_j < \sqrt{y_j},\, p_j \nmid k,\, j = 1, \ldots, i}} \frac{R_k(z_1)}{p_1 \ldots p_i} g_{\nu+i}\left(\frac{\log y_i}{\log z_1}\right) +$$

$$+ (-1)^r \sum_{\substack{z_1 \leqslant p_r < \ldots < p_1 < z \\ p_j < \sqrt{y_j},\, p_j \nmid k,\, j = 1, \ldots, r}} \frac{R_k(p_r)}{p_1 \ldots p_r} g_{\nu+r}\left(\frac{\log y_r}{\log p_r}\right) +$$

$$+ \sum_{1 \leqslant i \leqslant r} (-1)^i \sum_{\substack{z_1 \leqslant p_i < \ldots < p_1 < z \\ p_j < \sqrt{y_j},\, p_j \nmid k,\, j = 1, \ldots, i-1 \\ \sqrt{y_i} \leqslant p_i < y_i,\, p_i \nmid k}} \frac{R_k(p_i)}{p_1 \ldots p_i} g_{\nu+i}\left(\frac{\log y_i}{\log p_i}\right) + O\left(\frac{R_k(z) \log^2 z \log\log 3k}{\log^3 z_1}\right).$$

This holds independently of condition $H_k(M)$, and the O-constant is meant to be independent of r.

Proof. Apart from the remainder term the proof of Theorem 4 follows the same lines as in the proof of Theorem 1. Here, we start from Lemma 5.1, taking $\nu+1$, y/p, p for ν, y, z, respectively. Regarding the O-term we see by induction that it becomes (with the same O-constant as in Lemma 5.1)

$$O\left(\frac{\log\log 3k}{\log^2 z_1}\right) \left\{R_k(z) \log z + \sum_{1 \leqslant i \leqslant r-1} \sum_{z_1 \leqslant p_i < \ldots < p_1 < z} \frac{R_k(p_i) \log p_i}{p_1 \ldots p_i}\right\}.$$

Using now (2.6) and Mertens' formula

(5.17) $$\sum_{z_1 \leqslant p < z} \frac{1}{p} = \log \frac{\log z}{\log z_1} + O\left(\frac{1}{\log z_1}\right),$$

we obtain for the remainder the estimate

$$O\left(\frac{R_k(z)\log z\log\log 3k}{\log^2 z_1}\sum_{0\leqslant i\leqslant r-1}\frac{1}{i!}\Big(\sum_{z_1\leqslant p<z}\frac{1}{p}\Big)^i\right)=O\left(\frac{R_k(z)\log^2 z\log\log 3k}{\log^3 z_1}\right).$$

6. The main theorem.

THEOREM 5. *If y satisfies $\mathrm{H}_k(M)$ and $y\geqslant z$, we have*

$$A_k(M;z)\leqslant yR_k(z)\left(F\left(\frac{\log y}{\log z}\right)+c_4\frac{\log\log 3k}{(\log y)^{1/14}}\right),$$

$$A_k(M;z)\geqslant yR_k(z)\left(f\left(\frac{\log y}{\log z}\right)-c_4\frac{\log\log 3k}{(\log y)^{1/14}}\right).$$

This result should be compared with Selberg's in which the functions corresponding to $F(u)$ and $f(u)$ were found to be

$$F_0(u)=\Big(1-e^{-\gamma}\int_{u/2+1}^{\infty}\varrho(t)\,dt\Big)^{-1},\qquad f_0(u)=1-\frac{1}{u}\int_{u-1}^{\infty}|F_0(t)-1|\,dt,$$

for $u\geqslant 1$ resp. $u\geqslant 2$, see Ankeny and Onishi ([1]). It is possible to show(4) that

$$F_0(u)>F(u)\quad\text{ for }\quad u>2,$$

and, therefore,

$$f_0(u)<f(u)\quad\text{ for }\quad u>2.$$

Note that our $f(u)$ is positive exactly for $u>2$.

Proof. If $\sqrt{y}\leqslant z\leqslant y$, then by (5.5) and (3.9) our Theorem is true. Also, if $(\log y)^{1/14}/\log\log 3k$ is bounded by some absolute constant because of (4.1), (1.1) and (5.13) the Theorem holds. Therefore, we may suppose that

$$z\leqslant\sqrt{y},$$

and

$$\log y\geqslant c_3(\log\log 3k)^{14},\tag{6.1}$$

where c_3 is some sufficiently large constant, in particular that y is greater than a sufficiently large absolute constant.

We shall now apply Theorem 1 and Theorem 4 with

$$z_1=\exp\{(\log y)^{7/10}\}\tag{6.2}$$

(4) With a more detailed calculation for $3<u<4$.

and r satisfying

$$\frac{1}{3}\cdot\frac{(\log y)^{3/10}}{\log\log 3y}\leqslant\Big(\frac{3}{2}\Big)^r\leqslant\frac{3}{4}\cdot\frac{(\log y)^{3/10}}{\log\log 3y}.\tag{6.3}$$

If $z<z_1$ our Theorem has already been proved because of (4.2) and (5.13), hence the conditions of Theorems 1 and 4 may supposed to be fulfilled. We now form

$$(-1)^\nu\left\{A_k(M;z)-yR_k(z)g_\nu\left(\frac{\log y}{\log z}\right)\right\}$$

which has to be estimated from above by

$$O\left(yR_k(z)\frac{\log\log 3k}{(\log y)^{1/14}}\right)$$

according to definition (5.14). Having multiplied in Theorem 4 by y and in both Theorems by $(-1)^\nu$ we form the difference and will estimate each of the terms occurring on the right-hand side.

For the first term and the first sum on the right we apply (4.2). In the proof of Theorem 3 we already stated that $y_i=y/p_1\ldots p_i$ satisfies $\mathrm{H}_k(M_{p_1\ldots p_i})$. In (4.2) take z_1 for z and use the corresponding M's and y's. Then, because of (5.10), for the first term we get the remainder

$$O\left(yR_k(z_1)\frac{1}{\log y}\right).$$

The application to the sum presupposes

$$\log z_1\leqslant\frac{\log y_i}{2\log\log 3y_i}\quad\text{ for }\quad i=1,\ldots,r-1,\tag{6.4}$$

and contributes, since $y_i>p_i\geqslant z_1$,

$$O\left(\sum_{1\leqslant i\leqslant r-1}\ \sum_{z_1\leqslant p_i<\ldots<p_1<z}\frac{y}{p_1\ldots p_i}R_k(z_1)\frac{1}{\log z_1}\right)$$
$$=O\left(yR_k(z_1)\frac{1}{\log z_1}\sum_{1\leqslant i\leqslant r-1}\frac{1}{i!}\Big(\sum_{z_1\leqslant p<z}\frac{1}{p}\Big)^i\right).$$

That condition (6.4) is satisfied can be seen as follows. From $p_j<\sqrt{y_j}$ for $j=1,\ldots,i$ we first derive, for $j=1$, $p_1<y^{1/3}$, and hence

$$y_j>y^{(2/3)^j}\tag{6.5}$$

holds true for $j=1$. If (6.5) had already been proved for j, $1\leqslant j\leqslant i-1$, then $p_{j+1}<\sqrt{y_{j+1}}$ implies $p_{j+1}<y_j^{1/3}$, and hence

$$y_{j+1}=\frac{y_j}{p_{j+1}}>y_j^{2/3}>y^{(2/3)^{j+1}}.$$

For a proof of (6.4), therefore, the inequality

$$\log z_1\leqslant\frac{(2/3)^i\log y}{2\log\log 3y},\qquad i=1,\ldots,r-1,$$

suffices. It is even enough to check the case $i=r-1$. Here, by (6.3), we find

$$\frac{(2/3)^{r-1}\log y}{2\log\log 3y}=\frac{3\log y}{4\log\log 3y}\cdot\frac{1}{(3/2)^r}\geqslant(\log y)^{7/10}=\log z_1.$$

Next, take the last sum. Here, we have to deal with terms of the form

$$(6.6)\qquad (-1)^{\nu+i}\left\{A_k(M_{p_1\ldots p_i};p_i)-y_iR_k(p_i)g_{\nu+i}\left(\frac{\log y_i}{\log p_i}\right)\right\},$$

where $\sqrt{y_i}\leqslant p_i<y_i$. If $\nu+i$ is odd, by (5.5) $g_{\nu+i}\left(\frac{\log y_i}{\log p_i}\right)=0$, and hence (6.6) can be estimated from above by zero. If $\nu+i$ is even, then by (3.9), taking y_i and p_i for y and z, respectively, and by (5.5), (2.6) we get the upper estimate

$$O\left(\sum_{1\leqslant i\leqslant r}\ \sum_{z_1\leqslant p_i<\ldots<p_1<z}\frac{y}{p_1\ldots p_i}R_k(p_i)\log p_i\frac{\log\log y}{\log^2 z_1}\right)$$
$$=O\left(yR_k(z)\frac{\log z\log\log y}{\log^2 z_1}\sum_{1\leqslant i\leqslant r}\frac{1}{i!}\left(\sum_{z_1\leqslant p<z}\frac{1}{p}\right)^i\right).$$

We still have the error term of Theorem 4, which is

$$O\left(yR_k(z)\frac{\log^2 z\log\log 3k}{\log^3 z_1}\right).$$

So far, the remainder terms arising from all but the second sums in our identities have been

$$O\left(yR_k(z)\right)\left\{\frac{R_k(z_1)}{R_k(z)}\cdot\frac{1}{\log y}+\frac{R_k(z_1)}{R_k(z)}\cdot\frac{1}{\log z_1}\exp\left\{\sum_{z_1\leqslant p<z}\frac{1}{p}\right\}+\right.$$
$$\left.+\frac{\log z\log\log y}{\log^2 z_1}\exp\left\{\sum_{z_1\leqslant p<z}\frac{1}{p}\right\}+\frac{\log^2 z\log\log 3k}{\log^3 z_1}\right\}.$$

Using (2.6) and (5.17) we see that they are

$$O\left(yR_k(z)\right)\left\{\frac{1}{\log z_1}+\frac{\log^2 y}{\log^3 z_1}+\frac{\log^2 y\log\log y}{\log^3 z_1}+\frac{\log^2 y\log\log 3k}{\log^3 z_1}\right\},$$

and hence, by (6.2),

$$O\left(yR_k(z)\frac{\log\log y+\log\log 3k}{(\log y)^{1/10}}\right),$$

i.e. of sufficiently small order.

It remains to deal with the second sum. Let its factor $(-1)^{\nu+r}$ be chosen to be $+1$ such that (4.1) can be applied (the r-interval in (6.3) always contains both an even and an odd number). Then this O-term becomes, because of (5.10), apart from an O-constant

$$(6.7)\qquad y\sum_{\substack{z_1\leqslant p_r<\ldots<p_1<z\\ p_j<\sqrt{y_j},\,p_j\nmid k,\,j=1,\ldots,r}}\frac{R_k(p_r)}{p_1\ldots p_r}e^{-\frac{\log y_r}{\log p_r}}.$$

The sum over p_r is

$$U_r=\sum_{\substack{z_1\leqslant p_r<m_r\\ p_r\nmid k}}\frac{R_k(p_r)}{p_r}e^{-\frac{\log(y_{r-1}/p_r)}{\log p_r}},\qquad m_r=\min(p_{r-1},y_{r-1}^{1/3}).$$

For this sum we get, using Lemma 5.2, the estimate

$$U_r\leqslant R_k(p_{r-1})e^{-\frac{\log y_{r-1}}{\log p_{r-1}}}\theta,\qquad \theta=\frac{e}{3}\left(1+c_2\frac{\log\log 3k}{(\log y)^{2/5}}\right).$$

Introducing this in (6.7) we see that apart from the factor θ we get the same sum with $r-1$ instead of r as an upper estimate. Thus, we can repeat this procedure, using $p_0=z$, $y_0=y$ in the last step. Then we get in total

$$y\sum_{\substack{z_1\leqslant p_r<\ldots<p_1<z\\ p_j<\sqrt{y_j},\,p_j\nmid k,\,j=1,\ldots,r}}\frac{R_k(p_r)}{p_1\ldots p_r}e^{-\frac{\log y_r}{\log p_r}}\leqslant yR_k(z)e^{-\frac{\log y}{\log z}}\theta^r.$$

Now, it finally remains to show that this error term also is of sufficiently small order. However,

$$\theta^r = O\left(\frac{1}{(\log y)^{1/14}}\right)$$

follows from (6.3), since by (6.1) θ can be brought sufficiently close to $e/3$.

7. Applications. The most interesting and effective application of the linear sieve is to a set consisting of numbers in a short interval belonging to an arithmetic progression. In this case we define instead of $A_k(M; z)$ the following special function.

Let k and l be positive integers, $(k, l) = 1$, $1 \leqslant l \leqslant k$. Let x, h, z be real numbers

$$k < h \leqslant x, \qquad z \geqslant 2.$$

Then we are interested in

$$B_{k,l}(x, h, z) = |\{x-h < n \leqslant x;\ n \equiv l \bmod k,\ n \not\equiv 0 \bmod p\ \forall p < z\}|$$

(cf. N. I. Klimov [9]). This number is not changed by adding the condition $p \nmid k$, since no number $\equiv l \bmod k$ can be divisible by a prime divisor of k, because of $(k, l) = 1$. Therefore, $B_{k,l}(x, h, z) = A_k(M; z)$, where

$$M = \{x-h < n \leqslant x;\ n \equiv l \bmod k\}.$$

If $(d, k) = 1$, with a suitable l_d, we get

$$|M_d| = \left[\frac{\frac{x}{d} - l_d}{k}\right] - \left[\frac{\frac{x-h}{d} - l_d}{k}\right],$$

and hence $y = h/k$ satisfies $\mathrm{H}_k(M)$.

Therefore, by Theorem 5

$$f(u) - c_4\frac{\log\log 3k}{(\log h)^{1/14}} \leqslant \frac{B_{k,l}(x, kh, h^{1/u})}{hR_k(h^{1/u})} \leqslant F(u) + c_4\frac{\log\log 3k}{(\log h)^{1/14}}, \tag{7.1}$$

$$u \geqslant 1,\ 2^u \leqslant h \leqslant \frac{x}{k}$$

and by (3.9), using (5.5),

$$\frac{B_{k,l}(x, kh, h^{1/u})}{hR_k(h^{1/u})} \leqslant F(u) + c_1\frac{\log\log 3h}{u \log h}, \qquad 0 < u < 1,\ 2^u \leqslant h \leqslant \frac{x}{k}. \tag{7.2}$$

The upper estimate in (7.1) improves Klimov's result ([9]).

Let a be a non-negative integer and

$$x \geqslant h \geqslant \zeta \geqslant z \geqslant 2.$$

In order to count numbers with smaller prime factors, we follow P. Kuhn ([10], § 3) in defining *

$$C^a_{k,l}(x, h, \zeta, z) = \Big|\Big\{x-h < n \leqslant x;\ n \equiv l \bmod k,\ n \not\equiv 0 \bmod p\ \forall p < z,\ n \not\equiv 0 \bmod p^2\ \forall z \leqslant p < \zeta,\ \sum_{\substack{p|n\\ z\leqslant p<\zeta}} 1 \leqslant a\Big\}\Big|, \tag{7.3}$$

$$K = \{x-kh < n \leqslant x;\ n \equiv l \bmod k,\ n \not\equiv 0 \bmod p\ \forall p < z,\ n \not\equiv 0 \bmod p^2\ \forall z \leqslant p < \zeta\}.$$

The lower estimate was technically perfected by Y. Wang (cf. [18]), and is given here in a form due to W. Haneke (oral communication).

With suitable numbers l_p, $(k, l_p) = 1$, $1 \leqslant l_p \leqslant k$,

$$C^a_{k,l}(x, kh, \zeta, z) = |K| - \Big|\Big\{n \in K;\ \sum_{\substack{p|n\\ z\leqslant p<\zeta}} 1 > a\Big\}\Big|$$

$$\geqslant B_{k,l}(x, kh, z) - \sum_{z\leqslant p<\zeta}\ \sum_{\substack{x-kh<n\leqslant x\\ n\equiv 0 \bmod p^2\\ n\equiv l \bmod k}} 1 - \sum_{n\in K}\frac{1}{a+1}\sum_{\substack{p|n\\ z\leqslant p<\zeta}} 1$$

$$\geqslant B_{k,l}(x, kh, z) - \sum_{z\leqslant p<\zeta}\left(\frac{h}{p^2}+1\right) - \frac{1}{a+1}\sum_{\substack{z\leqslant p<\zeta\\ p\nmid k}} B_{k,l_p}\left(\frac{x}{p}, k\frac{h}{p}, z\right).$$

Setting

$$\zeta = h^{1/u}, \qquad z = h^{1/v},$$

we obtain by (7.1) and (7.2)

$$\frac{C^a_{k,l}(x, kh, h^{1/u}, h^{1/v})}{hR_k(h^{1/v})} \geqslant f(v) - c_4\frac{\log\log 3k}{(\log h)^{1/14}} - c_5\frac{h^{-1/v}+h^{1/u-1}}{R_k(h^{1/v})} - \tag{7.4}$$

$$-\frac{1}{a+1}\sum_{h^{1/v}\leqslant p<h^{1/u}}\frac{1}{p}\left\{F\left(v\frac{\log(h/p)}{\log h}\right) + c_6(u, v)\frac{\log\log 3k}{(\log h)^{1/14}}\right\},\ 1 < u \leqslant v,\ 2^v \leqslant h \leqslant \frac{x}{k},$$

where $c_6(u, v), \ldots$ are positive constants depending on u and v only.

We have (cf. (5.17))

$$\tau(w) = \sum_{h^{1/v}\leqslant p<w}\frac{1}{p} = \log\left(\frac{v\log w}{\log h}\right) + O\left(\frac{v}{\log h}\right), \qquad w \geqslant h^{1/v}. \tag{7.5}$$

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* **Added in proof:** In Acta Arithmetica 10(1965), pp. 387-397, B. V. Levin has investigated the counting function for a more general set M. However, compared with our applications he obtains slightly weaker results (cf. Teorema 1, p. 392 with (7.14), e.g.).

Setting

$$g(w) = F\left(v\frac{\log(h/w)}{\log h}\right), \qquad h^{1/v} \leqslant w \leqslant h^{1/u},$$

we get by (7.5) and (5.13)

$$\sum_{h^{1/v}\leqslant p<h^{1/u}} \frac{1}{p}F\left(v\frac{\log(h/p)}{\log h}\right) = \sum_{h^{1/v}\leqslant p<h^{1/u}} \frac{1}{p}g(p)$$

$$= \left\{\log\frac{v}{u}+O\left(\frac{v}{\log h}\right)\right\}g(h^{1/u}) - \int_{h^{1/v}}^{h^{1/u}} \tau(w)\,dg(w)$$

$$= \log\frac{v}{u}\,g(h^{1/u})+O\left(\frac{v}{\log h}\,g(h^{1/u})\right) - \int_{h^{1/v}}^{h^{1/u}} \log\left(\frac{v\log w}{\log h}\right)dg(w)$$

$$= \int_{h^{1/v}}^{h^{1/u}} \frac{g(w)}{\log w}\cdot\frac{dw}{w}+O\left(\frac{v}{\log h}F\left(v-\frac{v}{u}\right)\right).$$

Hence, it follows from (7.4), (2.7) and (7.5)

(7.6) $$\frac{C^a_{k,l}(x, kh, h^{1/u}, h^{1/v})}{hR_k(h^{1/v})} \geqslant f(v) - \frac{1}{a+1}\int_u^v F\left(v-\frac{v}{t}\right)\frac{dt}{t} - c_7(u,v)\frac{\log\log 3k}{(\log h)^{1/14}},$$

$$1 < u \leqslant v, \qquad 2^v \leqslant h \leqslant x/k.$$

THEOREM 6. *Let $r \geqslant 2$ be an integer and*

$$\varLambda_r = \sup_{u,v,a}\left(\frac{r+1-a}{u}+\frac{a}{v}\right),$$

where the sup *is taken over all real numbers u, v such that*

(7.7) $$1 < u \leqslant v,$$

(7.8) $$f(v) - \frac{1}{a+1}\int_u^v F\left(v-\frac{v}{t}\right)\frac{dt}{t} > 0,$$

and over all non-negative integers a.

Then, for every $\varepsilon > 0$, there is a number $x_0 = x_0(r, \varepsilon)$ depending on r and ε only such that for $x \geqslant x_0$, $x^{1-\frac{1}{\varLambda_r}-\varepsilon} \geqslant k$ there exist at least two integers n satisfying

(7.9) $$x - kx^{\frac{1}{\varLambda_r}+\varepsilon} < n \leqslant x,$$

(7.10) $$n \equiv l \bmod k, \qquad \Omega(n) \leqslant r,$$

$\Omega(n)$ denotes the total number (with multiplicities) of prime divisors of n.

Proof. For information we remark that according to (7.7)

$$\frac{r+1-a}{u}+\frac{a}{v} \leqslant \frac{r+1}{u} < r+1$$

whereas by taking $v = u = 2+\delta$, $0<\delta<1$, and by (5.9)

$$\varLambda_r \geqslant (r+1)/(2+\delta),$$

thus proving

$$(r+1)/2 \leqslant \varLambda_r \leqslant r+1.$$

We may suppose ε to be sufficiently small and $x_0(r,\varepsilon)$ large enough. Choosing $u = u(r,\varepsilon)$, $v = v(r,\varepsilon)$, $a = a(r,\varepsilon)$ satisfying (7.7), (7.8) and

(7.11) $$\frac{1}{1/\varLambda_r+\varepsilon} < \frac{r+1-a}{u}+\frac{a}{v}$$

and putting

$$h = x^{1/\varLambda_r+\varepsilon},$$

we find that the conditions of (7.6) are fulfilled, and therefore $C^a_{k,l}(x, kh, h^{1/u}, h^{1/v}) \geqslant 2$, i.e. there are at least two numbers n satisfying both (7.9) and (7.10). Let b denote the number of prime divisors of n being $< h^{1/u}$, then we have because of $u \leqslant v$ and $b \leqslant a$ (cf. (7.3))

$$h^{\frac{a}{v}+\frac{\Omega(n)-a}{u}} \leqslant h^{\frac{b}{v}+\frac{\Omega(n)-b}{u}} \leqslant n \leqslant x = h^{\frac{1}{1/\varLambda_r+\varepsilon}},$$

$$\Omega(n) \leqslant a+u\left(\frac{1}{1/\varLambda_r+\varepsilon}-\frac{a}{v}\right) < r+1,$$

by (7.11).

The most important special cases of Theorem 6 are obtained by choosing $k=1$ resp. $x = k^{\frac{1}{1-1/\varLambda_r-\varepsilon}}$.

THEOREM 7. *Let $r \geqslant 2$. Then, for $\varepsilon > 0$ and $x \geqslant x_0(r,\varepsilon)$ there exist at least two integers n in the interval*

$$x - x^{1/\varLambda_r+\varepsilon} < n \leqslant x$$

having at most r prime factors.

The best previous results in this direction were given by P. Kuhn ([10]) and particularly by Y. Wang ([18], [19]). According to the later evaluation of $\varLambda_r$ our exponent is better for every r.

THEOREM 8. *Let $r \geqslant 2$. Then, for $\varepsilon > 0$ and $k \geqslant k_0(r, \varepsilon)$ there exists an integer*

$$n \equiv l \bmod k, \quad 1 \leqslant n \leqslant k^{\frac{1}{1-1/\Lambda_r}+\varepsilon}$$

having at most r prime factors.

For applications it remains to determine Λ_r numerically. Define for $v \geqslant 4$ and integral $a \geqslant 0$

$$I(v) = \int_{v/(v-3)}^{v} F\left(v - \frac{v}{t}\right)\frac{dt}{t} - \frac{2e^{\gamma}}{v}\log\frac{v-3}{3},$$

$$w = w_a(v) = 1 + \exp\left\{-\frac{v}{2e^{\gamma}}\big((a+1)f(v) - I(v)\big)\right\},$$

and assume that

(7.12) $$w_a(v) < \frac{v}{v-3} \quad (\leqslant v).$$

Then, by (5.8),

$$(a+1)f(v) - \int_{w}^{v} F\left(v - \frac{v}{t}\right)\frac{dt}{t} = (a+1)f(v) - I(v) + \frac{2e^{\gamma}}{v}\log(w-1) = 0.$$

Thus, for sufficiently small $\varepsilon > 0$, putting $u = w + \varepsilon$ with v and a chosen as above, the conditions (7.7) and (7.8) are satisfied. Hence, making $\varepsilon \to 0$, we obtain

(7.13) $$\Lambda_r \geqslant \frac{r+1-a}{w_a(v)} + \frac{a}{v},$$

provided that $v \geqslant 4$ and (7.12) hold. For suitable choices of v and a we obtain numerical approximations to Λ_r.

As $r \to \infty$ we choose

$$v = \log r, \quad a = [2\log v],$$

and note that in view of (5.10)

$$f(v) = 1 + o(1), \quad I(v) = \log v + o(\log v),$$

$$(1 <) \quad w_a(v) \leqslant 1 + e^{-v} < 1 + \frac{1}{v} \leqslant \frac{v}{v-3},$$

and hence

$$\Lambda_r \geqslant r - c_8 \log\log r.$$

Thus, Λ_r is asymptotically equal to r as $r \to \infty$.

For all integers $r \geqslant 2$ we may put $v = 6$. By numerical calculations (5) one obtains

$$f(6) \geqslant 0.999895, \quad I(6) \leqslant 1.133056,$$

and, by using three places only,

$$w_1(6) \leqslant \frac{21}{17}, \quad w_2(6) \leqslant \frac{24}{23}.$$

Hence, (7.13) implies both

(7.14) $$\Lambda_r \geqslant \frac{34r+7}{42}, \quad \Lambda_r \geqslant \frac{23r-15}{24} \quad (r \geqslant 2),$$

thus superseding all previous results for relatively small values of r. (For larger values of r we choose a larger also.)

Taking $r = 2$ in (7.14) gives $\Lambda_2 \geqslant \frac{25}{14}$, thus proving Theorems 9 and 10 given in the introduction.

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(5) We are indebted to D. Grohne for many numerical calculations. In particular, he determined $\Lambda_2 = 1.7943\ldots$ Also, his computations suggested that the sieving limit be actually $u = 2$, as shown by the main theorem.

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Sur un théorème de Rényi

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1. Introduction. Désignons par $\omega(n)$ le nombre des diviseurs premiers de l'entier positif n et par $\Omega(n)$ le nombre total des facteurs dans la décomposition de n en facteurs premiers. Autrement dit, soient ω et Ω les fonctions de l'entier positif n définies de la façon suivante:

$$\omega(1) = \Omega(1) = 0$$

et, si $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, où $p_1, p_2, \dots, p_k$ sont des nombres premiers distincts et $\alpha_1, \alpha_2, \dots, \alpha_k$ des entiers > 0,

$$\omega(n) = k \quad \text{et} \quad \Omega(n) = \alpha_1+\alpha_2+\dots+\alpha_k.$$

Il est clair que l'on a toujours $\Omega(n) \geqslant \omega(n)$, l'égalité ayant lieu pour les entiers "quadratfrei".

Rényi a montré ([4]) que, pour chaque entier $q \geqslant 0$, l'ensemble des n pour lesquels on a $\Omega(n)-\omega(n) = q$ possède une densité d_q, la suite des nombres d_q étant déterminеé par le fait que, pour $|z| < 2$,

$$\sum_{q=0}^{+\infty} d_q z^q = \prod\Big(1-\frac{1}{p}\Big)\Big(1+\frac{1}{p-z}\Big) = \frac{6}{\pi^2}\prod\frac{1-z/(p+1)}{1-z/p},$$

où p parcourt la suite des nombres premiers (¹).

Pour $q = 0$, on retrouve le fait bien connu que l'ensemble des entiers positif "quadratfrei" possède une densité égale à $6/\pi^2$.

En ce qui concerne les entiers "quadratfrei", Landau a montré ([3]) que le théorème des nombres premiers, sous la forme $\overline{\omega}(x) \sim x/\log x$, entraîne le résultat suivant:

Si $Q(x)$ est le nombre de ces entiers au plus égaux à x, on a pour x infini:

$$Q(x) = \frac{6}{\pi^2}x + o[x^{1/2}].$$

(¹) Tout au long de cet article, dans toute somme ou tout produit portant sur une expression où figure p, il est entendu que p parcourt la suite des nombres premiers.