Then,

\[ Q \leq \log(\sqrt{2} + \frac{1}{2\omega}) + \frac{1}{\omega} \left( 1 + \frac{1}{2\omega} \right) \text{ for } \omega \gg \log 10. \]

For \( \omega \gg \sqrt{2} \omega (> \log 10) \) this function is decreasing, and for \( \omega = \sqrt{2} \omega \) it is \( < \frac{1}{\omega} \). This proves Theorem 2.

It remains to prove Theorem 1 for \( 1 < u < \epsilon^2 \). Obviously, for \( u < \epsilon^2 \), Theorem 1 is a consequence of Theorem 2. For \( u > \epsilon^2 \) however, the corresponding \( \omega \) being \( > 6.4 \), (4.1) gives for \( u < \epsilon^2 \)

\[ Q < 1.4 < 1 + \frac{4}{\log u}. \]

References


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An improvement of Selberg's sieve method 1

by

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The sieve method is concerned with counting in a finite set \( M \) of positive integers those elements which are not divisible by any prime \( p < x \) or rather with estimating the number \( A(M; x) \) of these elements from above and from below. If \( s \) is sufficiently large compared with the largest element in \( M \) the numbers counted in \( A(M; s) \) will have a restricted number of prime divisors, and it is this fact that makes estimates of \( A(M; s) \) interesting. The estimates, however, depend essentially upon only the number of elements in \( M \), say \( |M| \).

For relatively small values of \( s \) the classical sieve of Eratosthenes-Legendre is quite satisfactory (§ 1). However, the interest lies in larger values of \( s \) which were first treated successfully by Viggo Brun. Later, Buchstab was able to give improved estimates, starting from initial estimates, by using identities of Meissel's type (cf. (2.2)). For an upper estimate the sieve method was formulated generally and this estimate minimised by A. Selberg. He also gave a lower estimate which can be derived from his upper estimate by using Buchstab's method (cf. Ankeny-Onishi [1]). In applications a combinatorial argument of Kuhn led to further improvements. In a series of papers Y. Wang very successfully combined all methods mentioned above.

We propose in this paper a method (based on Theorems 1 and 4) which leads to a new two-sided estimate (Theorem 5) by employing only the simplest upper estimate of Selberg for \( s > |M| \) (cf. Corollary to Theorem 2) and extensions of the classical estimate (Theorem 3).

The new estimate improves Selberg's upper and lower estimate for \( s > |M| \), and cannot be further improved upon by Buchstab's method. In particular, the new lower estimate is positive already for \( s^2 + x > |M| \) (with arbitrary \( x > 0 \) and large \( s \)), which decides a question left open by A. Selberg ([15], p. 292). Furthermore, our estimate is completely uniform for all sets of a certain regular behaviour (condition \( H_0(M) \)). This condition restricts, however, our present presentation to what might be called the linear sieve. Exceptional primes are allowed which we com-
blue to a product $k$. Thereby, we may apply our results to numbers in a short interval belonging to a residue class mod $k$ and having at most $r$ prime factors (Theorems 6, 7, and 8). Here, we make use of Kuhn's method. Thus, in particular, we obtain the following theorems.

**Theorem 9.** For $e > 0$ and $x \geq x_0(e)$ there exist at least two integers $n$ in the interval

$$x - e^{10^{10}} < n \leq x$$

having at most two prime factors.

**Theorem 10.** For $e > 0$, $k \geq x_0(e)$, and $(k, l) = 1$, there exists an integer

$$n = 1 \mod k, \quad 1 \leq n \leq e^{10^{10}}$$

having at most two prime factors.

These problems respectively, were first discussed by Viggo Brun ([16]) and W. Fuchs ([18]). The best exponents, previously found are $\frac{1}{5}$ (Y. Wang [18]) and $\frac{1}{6}$ (S. Uchiyama [16]).

1. The sieve of Eratosthenes. Throughout the paper let $z$ denote a real number $\geq 2$, $k$ a positive integer, and (1)

$$P_k(z) = \prod_{p \leq z} \left(1 - \frac{1}{p}\right).$$

Generalizing the terminology of the introduction we set

$$A_k(M; z) = \sum_{n \leq z \leq M} \left\lvert \frac{A_k(M; z)}{M} \right\rvert,$$

and, we denote by $M_d$ for any positive integer $d$, the set of all integers $m$ such that $m \equiv \chi(M; n \equiv 0 \mod p)$ with arbitrary integers $a_p$ can be reduced to the case $a_p = 0$ by a suitable translation of $M$ (in view of the Chinese Remainder Theorem). Let $y$ denote a real number $> 1$. For given $M$ and $k$ we impose upon $y$ the condition

$$H_k(M) = \left| M_d - \frac{y}{d} \right| \leq 1, \quad \forall (d, k) = 1.$$

Clearly, the existence of $y$ presupposes a certain regular distribution of the numbers in $M$.

Under this assumption Legendre's formula or the sieve of Eratosthenes in the form of

$$A_k(M; z) = \sum_{n \leq M} \sum_{p \leq \sqrt{M}} \mu(d) = \sum_{p \leq \sqrt{M}} \mu(d) \sum_{n \leq M_d} 1$$

(*) An empty product shall be one, similarly an empty sum shall be zero.
We now proceed by induction with respect to \( r \). Applying (2.2) for both \( x \) and \( x_i \), we get by subtraction

\[
A_k(M; q) = A_k(M; s_i) - \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p < q}} A_k(M, p) - \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p > q}} A_k(M, p),
\]

i.e. (2.1) for \( r = 1 \).

Suppose (2.1) had been proved for \( r \). Then, in the first sum of (2.4), we apply (2.1) taking \( M, y/p, p \) for \( M, y, x \), respectively. We also replace \( p_i \) by \( p_{i+1} \) and \( p \) by \( p \), which changes \( y/p \) to \( y/p_1 \) and \( y/p \) becomes \( y_1 \).

Thus,

\[
\sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p < r}} A_k(M; p) = \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p < r}} A_k(M, p; x_i) + \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p < r}} A_k(M, p_{i+1}; x_i) + \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p > r}} A_k(M, p_{i+1}; x_i) + \sum_{\substack{p < q^{1/3}\sqrt{\phi(p)}\sqrt{\phi(q)} \\ p > r}} A_k(M, p; x_i).
\]

Using this in (2.4) we obtain (2.1) for \( r+1 \).

For later use we collect some estimates for \( R_k(z) \) starting with the well-known result of Mertens

\[
R_k(z) = e^{-\gamma} \log z + O \left( \frac{1}{\log^2 z} \right),
\]

where \( \gamma \) denotes Euler's constant. This implies

\[
\frac{R_k(w)}{R_k(z)} = \log \frac{z}{w} + O \left( \frac{\log \log \log \log w}{\log \log w} \right), \quad z \gg w \gg 2.
\]

**Lemma 2.1.** For \( z \gg w \gg 2 \)

\[
\frac{R_k(w)}{R_k(z)} = \frac{\log z}{\log w} + O \left( \frac{\log \log \log \log w}{\log \log w} \right).
\]

Proof. In view of

\[
\frac{R_k(w)}{R_k(z)} = \frac{R_k(w)}{R_k(z)} \prod_{\Pi \leq w} \left( 1 - \frac{1}{p_i} \right).
\]

(2.5) implies (2.6). Taking \( w = 2 \), (2.7) follows from (2.6). Because of (2.6) a proof of (2.8) is required only in case that

\[
\log \log 2k < \log w
\]

holds true. Let \( \nu(k) \) denote the number of different prime factors of \( k \). Then

\[
1 \gg \prod_{\Pi \leq w} \left( 1 - \frac{1}{p_i} \right) \gg \left( 1 - \frac{1}{w} \right)^{\nu(k)} \gg 1 - \frac{\nu(k)}{w}.
\]

Hence, by \( \nu(k) = O(\log k) \) and (2.10)

\[
\prod_{\Pi \leq w} \left( 1 - \frac{1}{p_i} \right) = 1 + O \left( \frac{\log k}{w} \right) = 1 + O \left( \frac{1}{\log w} \right).
\]

This combined with (2.9) and (2.5) yields (2.8).

5. Selberg's upper estimate. For positive real \( x \) we define

\[
S_\varphi(z, x) = \sum_{\Pi \leq z} \frac{1}{\varphi(n)} \varphi(x), \quad T(x, z) = \sum_{\Pi \leq z} \frac{1}{\varphi(n)},
\]

where \( p(n) \) denotes the greatest prime divisor of \( n, p(1) = 1 \).

**Lemma 3.1.**

\[
S_\varphi(z, x) \gg \frac{d}{d'} S_{\varphi^k} \left( \frac{x}{x'} \right), \quad \text{if} \quad d(P[x]),
\]

(3.2)

\[
R_k(z)S_\varphi(z, x) \gg R_k(z)S_\varphi(z, x),
\]

(3.3)

\[
S_\varphi(z, x) \gg T(x, z).
\]

Proof. We have

\[
S_\varphi(z, x) = \sum_{\Pi \leq z} \sum_{\Pi \leq w} \frac{1}{\varphi(n)} = \sum_{\Pi \leq z} \frac{1}{\varphi(n)} \sum_{\Pi \leq w} \frac{1}{\varphi(m)}
\]

\[
= \sum_{\Pi \leq z} \frac{1}{\varphi(n)} S_\varphi \left( \frac{n}{m} \right).
\]

(3.4)
If \( dP_k(x) \) it follows

\[
S_k(x, \varepsilon) \geq \sum_{d \leq x} \frac{1}{\varphi(d) S_k(\varepsilon, x, x)} S_k(x, x) = \frac{d}{\varphi(d) S_k(\varepsilon, x, x)}
\]

which proves (3.1).

On the other hand we derive from (3.4)

\[
S_1(x, \varepsilon) \leq S_0(\varepsilon, x) \prod_{p < \varepsilon} \left(1 + \frac{1}{p-1}\right) = S_0(\varepsilon, x) \frac{E_0(\varepsilon)}{\varpi(\varepsilon)}.
\]

Let \( q(n) \) denote the largest squarefree divisor of \( n \). Then,

\[
S_1(x, \varepsilon) = \sum_{\ell \leq x} \frac{1}{\ell} \prod_{n \mid q(n)} \frac{1}{n} = \sum_{\ell \leq x} \frac{1}{\ell} \sum_{1 \leq m \leq \ell, m \mid q(\ell)} \frac{1}{m} = T(\ell, \varepsilon).
\]

**Theorem 2.** If \( y \) satisfies \( \Omega_k(M) \), we have

\[
A_k(M; x) \leq \frac{yR_k(x)}{B_k(\xi, x) T(\xi, \varepsilon)} + \Psi_k(\xi, x)
\]

for arbitrary values of \( \xi > 1 \).

This is Selberg's well-known upper estimate in a form convenient for our purposes. Note that the \( k \) occurs in \( B_k(\varepsilon) \) only, and that for \( T \) and \( \Psi \) one can obtain asymptotic expansions.

**Proof.** Let \( \varepsilon > 1 \). For positive integers \( d \) we define

\[
\lambda_d = \mu(d) \frac{d}{\varphi(d) S_k(\varepsilon, x)}
\]

Because of \( \xi > 1 \) we have \( S_k(\varepsilon, x, x) \geq 1 \), and

\[
\lambda_d = 0 \quad \text{for} \quad d > \xi.
\]

If \( d \mid P_k(x) \), \( 1 \leq t \leq \xi \), then

\[
\sum_{d \mid P_k(x)} \lambda_d = \sum_{d \mid P_k(x)} \sum_{m \mid d} \frac{\mu(m)}{\varphi(m)} S_k(m, x) = \mu(\ell) \frac{1}{S_k(\ell, x)} \sum_{m \mid \ell} \sum_{m \mid \ell} \frac{1}{\varphi(m)} 
\]

\[
= \frac{1}{\varphi(\ell)S_k(\ell, x)} \sum_{m \mid \ell} \sum_{m \mid \ell} \mu(m) = \frac{1}{\varphi(\ell)S_k(\ell, x)} \sum_{m \mid \ell} \mu(m) = \frac{1}{\varphi(\ell)S_k(\ell, x)} \mu(\ell).
\]

By (3.1)

\[
(3.8) \quad |\lambda_d| \leq \mu^2(d), \quad \text{if} \quad d \mid P_k
\]

Now, since \( \xi = 1 \)

\[
A_k(M; x) \leq \sum_{d \mid P_k} \left( \frac{\lambda_d}{d} \right)^2 \leq \sum_{d \mid P_k} \lambda_d \lambda_{d_2} \sum_{e \mid d d_2} 1,
\]

where \([d_1, d_2]\) denotes the least common multiple of \( d_1 \) and \( d_2 \). Here, by \( \Omega_k(M) \),

\[
\sum_{e \mid d d_2} 1 = \frac{y}{d_1, d_2} + \theta = \frac{y}{d_1, d_2} \sum_{e \mid d d_2} \varphi(e) + \theta, \quad |\theta| \leq 1.
\]

Hence, by (3.6), (3.8) and (3.7)

\[
A_k(M; x) \leq y \sum_{1 \leq d_1, d_2 \leq \xi} \lambda_{d_1} \lambda_{d_2} \sum_{e \mid d_1 d_2} \varphi(e) + \sum_{1 \leq d_1, d_2 \leq \xi} \lambda_{d_1} \lambda_{d_2}
\]

\[
\leq y \sum_{1 \leq d_1, d_2 \leq \xi} \varphi(e) \left( \sum_{e \mid d_1 d_2} \frac{\lambda_{d_1}}{d_1} + \left( \sum_{e \mid d_1 d_2} \mu(e) \right)^2 \leq \frac{y}{S_k(\xi, x)} + \Psi_k(\xi, x).
\]

Using herein (3.2) and (3.3) we obtain (3.5).

The simplest case of Theorem 2 takes the following form.

**Corollary.** If \( y \) satisfies \( \Omega_k(M) \) we have

\[
(3.9) \quad A_k(M; x) \leq \frac{\Omega_k(x) \left[ 23 \log x + 1 \log \log \log 3y \right]}{y \log y} + \frac{1}{\mu(\xi)}
\]

provided that \( x > \sqrt{y} \).

**Proof.** Taking

\[
\xi^2 = \frac{y}{1 + \log y} \quad (> 1)
\]

Theorem 2 can be applied. Since \( x > \xi \),

\[
T(\xi, \varepsilon) = \sum_{\ell \leq \xi} \frac{1}{\ell} \geq \frac{e^{-\gamma}}{\xi}
\]

by an inequality of Rosser and Schoenfeld ([12], p. 71).
Hence, from (3.5) estimating $\mathcal{P}(z, \delta)$ from above trivially by $z$ and using (2.5) and (2.7) we obtain

$$A_4(M; z) \leq yR_0(z) \left( \frac{z^2}{E_1(z)} + \frac{\delta^2}{yR_0(z)} \right) \leq yR_0(z) \left( 2z^2 \log z \log y + \frac{z^2 \log 3y}{\log^2 y} \right).$$

4. Extensions of the classical estimate.

**Theorem 3.** If $y$ satisfies $H_4(M)$, we have

$$A_4(M; z) \leq yR_0(z) \left( 1 + O(e^{-\frac{\log z}{\log y}}) \right) \quad \text{for} \quad 1 \leq \log z \leq \log y,$$

and

$$A_4(M; z) \leq yR_0(z) \left( 1 + O\left( \frac{1}{\log y} \right) \right) \quad \text{for} \quad \log z \leq \frac{\log y}{2\log \log 3y}.$$

A similar estimate which is not sufficient for our purpose is due to Barban ([2]).

**Proof.** Using (2.7) we see from (1.1) that Theorem 3 holds true if $z$ is bounded, so that we may assume

$$z > e_{\delta},$$

where $e_{\delta}$ denotes a suitable absolute constant. Then, from A. I. Vinogradov's result ([17]) we infer

$$\mathcal{P}(x, z) = O(xe^{-\frac{\log z}{\log y}}).$$

From the identity

$$T(x, z) = \frac{\mathcal{P}(x, z)}{x} + \int_1^x \frac{\mathcal{P}(u, z)}{u} \frac{du}{w^z},$$

since

$$\lim_{z \to \infty} T(x, z) = \sum_{n=1}^{\infty} \frac{1}{n} \prod_{p \leq x} \left( 1 - 1/p \right) = \frac{1}{E_1(x)},$$

it follows

$$\frac{1}{E_1(x)} = \int_1^x \frac{\mathcal{P}(u, z)}{u} \frac{du}{w^z}.$$

Since $T(x, z)$ is non-decreasing in $x$ and so, by (4.4), is $\leq 1/E_1(x)$, we find

$$\frac{1}{E_1(x)} - T(x, z) \leq \int_1^x \frac{\mathcal{P}(u, z)}{u} \frac{du}{w^z} = O \left( \frac{1}{E_1(z)} \right) = O \left( \frac{z^2 \log z}{\log^2 y} \right).$$

In view of (4.3), Taking

$$z = \frac{y}{\log z} \quad (y > \frac{1}{\log z}),$$

we observe

$$T(x, z) \geq \sum_{l \leq \log y \log \log \log \log y} \frac{1}{l},$$

and therefore

$$\mathcal{P}(x, z) \leq 1 + O\left( e^{-\frac{\log z}{\log y}} \right).$$

Now, from Theorem 2 we obtain by (4.3) and (2.7) the first part of Theorem 3.

Next, since $z > e_{\delta}$, the upper estimate of (4.2) follows from (4.1). Therefore, it is sufficient to prove the corresponding estimate from below.

By (2.2) and (2.3)

$$A_4(M; z) \geq yR_0(z) = A_4(M; e_{\delta}) = yR_0(e_{\delta}) = \sum_{p \leq e_{\delta}} \left( \delta A_4(M; p) - \frac{y}{p} R_0(p) \right).$$

Notice that the condition $H_4(M)$ for $y$ implies $H_4(M)$ for $y/p$, $p \nmid L$, since $(M_p)_y = M_{y^p}$. Hence, using (1.1) at $z = e_{\delta}$ and (4.1) we have

$$A_4(M; z) \leq yR_0(z) \geq \Omega(1) - \frac{y}{p} \sum_{p \leq e_{\delta}} \left( \delta A_4(M; p) - \frac{y}{p} R_0(p) \right) = O\left( e^{-\frac{\log z}{\log y}} \right).$$

Because of (2.7) now the proof of (4.2) has been completed.

5. Basic functions and approximate identities. Let

$$\omega(u) = \begin{cases} 1, & 0 < u < 2, \\ 0, & u \geq 2, \end{cases} \quad g(u) = 1, \quad (u-1)g(u) = -g(u-1), \quad u \geq 2.$$

Both functions have been investigated by several authors. De Bruijn ([4], [5]) proved

$$\omega(u) = e^{u} + O(e^{-u}), \quad g(u) = O(e^{-u}), \quad u \geq 1, \quad u > 0.$$

(*) At the point $u = 2$ the right-hand derivative has to be taken.

(*) For our convenience we have shifted the argument of $g(u)$ by 1.
With these functions we set
\[ F(u) = e^u (\phi(u) + \varphi(u)/u), \quad u > 0, \]
\[ f(u) = e^u (\varphi(u) - \varphi(u)/u), \quad u > 0. \]

By (5.1) and (5.3) we have
\[ F(u) = 2e^u, \quad f(u) = 0, \quad 0 < u \leq 2; \]
\[ \int uF(u) = f(u-1), \quad (uf(u))' = F(u-1), \quad u \geq 2, \]
and hence
\[ \int f(t-1)dt = uF(u) - vF(v), \quad 2 \leq v \leq u. \]

Taking \( v = 2 \), because of (5.5) we obtain
\[ uF(u) = 2F(2) + \int f(t-1)dt = 2e^2, \quad 2 \leq u \leq 3, \]
and hence
\[ F(u) = 2e^u, \quad 0 < u \leq 3. \]

Thus,
\[ uf(u) = 2f(2) + \int f(t-1)dt = 2e^2 \log(u-1), \quad 2 \leq u \leq 4. \]

Also, by (5.3)
\[ F(u) = 1 + O(e^{-u}), \quad f(u) = 1 + O(e^{-u}), \quad u \geq 1, \]
and, by (5.4),
\[ F(u) - f(u) = 2e^u \varphi(u) > 0, \quad u > 0. \]

If \( F(u) = 0 \) were possible, take the smallest value \( u_0 \) of this kind. Then, because of (5.6),
\[ F(u_0) = 0, \quad u_0 > 3, \quad F(u) < 0 \quad \text{for} \quad 0 < u < u_0. \]

However, by (5.6) and (5.11) with suitable numbers \( u_0 \) and \( u_1 \) satisfying
\[ u_0 - 1 < u < u_1, \quad u_1 - 1 < u < u_1, \]
we derive
\[ 0 = uf'(u_0) = f(u_0 - 1) - f(u_0) < f(u_1 - 1) - f(u_1) = -f'(u_1) \]
\[ = \frac{f(u_1 - 1) - F(u_1 - 1)}{u_1} < \frac{F(u_1) - F(u_1 - 1)}{u_1} = \frac{F(u)}{u}, \]
which contradicts (5.12). Hence \( F(u) < 0 \) for \( u > 0 \), and therefore with (5.6) and (5.11) again
\[ uf'(u) = F(u-1) - f(u) > F(u-1) - F(u) > 0, \quad u > 2. \]

Combining these results with (5.10) we get
\[ F(u) \text{ is monotonically decreasing towards 1,} \]
\[ f(u) \text{ is monotonically increasing towards 1.} \]

The following notation will be convenient
\[ g_r(u) = \begin{cases} F(u), & \text{if } v \equiv 0 \mod 2, \quad u > 0; \\ f(u), & \text{if } v \equiv 1 \mod 2, \quad u > 0. \end{cases} \]

**Lemma 5.1.** For \( 2 \leq z_1 \leq z \leq \sqrt{v} \) and both values of \( r \) we have
\[ R_k(z)v \log z \frac{\log y}{\log z} \int_{v/2}^{z/2} \frac{R_k(p)}{p} \frac{\log y}{\log p} + o\left(\frac{R_k(z)\log z\log\log z^k}{\log^2 z_1}\right). \]

**Proof.** By (2.3) and (2.8), if \( z_1 \leq w \leq z \),
\[ R_k(z)v \log z \frac{\log y}{\log z} \int_{v/2}^{z/2} \frac{R_k(p)}{p} \frac{\log y}{\log p} + o\left(\frac{R_k(z)\log z\log\log z^k}{\log^2 z_1}\right). \]

We set
\[ g(w) = g_{s+1}\left(\frac{\log y}{\log w}\right), \quad z_1 \leq w \leq z. \]
Because of \( w < V y \) the argument of \( g, \alpha \) is \( \geq 1 \). Therefore, using (5.13), \( g(w) \) is monotonic, uniformly bounded and continuous in \( s \leq w \leq \alpha \). Hence, by (5.15) and (5.7),

\[
\frac{1}{E(s)} \sum_{p \leq s} \frac{R_k(p)}{p} e^{\frac{\log y}{\log p}} = \sigma(s)g(s) - \int_{s_1}^{s} \sigma(w)dg(w)
\]

\[
= \frac{\log s}{\log s_1} g(s) + O\left( \frac{\log s \log \log 3k}{\log s_1} \right) - \log s \int_{s_1}^{s} \left( \frac{1}{\log s_1} - \frac{1}{\log w} \right) dg(w)
\]

\[
= \log s \frac{g(s)}{\log s_1} \frac{dw}{\log w} + O\left( \frac{\log s \log \log 3k}{\log s_1} \right)
\]

\[
= \log s \frac{\log y}{\log s_1} \frac{\log y}{\log s} \frac{\log y}{\log s_1} + O\left( \frac{\log s \log \log 3k}{\log s_1} \right).
\]

Using (2.8) our lemma follows.

With a similar proof we get

**Lemma 5.2.** For \( 2 \leq s \leq \alpha \leq y \) we have

\[
\sum_{p \leq s} \frac{R_k(p)}{p} e^{\frac{\log y}{\log p}} \leq R_k(s) e^{\frac{\log y}{\log s}} \left( 1 + \frac{\log y \log \log 3k}{\log s_1} \right),
\]

where \( m = \min(s, y^{10}) \).

**Proof.** Using (5.15) as before and setting \( g(w) = e^{\frac{\log y}{\log p}} \), which also is monotonic, continuous, and bounded by \( g(w) \) in \( s_1 \leq w \leq m \), we obtain

(5.16)

\[
\frac{1}{R_k(s)} \sum_{p \leq s} \frac{R_k(p)}{p} e^{\frac{\log y}{\log p}} = \log s \frac{g(s)}{\log s_1} \frac{dw}{\log w} + O\left( \frac{\log y \log \log 3k}{\log s_1} \right).
\]

Improvement of Selberg's view method

Noting that

\[
\log e \frac{1}{\log \alpha} \leq \log e \frac{1}{\log y} \frac{1}{3},
\]

we find that the left-hand side of (5.16) is

\[
\leq e^{\frac{1}{\log y}} \left( 1 + \frac{\log y \log \log 3k}{\log s_1} \right).
\]

**Theorem 4.** Let \( 2 \leq s_1 \leq \alpha \leq V y \) and set

\[
y_j = \frac{y}{p_1 \cdots p_i}, \quad j = 1, 2, \ldots
\]

Then, for all positive integers \( r \) and both values of \( y \), we have

\[
R_k(s) g_j \left( \frac{\log y}{\log s} \right) \leq R_k(s_1) g_j \left( \frac{\log y}{\log s} \right) + \sum_{1 \leq i < r-1} \sum_{1 \leq j \leq p_i \cdots p_i} \frac{R_k(p_i \cdots p_i)}{p_1 \cdots p_i} g_j \left( \frac{\log y}{\log s} \right) + \sum_{r \leq i \leq r} \frac{R_k(p_i \cdots p_i)}{p_1 \cdots p_i} g_j \left( \frac{\log y}{\log s} \right) + O \left( \frac{R_k(s) \log y^2 \log \log 3k}{\log s_1} \right).
\]

This holds independently of condition \( H_s(M) \), and the O-constant is meant to be independent of \( r \).

**Proof.** Apart from the remainder term the proof of Theorem 4 follows the same lines as in the proof of Theorem 1. Here, we start from Lemma 5.1, taking \( r+1 \), \( y/p \), \( p \) for \( r, y, s \), respectively. Regarding the O-term we see by induction that it becomes (with the same O-constant as in Lemma 5.1)

\[
O \left( \frac{\log y \log \log 3k}{\log s_1} \right) \left( R_k(s) g_j \left( \frac{\log y}{\log s} \right) + \sum_{1 \leq i < r-1} \sum_{1 \leq j \leq p_i \cdots p_i} \frac{R_k(p_i \cdots p_i)}{p_1 \cdots p_i} g_j \left( \frac{\log y}{\log s} \right) + O \left( \frac{R_k(s) \log y^2 \log \log 3k}{\log s_1} \right) \right).
\]

Using now (2.6) and Mertens' formula

\[
\sum_{p \leq s} \frac{1}{p} = \log \frac{\log s}{\log s_1} + O \left( \frac{1}{\log s_1} \right),
\]

(5.17)
we obtain for the remainder the estimate
\[
O\left( \frac{R_k(z) \log z \log \log 3k}{\log^2 x} \sum_{\sigma \leq \alpha < 1} \frac{1}{s_1} \left( \sum_{\substack{1 \leq a \leq p \leq x}} \frac{1}{s_2} \sum_{\substack{2 \leq j \leq p \leq x}} \right) \right) = O\left( \frac{R_k(z) \log^2 z \log \log 3k}{\log^2 x} \right).
\]

6. The main theorem.

Theorem 5. If \( y \) satisfies \( E_4(M) \) and \( y \geq \alpha \), we have

\[
A_4(M; z) = yR_k(z) \left( F \left( \frac{\log y}{\log z} \right) + c_1 \left( \frac{\log \log 3k}{\log z} \right)^{1/2} \right),
\]

\[
A_4(M; z) \leq yR_k(z) \left( f \left( \frac{\log y}{\log z} \right) - c_2 \left( \frac{\log \log 3k}{\log z} \right)^{1/2} \right).
\]

This result should be compared with Selberg's in which the functions corresponding to \( F(u) \) and \( f(u) \) were found to be

\[
F_s(u) = \left( 1 - e^{-r} \right) \sum_{n \leq x} \phi(t) dt,
\]

\[
f_s(u) = 1 - \frac{1}{u} \sum_{n \leq x} \left( F_s(t) - 1 \right) dt,
\]

for \( u \geq 1 \) resp. \( u \geq 2 \), see Ankeny and Onishi ([1]). It is possible to show(*) that

\[
F_s(u) > F(u) \quad \text{for} \quad u > 2,
\]

and, therefore,

\[
f_s(u) < f(u) \quad \text{for} \quad u > 2.
\]

Note that our \( f(u) \) is positive exactly for \( u > 2 \).

Proof. If \( y \leq z \leq y \), then by (5.5) and (5.9) our Theorem is true. Also, if \( (\log y)^{\frac{1}{2}} \log \log 3 \) is bounded by some absolute constant because of (4.1), (1.1) and (5.13) the Theorem holds. Therefore, we may suppose that

\[
z \leq y_{\gamma},
\]

and

\[
(6.1) \quad \log y \geq c_4 (\log \log 3k)^{1/2}
\]

where \( c_4 \) is some sufficiently large constant, in particular that \( y \) is greater than a sufficiently large absolute constant.

We shall now apply Theorem 1 and Theorem 4 with

\[
(6.2) \quad z_1 = \exp ((\log y)^{\eta_{10}})
\]

and \( r \) satisfying

\[
(6.3) \quad \frac{1}{3} (\log y)^{\eta_{10}} \leq \frac{3}{2} \leq \frac{3}{4} (\log y)^{\eta_{10}}
\]

If \( z < z_1 \), our Theorem has already been proved because of (4.2) and (5.13), hence the conditions of Theorems 1 and 4 may supposed to be fulfilled. We now form

\[
(-1)^j A_4(M; z) - yR_k(z) \left( \frac{\log y}{\log z} \right)
\]

which has to be estimated from above by

\[
O \left( yR_k(z) \left( \frac{\log \log 3k}{\log y} \right)^{1/2} \right)
\]

according to definition (5.14). Having multiplied in Theorem 4 by \( y \) and in both Theorems by \( (-1)^j \) we form the difference and will estimate each of the terms occurring on the right-hand side.

For the first term and the first sum on the right we apply (4.2). In the proof of Theorem 3 we already stated that \( y_i = y/p_1 \ldots p_r \) satisfies \( E_4(M_{p_1 \ldots p_r}) \). In (4.2) take \( s_i \) for \( s \) and use the corresponding \( M \)'s and \( y_i \)'s. Then, because of (5.10), for the first term we get the remainder

\[
O \left( yR_k(z_1) \frac{1}{\log y} \right).
\]

The application to the sum presupposes

\[
\log z_1 \leq \frac{\log y_i}{2 \log \log 3y_i} \quad \text{for} \quad i = 1, \ldots, r-1,
\]

and contributes, since \( y_i > p_i \geq z_i \),

\[
O \left( \sum_{1 \leq i \leq r-1} y \sum_{p_1 \ldots p_r} \frac{1}{p_1 \ldots p_r \log z_1} \right) \frac{1}{\log z_1} = O \left( yR_k(z_1) \frac{1}{\log z_1} \sum_{1 \leq i \leq r-1} \frac{1}{s_{p_1 \ldots p_r}} \right).
\]

That condition (6.4) is satisfied can be seen as follows. From \( p_j < \sqrt{y_{\gamma}} \) for \( j = 1, \ldots, i \) we first derive, for \( j = 1, p_j < y_{\gamma}^{10} \), and hence

\[
y_i > y_{\gamma}^{10 \eta_{10}}
\]
holds true for \( j = 1 \). If (6.5) had already been proved for \( j, 1 \leq j \leq i-1 \), then \( p_{i+1} < y_{i+1} \) implies \( p_{i+1} < y_{i+1}^{2l} \), and hence
\[
y_{i+1} = \frac{y_i}{p_{i+1}} > y_{i+1}^{2l} > y_{i+1}^{2l+1}.
\]
For a proof of (6.4), therefore, the inequality
\[
\log z_i \leq \frac{(2/3)^{i-1} \log y_i}{2 \log \log 3y}, \quad i = 1, \ldots, r-1,
\]
suffices. It is even enough to check the case \( i = r-1 \). Here, by (6.3), we find
\[
\frac{(2/3)^{r-1} \log y_i}{2 \log \log 3y} = \frac{3 \log y_i}{4 \log \log 3y}, \quad \left(\frac{2/3}{r-1}\right) \leq \left(\frac{\log y_i}{\log \log 3y}\right)^{1/2} = \log z_i.
\]
Next, take the last sum. Here, we have to deal with terms of the form
\[
(1-1)^{r+1} \sum_{s \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} \sum_{p_1 \cdots p_r} \frac{y_i R_k(p_i) g_{s+1}^{\left(\log y_i / \log \log y\right)}}{p_1 \cdots p_r}
\]
where \( y_i R_k(p_i) \leq y_i < y \). If \( r+i \) is odd, by (5.5) \( g_{s+1}^{\left(\log y_i / \log \log y\right)} = 0 \), and hence (6.6) can be estimated from above by zero. If \( r+i \) is even, then by (3.9), taking \( y_i \) and \( p_i \) for \( y \) and \( z \), respectively, and by (5.3), (2.6) we get the upper estimate
\[
O\left(\sum_{s_1 \leq c_1} \sum_{p_2} \sum_{p_1} \frac{y_i R_k(p_i) g_{s_1}^{\log y_i / \log \log y}}{p_1 \cdots p_r}ight) = O\left(y R_k(z) \frac{\log \log y_i}{\log z_i} \sum_{s_1 \leq c_1} \sum_{p_2} \frac{1}{p_1} \left(\sum_{p_1} \frac{1}{p_1}\right)^i\right).
\]
We still have the error term of Theorem 4, which is
\[
O\left(y R_k(z) \frac{\log \log y_i}{\log z_i} \log\log 3k / \log z_i\right).
\]
So far, the remainder terms arising from all but the second sums in our identities have been
\[
O\left(y R_k(z) \frac{R_k(z)}{\log y_i} + \frac{R_k(z)}{\log z_i} \exp \left(\sum_{s_1 \leq c_1} \sum_{p_2} \frac{1}{p_1} \left(\sum_{p_1} \frac{1}{p_1}\right)^i\right) \right.
\]
\[
+ \log \log y_i \exp \left(\sum_{s_1 \leq c_1} \sum_{p_2} \frac{1}{p_1} \left(\sum_{p_1} \frac{1}{p_1}\right)^i\right) \log^2 \log 3k / \log z_i.
\]
Using (2.6) and (5.17) we see that they are
\[
O\left(\frac{y R_k(z)}{\log y_i} \frac{\log^2 y_i}{\log z_i} + \frac{\log^2 y_i}{\log z_i} \log \log 3k / \log z_i\right),
\]
and hence, by (6.2),
\[
O\left(\frac{y R_k(z) \log \log y_i + \log \log 3k}{\log y_i}\right),
\]
\[\text{i.e. of sufficiently small order.}\]
It remains to deal with the second sum. Let its factor \((-1)^{r+1}\) be chosen to be \(+1\) such that (4.1) can be applied (the \(r\)-interval in (6.3) always contains both an even and an odd number). Then this \(O\)-term becomes, because of (5.10), apart from an \(O\)-constant
\[
\sum_{s_1 \leq c_1} \sum_{p_2} \sum_{p_1} \frac{R_k(p_i) \log y_i}{p_1 \cdots p_r}
\]
The sum over \( p_r \) is
\[
U_r = \sum_{s_1 \leq c_1} \sum_{p_2} \frac{R_k(p_i) \log y_i}{p_1 \cdots p_r}, \quad m_r = \min\{p_{r-1}, y_1^{2l}\}
\]
For this sum we get, using Lemma 5.3, the estimate
\[
U_r \leq R_k(p_{r-1}) \frac{\log y_i}{\log y_i}, \quad \theta = \frac{1 + c_2 (\log y_i) / \log 3k}{3}
\]
Introducing this in (6.7) we see that apart from the factor \( \theta \) we get the same sum with \( r-1 \) instead of \( r \) as an upper estimate. Thus, we can repeat this procedure, using \( p_0 = s, y_0 = y \) in the last step. Then we get in total
\[
\sum_{s_1 \leq c_1} \sum_{p_2} \frac{R_k(p_i) \log y_i}{p_1 \cdots p_r} \leq y R_k(z) \frac{\log y_i}{\log \log 3k}.
\]
Now, it finally remains to show that this error term also is of sufficiently small order. However,
$$\theta = O\left(\frac{1}{(\log y)^{1/2}}\right)$$
follows from (6.3), since by (6.1) \(\theta\) can be brought sufficiently close to \(\varepsilon/3\).

7. Applications. The most interesting and effective application of the linear sieve is to a set consisting of numbers in a short interval belonging to an arithmetic progression. In this case we define instead of \(A_k(M;z)\) the following special function.

Let \(k\) and \(l\) be positive integers, \((k, l) = 1, 1 \leq l \leq k\). Let \(x, h, z\) be real numbers
$$k < h \leq x, \quad z \geq 2.$$

Then we are interested in
$$B_{x,h}(x, h, z) = \left\{x - kh < n \leq x; \: n = \lambda x + \lambda \mod k, \: \lambda \neq 0 \mod \lambda \mod \lambda \mod \lambda < z\right\}$$
(cf. N. I. Klinov [9]). This number is not changed by adding the condition \(p \lambda k\), since no number \(\lambda x + \lambda \mod k\) can be divisible by a prime divisor of \(k\), because of \((k, l) = 1\). Therefore, \(B_{x,h}(x, h, z) = A_k(M;z)\), where
$$M = \{x - k < n \leq x; \: n = \lambda x + \lambda \mod k\}.$$

If \((d, k) = 1\), with a suitable \(\lambda_k\), we get
$$|M| = \frac{x}{d} - \frac{1}{d}$$
and hence \(y = h/k\) satisfies \(\lambda_k(M)\).

Therefore, by Theorem 5

(7.1) \[ f(x) - a_k \log \log k \frac{1}{\nu} \leq B_{x,kh}(x, kh, h^{1/\nu}) \leq f(x) + a_k \log \log k \frac{1}{\nu}, \]

\[ u \geq 1, \quad 2^u \leq h \leq \frac{x}{k}, \]

and by (3.9), using (5.5),

(7.2) \[ B_{x,kh}(x, kh, h^{1/\nu}) \leq f(x) + a_k \log \log k \frac{1}{\nu} \log h, \quad 0 < u < 1, \quad 2^u \leq h \leq \frac{x}{k}. \]

The upper estimate in (7.1) improves Klinov’s result ([9]).

Let \(a\) be a non-negative integer and
$$x \geq h \geq \zeta \geq z \geq 2.$$
Setting
\[ g(w) = \frac{\log(h|w|)}{\log h}, \quad h^{1-u} \leq w \leq h^{1+u}, \]
we get by (7.5) and (5.13)
\[
\sum_{h^{1-u}<w<h^{1+u}} \frac{1}{p} F\left(\frac{\log(h|p|)}{\log h}\right) = \sum_{h^{1-u}<w<h^{1+u}} \frac{1}{p} g(p)
\]
\[
= \left(\log \frac{v}{u} + O\left(\frac{v}{\log h}\right)\right) g(h^{1+u}) - \int_{h^{1+u}}^{h^{1+u}} \tau(w) dg(w)
\]
\[
= \log \frac{v}{u} g(h^{1+u}) + O\left(\frac{v}{\log h} g(h^{1+u})\right) - \int_{h^{1+u}}^{h^{1+u}} \log \left(\frac{v}{\log h}\right) dg(w)
\]
\[
= \int_{h^{1+u}}^{h^{1+u}} \frac{g(w) \, dw}{\log w} + g(h^{1+u}) = \int_{h^{1+u}}^{h^{1+u}} \frac{w \, dw}{\log w} + O\left(\frac{\log \log h}{\log h}\right)
\]

Hence, it follows from (7.4), (2.7) and (7.5)

(7.6) \[
C_{\mathbb{Z}}^2(x, \mathbb{H}, h^{1+u}, h^{1+u}) \geq f(v) - \frac{1}{a+1} \left(\frac{v}{a} \right) \int_{h^{1+u}}^{h^{1+u}} \frac{\log\log h}{h^{1+u}}
\]

\[ 1 < u \leq v, \quad 2v \leq h \leq x/k. \]

**Theorem 6.** Let \( r \geq 2 \) be an integer and

\[ A_r = \sup_{u, v} \left(\frac{r+1-a}{u} + \frac{a}{v}\right), \]

where the sup is taken over all real numbers \( u, v \) such that

(7.7) \[
1 < u \leq v,
\]

(7.8) \[
f(v) = \frac{1}{a+1} \int_{h^{1+u}}^{h^{1+u}} \frac{v \, dt}{t} > 0,
\]

and over all non-negative integers \( a \).

Then, for every \( \epsilon > 0 \), there is a number \( a_0 = a_0(r, \epsilon) \) depending on \( r \) and \( \epsilon \) only such that for \( x \geq a_0, x^{1-\frac{a}{a_0}} \geq h \) there exist at least two integers \( n \) satisfying

(7.9) \[
\frac{1}{x} < h^{1+u} < x \leq \frac{x}{k^{1+u}}.
\]

(7.10) \[ n \equiv 1 \mod h, \quad \Omega(n) \leq r, \]

\( \Omega(n) \) denotes the total number (with multiplicities) of prime divisors of \( n \).

**Proof.** For information we remark that according to (7.7)

\[
\frac{r+1-a}{u} + \frac{a}{v} \leq \frac{r+1}{u} < r+1
\]

whereas by taking \( v = u = 2+\delta, 0 < \delta < 1 \), and by (5.9)

\[ A_r \geq (r+1)/(2+\delta), \]

thus proving

\[ (r+1)/2 \leq A_r \leq r+1. \]

We may suppose \( \epsilon \) to be sufficiently small and \( a_0(r, \epsilon) \) large enough. Choosing \( u = u(r, \epsilon), v = v(r, \epsilon), a = a(r, \epsilon) \) satisfying (7.7), (7.8) and

(7.11) \[
\frac{1}{A_r + \epsilon} < \frac{r+1-a}{u} + \frac{a}{v}
\]

and putting

\[ k = x^{1/\epsilon}, \]

we find that the conditions of (7.6) are fulfilled, and therefore

\[ C_{\mathbb{Z}}^2(x, \mathbb{H}, h^{1+u}, h^{1+u}) \geq 2, \]

i.e. there are at least two numbers \( n \) satisfying both (7.4) and (7.10). Let \( k \) denote the number of prime divisors of \( n \) being

\[ < h^{1+u}, \]

then we have because of \( u \leq v \) and \( b < a \) (cf. (5.3))

\[
\frac{a}{b} \leq \frac{a}{n} \leq \frac{b}{n} \leq x = k^{1+u},
\]

\[ \Omega(n) \leq a + u \left(\frac{1}{1/A_r + \epsilon} - \frac{a}{v}\right) < r+1, \]

by (7.11).

The most important special cases of Theorem 6 are obtained by choosing \( k = 1 \) resp. \( x = k^{1/\epsilon} \).

**Theorem 7.** Let \( r \geq 2 \). Then, for \( \epsilon > 0 \) and \( x \equiv a_0(r, \epsilon) \) there exist at least two integers \( n \) in the interval

\[ x-a^{1-\frac{a}{a_0}} \leq n \leq x \]

having at most \( r \) prime factors.

The best previous results in this direction were given by P. Kuhn (III0) and particularly by Y. Wang ([18], [19]). According to the later evaluation of \( A_r \) our exponent is better for every \( r \).
THEOREM 8. Let $r \geq 2$. Then, for $\varepsilon > 0$ and $k \geq k_0(r, \varepsilon)$ there exists an integer

$$n = \lfloor \text{mod } k \rfloor, \quad 1 \leq n \leq k^{1/\varepsilon(r+\varepsilon)},$$

having at most $r$ prime factors.

For applications it remains to determine $A_r$ numerically. Define for $a \geq 4$ and integral $a \geq 0$

$$I(v) = \int_{v}^{\infty} \left( \frac{v}{a-1} \right)^{1/2} \frac{2^{\varepsilon v} \log v - 3}{v^{3/2}} \, dv,$$

$$w = w_a(v) = 1 + \exp \left( \frac{v}{2^{\varepsilon a} (a+1)f(v) - I(v)} \right),$$

and assume that

$$(7.12) \quad w_a(v) \leq \frac{v}{v-3} \quad (\leq e).$$

Then, by (5.8),

$$\left( a+1 \right) f(v) = \int_{v}^{\infty} \left( \frac{v}{a-1} \right)^{1/2} \frac{2^{\varepsilon v} \log v - 3}{v^{3/2}} \, dv = \left( a+1 \right) f(v) - I(v) + \frac{2^{\varepsilon v} \log v - 1}{v} = 0.$$

Thus, for sufficiently small $\varepsilon > 0$, putting $u = v + \varepsilon$ with $v$ and $a$ chosen as above, the conditions (7.1) and (7.8) are satisfied. Hence, making $\varepsilon \to 0$, we obtain

$$(7.13) \quad A_r \sim \frac{r+1-a}{a} \frac{a}{v},$$

provided that $r \geq 4$ and (7.12) hold. For suitable choices of $v$ and $a$ we obtain numerical approximations to $A_r$.

As $r \to \infty$ we choose

$$v = \log r, \quad a = [2 \log v],$$

and note that in view of (5.10)

$$f(v) = 1 + o(1), \quad I(v) = \log v + o(\log v),$$

$$w_a(v) \leq 1 + e^{-v} < 1 + \frac{v}{v-3},$$

and hence

$$A_r \geq r - c_0 \log \log r.$$

Thus, $A_r$ is asymptotically equal to $r$ as $r \to \infty$.

For all integers $r \geq 2$ we may put $\varepsilon = 6$. By numerical calculations we find

$$f(6) \geq 0.9999805, \quad I(6) \leq 1.133056,$$

and, by using three places only,

$$w_6(6) = \frac{21}{17}, \quad w_6(6) = \frac{24}{23}.$$

Hence, (7.13) implies both

$$(7.14) \quad A_r \geq \frac{34r+1}{42}, \quad A_r \geq \frac{23r-15}{24} \quad (r \geq 2),$$

thus superseding all previous results for relatively small values of $r$.

(For larger values of $r$ we choose a larger also.)

Taking $r = 2$ in (7.14) gives $A_2 \geq 25$ thus proving Theorems 9 and 10 given in the introduction.

References


Sur un théorème de Rényi
par
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1. Introduction. Désignons par $\omega(n)$ le nombre des diviseurs premiers de l'entier positif $n$ et par $\Omega(n)$ le nombre total des facteurs dans la décomposition de $n$ en facteurs premiers. Autrement dit, soient $\omega$ et $\Omega$ les fonctions de l'entier positif $n$ définies de la façon suivante:

$$\omega(1) = 1 = \Omega(1) = 0$$

et, si $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, où $p_1, p_2, \ldots, p_k$ sont des nombres premiers distincts et $a_1, a_2, \ldots, a_k$ des entiers $> 0$,

$$\omega(n) = k \quad \text{et} \quad \Omega(n) = a_1 + a_2 + \cdots + a_k.$$

Il est clair que l'on a toujours $\Omega(n) \geq \omega(n)$, l'égalité ayant lieu pour les entiers "quadratfrei".

Rényi a montré ([4]) que, pour chaque entier $q \geq 0$, l'ensemble des $n$ pour lesquels on a $\Omega(n) - \omega(n) = q$ possède une densité $d_q$, la suite des nombres $d_q$ étant déterminée par le fait que, pour $|x| < 2$,

$$\sum_{y=0}^{+\infty} d_q x^y = \prod_{p \geq 2} \left(1 - \frac{1}{p}\right)^{1 + \frac{1}{p - 1}} \frac{6}{\pi^2} \prod_{p \geq 2} \frac{1 - x/(p+1)}{1 - x/p},$$

où $p$ parcourt la suite des nombres premiers ($^1$).

Pour $q = 0$, on retrouve le fait bien connu que l'ensemble des entiers positif "quadratfrei" possède une densité égale à $6/\pi^2$.

En ce qui concerne les entiers "quadratfrei", Landau a montré ([3]) que le théorème des nombres premiers, sous la forme $\omega(n) \sim \log n$, entraîne le résultat suivant:

Si $Q(x)$ est le nombre de ces entiers au plus égaux à $x$, on a pour $x$ infini:

$$Q(x) = \frac{6}{\pi^2} x^2 + o(x^2).$$

($^1$) Tout au long de cet article, dans toute somme ou tout produit portant sur une expression à figure $p$, il est entendu que $p$ parcourt la suite des nombres premiers.

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