Wegen (8) besteht

(11) $$a_{p_2}^{(i_2)}+a_{m_1}^{(i_2)}+a_{m_2}^{(i_2)}+\ldots+a_{m_z}^{(i_2)} \geqslant a_{m_1}^{(i_2)} \geqslant a_{[n/2]+1}.$$

Aus (9), (10) und (11) folgt, daß

$$a_{[n/2]} \geqslant a_{[n/2]+1},$$

also sind wir aus (4) zu einem Widerpsruch gelangt, und damit ist der Satz bewiesen.

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On primes in arithmetic progressions

by

J. H. van Lint (Eindhoven) and H.-E. Richert (Marburg)

As usual, for a real number x and coprime positive integers k and l we denote by $\pi(x, k, l)$ the number of primes $p \leqslant x$ for which $p \equiv l \bmod k$ The aim of this paper is to prove the following theorem of the Brun-Titchmarsh type.

Theorem 1. *If x and y are real numbers, k and l integers satisfying*

$$1 \leqslant k < y \leqslant x, \quad (k, l) = 1,$$

then

$$\pi(x, k, l)-\pi(x-y, k, l) < \frac{y}{\varphi(k)\log\sqrt{y/k}}\left(1+\frac{4}{\log\sqrt{y/k})}\right).$$

The only conditions we impose on the parameters y, k and l are the natural ones. In particular, we do not need the assumption $k = O(x^\delta)$ with $\delta < 1$ frequently used in this connection, although our estimates contain explicit numerical constants. Note however, that even if we had replaced the term $4/(\log\sqrt{y/k})$ in Theorem 1 by a corresponding O-term the result would have been superior to the best estimate hitherto known in this direction (Klimov [1], p. 182, where $2\log\log y$ occurs instead of the above constant 4). These improvements have been made possible mainly by a more careful treatment of the remainder term in the Selberg sieve. In our proofs we are more concerned with a convenient presentation than obtaining sharp estimates for the constants throughout the paper, for example the constant 4 in the remainder term of Theorem 1 can be replaced by 3 by a more refined argument.

We shall also prove

Theorem 2. *If x and y are real numbers, k and l integers satisfying*

$$1 \leqslant k < y \leqslant x, \quad (k, l) = 1,$$

then

$$\pi(x, k, l)-\pi(x-y, k, l) < \frac{3y}{\varphi(k)\log(y/k)}.$$

If we take $y = x$ in the above theorems we obtain the following estimates for $\pi(x, k, l)$.

Corollary 1. *For* $1 \leqslant k < x$, $(k, l) = 1$ *we have*

$$\pi(x, k, l) < \frac{x}{\varphi(k)\log\sqrt{x/k}}\left(1+\frac{4}{\log\sqrt{x/k}}\right).$$

Corollary 2. *For* $1 \leqslant k < x$, $(k, l) = 1$ *we have*

$$\pi(x, k, l) < \frac{3x}{\varphi(k)\log(x/k)},$$

(cf. Prachar [2], p. 44)

1. Defining(1)

$$H_k(x) = \sum_{\substack{1\leqslant n\leqslant x\\ (n,k)=1}} \mu^2(n)\frac{\sigma(n)}{\varphi(n)}, \qquad S_k(x) = \sum_{\substack{1\leqslant n\leqslant x\\ (n,k)=1}} \frac{\mu^2(n)}{\varphi(n)}, \qquad x > 0,$$

we have

$$\text{(1.1)}\qquad S_k(x) = \sum_{d|K}\sum_{\substack{1\leqslant n\leqslant x\\ (n,k)=1\\ (n,K)=d}} \frac{\mu^2(n)}{\varphi(n)} = \sum_{d|K}\frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{1\leqslant m\leqslant x/d\\ (m,k)=(m,K/d)=(m,d)=1}} \frac{\mu^2(m)}{\varphi(m)}$$

$$= \sum_{d|K}\frac{\mu^2(d)}{\varphi(d)} S_{Kk}\left(\frac{x}{d}\right) \quad \text{for } (K, k) = 1.$$

Hence

$$S_k(x) \geqslant \sum_{d|K}\frac{\mu^2(d)}{\varphi(d)} S_{Kk}\left(\frac{x}{K}\right) = \frac{K}{\varphi(K)} S_{Kk}\left(\frac{x}{K}\right), \qquad (K, k) = 1,$$

and therefore

$$\text{(1.2)}\qquad H_k(x) = \sum_{\substack{1\leqslant K\leqslant x\\ (K,k)=1}} \mu^2(K)\frac{K}{\varphi(K)} \sum_{\substack{1\leqslant m\leqslant x/K\\ (m,k)=(m,K)=1}} \frac{\mu^2(m)}{\varphi(m)}$$

$$= \sum_{\substack{1\leqslant K\leqslant x\\ (K,k)=1}} \mu^2(K)\frac{K}{\varphi(K)} S_{Kk}\left(\frac{x}{K}\right) \leqslant S_k(x) \sum_{\substack{1\leqslant K\leqslant x\\ (K,k)=1}} \mu^2(K).$$

Taking $k = 1$ in (1.1) we find

$$\text{(1.3)}\qquad S_1(x) \leqslant \sum_{d|K}\frac{\mu^2(d)}{\varphi(d)} S_K(x) = \frac{K}{\varphi(K)} S_K(x),$$

(1) An empty sum shall be zero, an empty product one.

and denoting by $q(n)$ the greatest squarefree divisor of n

$$\text{(1.4)}\qquad S_1(x) = \sum_{1\leqslant n\leqslant x}\frac{\mu^2(n)}{n}\prod_{p|n}\left(1-\frac{1}{p}\right)^{-1} = \sum_{1\leqslant q(n)\leqslant x}\frac{1}{n} \geqslant \sum_{1\leqslant n\leqslant x}\frac{1}{n} \geqslant \log x.$$

From these inequalities we infer

$$\text{(1.5)}\qquad S_K(x) \geqslant \frac{\varphi(K)}{K}\log x.$$

2. Let(2)

$$A_{k,l}(x, y, z, K) = \left|\left\{x-y < n \leqslant x,\ n \equiv l \bmod k,\ \left(n, P_K(z)\right) = 1\right\}\right|,$$
$$1 < y \leqslant x, \quad z > 1, \quad (k, l) = 1, \quad k|K,$$

where

$$P_K(z) = \prod_{\substack{p\leqslant z\\ p\nmid K}} p.$$

Lemma 1. *We have*

$$A_{k,l}(x, y, z, K) \leqslant \frac{y}{kS_K(z)} + \frac{H_K^2(z)}{S_K^2(z)}.$$

Proof. We put

$$\lambda_d = \mu(d)\frac{d}{\varphi(d)}\cdot\frac{S_{Kd}(z/d)}{S_K(z)},$$

i.e.

$$\text{(2.1)}\qquad \lambda_d = 0 \quad \text{for} \quad d > z.$$

For $1 \leqslant t \leqslant z$, $t|P_K$, where $P_K = P_K(z)$, we find

$$\text{(2.2)}\qquad \sum_{\substack{d|P_K\\ d\equiv 0 \bmod t}}\frac{\lambda_d}{d} = \sum_{\substack{m|P_K\\ (m,t)=1}}\frac{\mu(tm)}{\varphi(tm)}\cdot\frac{S_{Ktm}(z/tm)}{S_K(z)}$$

$$= \frac{\mu(t)}{\varphi(t)S_K(z)}\sum_{\substack{m|P_K\\ (m,t)=1}}\frac{\mu(m)}{\varphi(m)}\sum_{\substack{1\leqslant S\leqslant z/tm\\ (s,Ktm)=1}}\frac{\mu^2(s)}{\varphi(s)}$$

$$= \frac{\mu(t)}{\varphi(t)S_K(z)}\sum_{\substack{1\leqslant sm\leqslant z/t\\ (sm,Kt)=1}}\frac{\mu^2(sm)}{\varphi(sm)}\sum_{m|sm}\mu(m) = \frac{\mu(t)}{\varphi(t)S_K(z)},$$

(2) By $|M|$ we denote the number of elements of the set M.

and because of (2.1)

$$\sum_{d|P_K}|\lambda_d| = \sum_{\substack{1\leqslant d\leqslant Z\\(d,K)=1}}\mu^2(d)\frac{d}{\varphi(d)}\cdot\frac{1}{S_K(z)}\sum_{\substack{1\leqslant m\leqslant z/d\\(m,Kd)=1}}\frac{\mu^2(m)}{\varphi(m)}$$

$$= \frac{1}{S_K(z)}\sum_{\substack{1\leqslant n\leqslant z\\(n,K)=1}}\frac{\mu^2(n)}{\varphi(n)}\sum_{d|n}d = \frac{H_K(z)}{S_K(z)}.$$

Since $\lambda_1 = 1$, we obtain

$$A_{k,l}(x,y,z,K) \leqslant \sum_{\substack{x-y<n\leqslant x\\n\equiv l \bmod k}}\Big(\sum_{\substack{d|n\\d|P_K}}\lambda_d\Big)^2 = \sum_{\substack{d_\nu|P_K\\\nu=1,2}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{x-y<n\leqslant x\\n\equiv l \bmod k\\n\equiv 0 \bmod [d_1,d_2]}}1$$

$$\leqslant \frac{y}{k}\sum_{\substack{d_\nu|P_K\\\nu=1,2}}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}+\sum_{\substack{d_\nu|P_K\\\nu=1,2}}|\lambda_{d_1}\lambda_{d_2}|$$

$$= \frac{y}{k}\sum_{\substack{d_\nu|P_K\\\nu=1,2}}\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\sum_{\substack{t|d_1\\t|d_2}}\varphi(t)+\Big(\sum_{d|P_K}|\lambda_d|\Big)^2$$

$$\leqslant \frac{y}{k}\sum_{\substack{t|P_K\\1\leqslant t\leqslant z}}\varphi(t)\Big(\sum_{\substack{d|P_K\\d\equiv 0 \bmod t}}\frac{\lambda_d}{d}\Big)^2+\frac{H_K^2(z)}{S_K^2(z)}.$$

Using (2.2) we see that the first term equals $y/kS_K(z)$.

Let $p(k)$ denote the greatest prime divisor of k, $p(1) = 1$.

LEMMA 2. *For $x \geqslant 10^3$, $p(k) \leqslant x$ we have*(³)

$$T_k(x) = \sum_{\substack{1\leqslant n\leqslant x\\(n,k)=1}}1 < \frac{15}{2}\cdot\frac{\varphi(k)}{k}x.$$

Proof. Let $z > 1$,

$$k_1 = (k, P_1(z)),$$

and

$$K = P_{k_1}(z),$$

i.e.

$$k_1 = P_K(z).$$

Then,

$$T_{k_1}(x) = A_{1,1}(x,x,z,K).$$

(³) Here, we can prove that, for all values of x, $\frac{15}{2}$ can be replaced by 5.

Therefore, since $k_1|k$, by Lemma 1, (1.5) and (1.2)

$$T_k(x) \leqslant T_{k_1}(x) \leqslant \frac{x}{\prod_{\substack{p\nmid k\\p\leqslant z}}(1-1/p)\log z}+z^2,$$

and since $p(k) \leqslant x$

$$\frac{k}{\varphi(k)}\cdot\frac{T_k(x)}{x} \leqslant \frac{1}{\prod_{p\leqslant x}(1-1/p)}\Big(\frac{1}{\log z}+\frac{z^2}{x}\Big).$$

We remark that (cf. [3], (3.31))

$$\prod_{p\leqslant x}\Big(1-\frac{1}{p}\Big)^{-1} \leqslant e^\gamma\log(2x),$$

where γ is Euler's constant. We now take $z = (2x)^{1/3}$, and obtain

$$\frac{k}{\varphi(k)}\cdot\frac{T_k(x)}{x} \leqslant e^\gamma\Big(3+2\frac{\log(2x)}{(2x)^{1/3}}\Big).$$

Here, the right-hand side is decreasing for $2x > e^3$, and for $x = 10^3$ it is $< \frac{15}{2}$.

LEMMA 3. *For $z \geqslant 10^3$, and h even*(⁴)

$$\frac{H_h^2(z)}{S_h^2(z)} < 24\frac{h}{\varphi(h)}\cdot\frac{z^2}{\log^2 z}.$$

Proof. By Cauchy's inequality

$$H_h^2(z) \leqslant T_h(z)J_h(z),$$

where

$$J_h(z) = \sum_{\substack{1\leqslant n\leqslant z\\(n,h)=1}}\mu^2(n)\frac{\sigma^2(n)}{\varphi^2(n)}.$$

Since $2|h$ and

$$\frac{\sigma^2(n)}{\varphi^2(n)} = \sum_{d|n}\frac{4^{\nu(d)}d}{\varphi^2(d)} \quad \text{for} \quad \mu(n) \neq 0,$$

(⁴) This lemma is true without any restriction on z and h. We can also show that the estimate holds with an absolute constant instead of $h/\varphi(h)$.

we have[5]

$$J_h(z) \leqslant J_2(z) = \sum_{\substack{1\leqslant n\leqslant z\\(n,2)=1}} \mu^2(n) \sum_{d|n} \frac{4^{\nu(d)}d}{\varphi^2(d)}$$

$$= \sum_{\substack{1\leqslant d\leqslant z\\(d,2)=1}} \mu^2(d)\frac{4^{\nu(d)}d}{\varphi^2(d)} \sum_{\substack{1\leqslant m\leqslant z/d\\(m,2d)=1}} \mu^2(m) \leqslant z\prod_{p>2}\left(1+\frac{4}{(p-1)^2}\right) < \frac{16}{5}z.$$

Therefore, by Lemma 2 and (1.5)

$$\frac{H_h^2(z)}{S_h^2(z)} \leqslant \frac{\frac{15}{2}\cdot\frac{\varphi(h)}{h}z\frac{16}{5}z}{\frac{\varphi^2(h)}{h^2}\log^2 z} = 24\frac{h}{\varphi(h)}\cdot\frac{z^2}{\log^2 z}.$$

Here, we required $p(h) \leqslant z$. However, since both $H_h(z)$ and $S_h(z)$ are independent of the prime factors of h which are $> z$ while the right-hand side is increased by these prime factors, this condition is no longer necessary.

3. Proof of Theorem 1. Let

$$\Delta(x, y, k, l) = \pi(x, k, l)-\pi(x-y, k, l)$$

and

$$h = 2k/(2, k).$$

Then, with a suitable number l_1, we have

$$\Delta(x, y, k, l) \leqslant \Delta(x, y, h, l_1)+1,$$

since for odd values of k the numbers $mk+l$ are alternately even, and at most one of the even terms is prime.

By definition, $A_{h,l_1}(x, y, z, h)$ counts at least the primes p in the interval $x-y < n \leqslant x$ which satisfy $p > z$ an $p \equiv l_1 \bmod h$. Hence, using Lemma 1 and (1.3), we find

$$(3.1)\qquad \Delta(x, y, k, l) \leqslant \frac{y}{\varphi(k)S_1(z)}+\frac{H_h^2(z)}{S_h^2(z)}+\pi(z, h, l_1)+1 \quad \text{for any } z > 1.$$

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(5) Here, we use $\prod_p\left(1+\frac{4}{(p-1)^2}\right) = 15.9396\ldots$ Actually, $J_2(z) < \frac{5}{3}z$ for all values of z.

We put

$$u = \sqrt{y/k}.$$

As $2|h$, we have by (1.4) and Lemma 3 for $z \geqslant 10^3$

$$(3.2)\qquad \log\sqrt{\frac{y}{k}}\left\{\frac{\varphi(k)\log\sqrt{y/k}}{y}\Delta(x, y, k, l)-1\right\}$$
$$\leqslant \log u\left\{\frac{\log u}{\log z}-1+48\frac{\log u}{u^2}\cdot\frac{z^2}{\log^2 z}+\frac{\log u}{u^2}z\right\}.$$

Choosing

$$\log z = \log u-2$$

the right-hand side of (3.2) becomes

$$\log u\left\{\frac{2}{\log u-2}+\frac{48}{e^4}\cdot\frac{\log u}{(\log u-2)^2}+\frac{\log u}{e^2 u}\right\}.$$

This function is decreasing in u, and for $u = e^{10}$ it is < 4 which proves Theorem 1 for $u \geqslant e^{10}$. The proof for the remaining values of u will be given in section 4 after the proof of Theorem 2.

4. Proof of Theorem 2. Taking $z = 2$ in (3.1) we obtain

$$Q = \frac{\varphi(k)\log\sqrt{y/k}}{y}\Delta(x, y, k, l) \leqslant \log u\left(\frac{1}{2}+\frac{2}{u^2}\right) < \frac{3}{2} \quad \text{for} \quad 1 < u \leqslant e^{2.9}.$$

Next, note that

$$\pi(z, h, l_1)+1 \leqslant \sum_{\substack{1\leqslant K\leqslant z\\(K,2)\leqslant 1}} \mu^2(K) \leqslant \frac{z-1}{2} \quad \text{for} \quad z \geqslant 10.$$

Hence, by (3.1), (1.4) and (1.2)

$$Q \leqslant \log u\left\{\frac{1}{\log z}+\frac{z^2}{4u^2}\right\} \quad \text{for} \quad z \geqslant 10.$$

Defining ω by

$$u = \frac{\omega}{\sqrt{2}}e^{\omega}$$

we choose

$$z = e^{\omega}.$$

Then,

$$Q \leqslant \frac{\log(\omega/\sqrt{2})+\omega}{\omega}\left\{1+\frac{1}{2\omega}\right\} \quad \text{for} \quad \omega \geqslant \log 10. \tag{4.1}$$

For $\omega \geqslant \sqrt{2e}$ ($>\log 10$) this function is decreasing, and for $\omega = \sqrt{2e}$ it is $<\frac{3}{2}$. This proves Theorem 2.

It remains to prove Theorem 1 for $1<u<e^{10}$. Obviously, for $u \leqslant e^8$, Theorem 1 is a consequence of Theorem 2. For $u > e^8$ however, the corresponding ω being >6.4, (4.1) gives for $u<e^{10}$

$$Q < 1.4 < 1+\frac{4}{\log u}.$$

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An improvement of Selberg's sieve method I

by

W. B. Jurkat and H.-E. Richert (Syracuse, New York)

The sieve method is concerned with counting in a finite set M of positive integers those elements which are not divisible by any prime $p<z$ or rather with estimating the number $A(M;z)$ of these elements from above and from below. If z is sufficiently large compared with the largest element in M the numbers counted in $A(M;z)$ will have a restricted number of prime divisors, and it is this fact that makes estimates of $A(M;z)$ interesting. The estimates, however, depend essentially upon only the number of elements in M, say $|M|$.

For relatively small values of z the classical sieve of Eratosthenes-Legendre is quite satisfactory (§ 1). However, the interest lies in larger values of z which were first treated successfully by Viggo Brun. Later, Buchstab was able to give improved estimates, starting from initial estimates, by using identities of Meissel's type (cf. (2.2)). For an upper estimate the sieve method was formulated generally and this estimate minimized by A. Selberg. He also gave a lower estimate which can be derived from his upper estimate by using Buchstab's method (cf. Ankeny-Onishi [1]). In applications a combinatorial argument of Kuhn led to further improvements. In a series of papers Y. Wang very successfully combined all methods mentioned above.

We propose in this paper a method (based on Theorems 1 and 4) which leads to a new two-sided estimate (Theorem 5) by employing only the simplest upper estimate of Selberg for $z^2>|M|$ (cf. Corollary to Theorem 2) and extensions of the classical estimate (Theorem 3).

The new estimate improves Selberg's upper and lower estimate for $z^2<|M|$, and cannot be further improved upon by Buchstab's method. In particular, the new lower estimate is positive already for $z^{2+\varepsilon}<|M|$ (with arbitrary $\varepsilon>0$ and large z), which decides a question left open by A. Selberg ([15], p. 292). Furthermore, our estimate is completely uniform for all sets of a certain regular behaviour (condition $\mathrm{H}_k(M)$). This condition restricts, however, our present presentation to what might be called the linear sieve. Exceptional primes are allowed which we com-