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UNIVERSITY OF ARIZONA, TUCSON, ARIZONA
ST. JOHN'S COLLEGE, CAMBRIDGE, ENGLAND

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On a problem of Sierpiński

(Extract from a letter to W. Sierpiński)

by

P. Erdös (Budapest)

Denote by μ_s the least integer so that every integer $> u_s$ is the sum of exactly s integers >1 which are pairwise relatively prime. Sierpiński ([3]) proved that $u_2 = 6$, $u_3 = 17$ and $u_4 = 30$ and he asks for a determination or estimation of u_s. Denote by $f_1(s)$ the smallest integer so that every $l > f_1(s)$ is the sum of s distinct primes; $f_2(s)$ is the smallest integer so that every $l > f_2(s)$ is the sum of s distinct primes or squares of primes where a prime and its square are not both used and $f_3(s)$ is the least integer so that every $l > f_3(s)$ is the sum of s distinct integers > 1 which are pairwise relatively prime. By definition $f_3(s) = u_s$. Clearly

$$f_3(s) \leqslant f_2(s) \leqslant f_1(s).$$

Let $p_1 = 2$, $p_2 = 3, \ldots$ be the sequence of consecutive primes. Put

$$A(s) = \sum_{i=1}^{s} p_i, \quad B(s) = \sum_{i=2}^{s+1} p_i.$$

Theorem. *$f_2(s) < B(s)+C$ where C is an absolute constant independent of s.*

First we prove two lemmas.

Lemma 1. *Let C_1 be a sufficiently large absolute constant. Then*

$$f_1(s) < A(s)+c_1 s\log s. \tag{1}$$

We shall first prove

$$f_1(s) < A(s)+c_1 s\log s\log\log s \tag{2}$$

and then we will outline the proof of (1).

Denote by $r_k(N)$ the number of representations of N as the sum of k odd primes. It easily follows from the well-known theorem of Hardy-Littlewood-Vinogradoff ([2], p. 198), that

$$r_3(N) > c_2 N^2/(\log N)^3. \tag{3}$$

The well-known theorem of Schnirelmann ([2], p. 52) states

$$(4)\qquad r_2(N) < \frac{c_3 N}{(\log N)^2}\prod_{p|N}\left(1+\frac{1}{p}\right) < \frac{c_4 N \log\log N}{(\log N)^2}.$$

(The last inequality of (4) follows from the prime number theorem, or from a more elementary result.)

From (4) we obtain that the number of solutions of

$$(5)\qquad N = p_{i_1}+p_{i_2}+p_{i_3}, \quad i_1 \leqslant s$$

is less than

$$(6)\qquad c_4 sN\log\log N/(\log N)^2.$$

From (6) and (3) we obtain by a simple calculation that if $N > c_1 s\log s\log\log s$ then

$$(7)\qquad N = p_u+p_v+p_w, \quad s < u < v < w$$

is solvable (since the number of solutions of $N = 2p+q$ is clearly $< cN/\log N$).

Consider now the integer

$$A(s)+t, \quad t > c_1 s\log s\log\log s.$$

Put

$$t_1 = \begin{cases} p_{s-2}+p_{s-1}+p_s+t & \text{if} \quad t \text{ is even,}\\ 2+p_{s-1}+p_s+t & \text{if} \quad t \text{ is odd.}\end{cases}$$

By (7)

$$t_1 = p_u+p_v+p_w, \quad s < u < v < w$$

is solvable. Thus $A(s)+t$ is the sum of s distinct primes which proves (2).

Now we outline the proof of (1). It is easy to see that (1) will follow if we can prove that for

$$(8)\qquad c_1 s\log s < N < c_1 s\log s\log\log s$$

the number of solutions $\psi(N)$ of (5) satisfies

$$(9)\qquad \psi(N) < c_4 sN/(\log N)^2.$$

But by the above mentioned theorem of Schnirelmann

$$(10)\qquad \psi(N) \leqslant \sum_{i=1}^{s} r_2(N-p_i) < \frac{c_3 N}{(\log N)^2}\sum_{i=1}^{s}\prod_{p|(N-p_i)}\left(1+\frac{1}{p}\right).$$

Now it can be proved that if N satisfies (8) then

$$(11)\qquad \sum_{i=1}^{s}\prod_{p|(N-p_i)}\left(1+\frac{1}{p}\right) < c_5 s.$$

We supress the proof of (11) since it is not quite short but uses fairly standard arguments and it is of no great importance for us to have Lemma 1 in the sharpest possible form. (9) follows immediately from (10) and (11). Hence (1) is proved and the proof of Lemma 1 is complete.

The estimation given by Lemma 1 is best possible (apart from the value of c_1), since considerations of parity shows that $B(s)-2$ can not be the sum of distinct primes and clearly

$$B(s) > A(s)+c_6 s\log s \quad (\text{since } p_s > c_7 s\log s).$$

Perhaps $f_1(s) = B(s)+o(s\log s)$ but this I have not been able to prove. It is easy to see though that

$$\limsup_{s=\infty}\big(f_1(s)-B(s)\big) = \infty$$

and probably

$$\lim_{s=\infty}\big(f_1(s)-B(s)\big) = \infty.$$

Lemma 2. *Put $a_k = p_k^2-p_k$, $k \geqslant 2$. Then there exists an absolute constant A so that every even integer greater than A is the sum of distinct a_k's.*

One can easily deduce Lemma 2 from a theorem of Cassels ([1]) (it easily follows from the results on Vinogradoff ([4]) that if $0<\alpha<1$ then $\binom{p}{2}\alpha \pmod 1$ has at least one limit point different from 0, thus the theorem of Cassels can be applied). An elementary and direct proof of Lemma 2 should be possible which would have the advantage of determining the best possible value of A. Such a proof would perhaps require a considerable amount of numerical calculation and I have not carried it out.

Now we are ready to prove our Theorem. We shall in fact show that for $s > s_0(c_1)$

$$(12)\qquad f_2(s) \leqslant B(s)+A.$$

Let now $n \geqslant B(s)+A$. If $n > A(s)+c_1 s\log s\log\log s$ then by Lemma 1 n is the sum of s distinct primes (we only use (2)). Thus we can assume

$$B(s)+A < n < A(s)+c_1 s\log s\log\log s.$$

Assume first $n = B(s)+2t$. Since $2t > A$, by Lemma 2

$$2t = a_{k_1}+\ldots+a_{k_r}, \quad k_1 < \ldots < k_r,$$

but $2t < c_1 s \log s \log\log s$ clearly implies that for $s > s_0 = s_0(c_1)$, $k_r \leqslant s$ (since $a_s = p_s^2 - p_s > c_1 s \log s \log\log s$). Thus

$$B(s)+2t = \sum_{i=2}^{s+1} p_i + \sum_{i=1}^{r} a_{k_i}$$

gives a representation of $B(s)+2t$ as the sum of s distinct primes or squares of primes where p and p^2 are not both used.

Assume next $n = B(s)+2t+1$. Then $n = A(s)+2t_1$, $2t_1 < cs\log s \times \log\log s$. Thus the same proof again gives that n is the sum of s distinct primes of squares of primes where p and p^2 are not both used. Thus (12) and hence our Theorem is proved (the cases $s \leqslant s_0$ can be ignored because of Lemma 1).

Finally we remark that $f_3(s) \geqslant B(s)-2$ since $B(s)-2$ can not be the sum of s distinct integers >1 which are pairwise relatively prime. To see this we only have to observe that by considerations of parity no even number can occur in such a representation.

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Further developments in the comparative prime-number theory V

(The use of "two-sided" theorems)

by

S. Knapowski (Poznań) and P. Turán (Budapest)

1. This paper means in this series a methodical digression; its aim is at the same time modest and pretentious. It is modest since we are going to prove a theorem which we proved in stronger form in a previous paper (see Knapowski-Turán [1]). It is still pretentious for the following reason. The second of us observed some years ago that several problems in the analytical number-theory can be reduced to the following "two-sided" theorem.

If m is a positive number, further

(1.1) $$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$$

and

(1.2) $$B \overset{\text{def}}{=} \min_{\lambda} \Big|\sum_{j=1}^{\lambda} b_j\Big| > 0,$$

then there is an integer ν satisfying

(1.3) $$m \leqslant \nu \leqslant m+n$$

such that

(1.4) $$\Big|\sum_{j=1}^{n} b_j z_j^{\nu}\Big| \geqslant \left(\frac{n}{8e(m+n)}\right)^n \frac{B}{2n}.$$

He had in mind further applications too, a typical one being the explicit numerical determination of an X such that for a suitable $2 \leqslant x_0 \leqslant X$ the difference $\pi(x) - \mathrm{Li}\,x$ would change sign at $x = x_0$ (Littlewood's problem). But he came soon to a conclusion that such an application can be expected only after having instead of the "two-sided" theorem (1.1)-(1.4) a "one-sided" one, assuring the existence of integers ν_1 and ν_2 in