On a problem of Sierpiński

(Extract from a letter to W. Sierpiński)

by

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Denote by \( \mu \) the least integer so that every integer \( > u_4 \) is the sum of exactly \( s \) integers \( > 1 \) which are pairwise relatively prime. Sierpiński ([3]) proved that \( u_2 = 6 \), \( u_3 = 17 \) and \( u_4 = 30 \) and he asks for a determination or estimation of \( u_4 \). Denote by \( f_i(s) \) the smallest integer so that every \( l > f_i(s) \) is the sum of \( s \) distinct primes; \( f_1(s) \) is the smallest integer so that every \( l > f_1(s) \) is the sum of \( s \) distinct primes or squares of primes where a prime and its square are not both used and \( f_4(s) \) is the least integer so that every \( l > f_4(s) \) is the sum of \( s \) distinct integers \( > 1 \) which are pairwise relatively prime. By definition \( f_1(s) = u_4 \). Clearly

\[ f_1(s) \leq f_4(s) \leq f_1(s). \]

Let \( p_1 = 2 \), \( p_2 = 3 \), \ldots be the sequence of consecutive primes. Put

\[ A(s) = \sum_{i=1}^{s} p_i, \quad B(s) = \sum_{i=2}^{s-1} p_i. \]

**Theorem.** \( f_1(s) < B(s) + C \) where \( C \) is an absolute constant independent of \( s \).

First we prove two lemmas.

**Lemma 1.** Let \( C \) be a sufficiently large absolute constant. Then

\[ f_1(s) < A(s) + C \log s. \]

We shall first prove

\[ f_1(s) < A(s) + C \log \log \log s \]

and then we will outline the proof of (1).

Denote by \( r_k(N) \) the number of representations of \( N \) as the sum of \( k \) odd primes. It easily follows from the well-known theorem of Hardy-Littlewood-Vinogradoff ([2], p. 198), that

\[ r_k(N) > \text{const} N^{2k} / (\log N)^k. \]

\[ (3) \]
The well-known theorem of Schinzel and Sierpinski ([2], p. 52) states

\[ r_s(N) < \frac{c_s N}{(\log N)^2} \prod_{p \leq N} \left(1 + \frac{1}{p}\right) < \frac{c_s N \log \log N}{(\log N)^2}. \]

(The last inequality of (4) follows from the prime number theorem, or from a more elementary result.)

From (4) we obtain that the number of solutions of

\[ N = P_1 + P_2 + P_3, \quad \delta_1 < \delta \]

is less than

\[ c_s N \log \log N/(\log N)^2. \]

From (6) and (3) we obtain by a simple calculation that if \( N > c_s \log \log \log s \) then

\[ N = P_u + P_v + P_w, \quad s < u < v < w \]

is solvable (since the number of solutions of \( N = 2p + q \) is clearly \( < cN/\log N \)).

Consider now the integer

\[ A(s) + \delta_1, \quad \delta > c_s \log \log \log s. \]

Put

\[ t_1 = \begin{cases} \frac{p_{s-1} + p_{s-1} + p_s + t}{2} & \text{if } \delta \text{ is even}, \\ \frac{2p_s + p_s + t}{3} & \text{if } \delta \text{ is odd}. \end{cases} \]

By (7)

\[ t_1 = \frac{p_u + p_v + p_w}{3}, \quad s < u < v < w \]

is solvable. Thus \( A(s) + \delta \) is the sum of \( s \) distinct primes which proves (2).

Now we outline the proof of (1). It is easy to see that (1) will follow if we can prove that for

\[ c_s \log s < N < c_s \log \log \log s \]

the number of solutions \( \psi(N) \) of (5) satisfies

\[ \psi(N) < c_s A(N)/(\log N)^{2}. \]

But by the above mentioned theorem of Schinzel and Sierpinski

\[ \psi(N) \leq \sum_{N < p} r_s(N - p) < \frac{c_s N}{(\log N)^2} \sum_{N < p} \prod_{p \leq N} \left(1 + \frac{1}{p}\right). \]

Now it can be proved that if \( N \) satisfies (8) then

\[ \sum_{j=1}^{s} \prod_{p \leq N} \left(1 + \frac{1}{p}\right) < c_s \varepsilon. \]

We suppress the proof of (11) since it is not quite short but uses fairly standard arguments and it is of no great importance for us to have Lemma 1 in the sharpest possible form. (9) follows immediately from (10) and (11). Hence (1) is proved and the proof of Lemma 1 is complete.

The estimation given by Lemma 1 is best possible (apart from the value of \( c_s \)), since considerations of parity shows that \( B(s) - 2 \) cannot be the sum of distinct primes and clearly

\[ B(s) > A(s) + c_s \log \log \log s \quad \text{(since } p_n > c_s \log \log \log s). \]

Perhaps \( f_s(s) = B(s) + o(\log \log \log s) \) but this I have not been able to prove. It is easy to see though that

\[ \limsup_{s \to \infty} (f_s(s) - B(s)) = \infty \]

and probably

\[ \lim_{s \to \infty} (f_s(s) - B(s)) = \infty. \]

**Lemma 2.** Put \( a_k = p_{k^2} - p_k, k \geq 2 \). Then there exists an absolute constant \( A \) so that every even integer greater than \( A \) is the sum of distinct \( a_k \)'s.

One can easily deduce Lemma 2 from a theorem of Cassels (11) (it is also implied from the results of Vinogradoff (14)) if \( 0 < a < 1 \) then \( \left(\frac{a}{2}\right)^a \pmod{1} \) has at least one limit point different from 0, thus the theorem of Cassels can be applied. An elementary and direct proof of Lemma 2 should be possible which would have the advantage of determining the best possible value of \( A \). Such a proof would perhaps require a considerable amount of numerical calculation and I have not carried it out.

Now we are ready to prove our Theorem. We shall in fact show that for \( s > s_s(c) \)

\[ f_s(s) < B(s) + A. \]

Let now \( s > B(s) + A \). If \( s > A(s) + c_s \log \log \log s \) then by Lemma 1 \( s \) is the sum of \( s \) distinct primes (we only use (2)). Thus we can assume

\[ B(s) + A < s < A(s) + c_s \log \log \log s. \]

Assume first \( s = B(s) + 2t \). Since \( 2t > A \), by Lemma 2

\[ 2t = a_{k_1} + \ldots + a_{k_t}, \quad k_1 < \ldots < k_t, \]
but $2t < c_1 \log \log \log s$, clearly implies that for $s > s_0 = s_0(c_1) = n + 1$, $n < s$ (since $a_0 = p_1 - p_s > c_1 \log \log \log s$). Thus

$$B(s) + 2t = \sum_{i=2}^{a_0} p_i + \sum_{i=1}^{r} a_{b_i}$$

gives a representation of $B(s) + 2t$ as the sum of $s$ distinct primes or squares of primes where $p$ and $p^2$ are not both used.

Assume next $n = B(s) + 2t + 1$. Then $n = B(s) + 2t + 1$, $2t < c_1 \log \log \log s$. Thus the same proof again gives that $n$ is the sum of $s$ distinct primes of squares of primes where $p$ and $p^2$ are not both used. Thus (12) and hence our Theorem is proved (the cases $s < s_0$ can be ignored because of Lemma 1).

Finally we remark that $f_1(s) \geq B(s) - 2$ since $B(s) - 2$ can not be the sum of $s$ distinct integers $> 1$ which are pairwise relatively prime. To see this we only have to observe that by considerations of parity no even number can occur in such a representation.

References


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Further developments in the comparative prime-number theory V
(The use of "two-sided" theorems)

by S. Knapowski (Poznań) and P. Turán (Budapest)

1. This paper means in this series a methodological digression; its aim is at the same time modest and pretentious. It is modest since we are going to prove a theorem which we proved in stronger form in a previous paper (see Knopowski-Turán [1]). It is still pretentious for the following reason. The second of us observed some years ago that several problems in the analytical number-theory can be reduced to the following "two-sided" theorem.

If $m$ is a positive number, further

$$1 = |a_1| > |a_2| > \ldots > |a_n|$$

and

$$B \equiv \min_{\nu} \left| \sum_{i=1}^{\nu} a_i \right| > 0,$$

then there is an integer $\nu$ satisfying

$$m \leq \nu \leq m + n$$

such that

$$\left| \sum_{i=1}^{n} a_i \right| \geq \left( \frac{m}{\delta(m+n)} \right)^\nu B \frac{2^m}{\nu}.$$

He had in mind further applications too, a typical one being the explicit numerical determination of an $X$ such that for a suitable $2 \leq a_0 \leq X$ the difference $\pi(x) - Lx$ would change sign at $x = x_0$ (Littlewood's problem). But he came soon to a conclusion that such an application can be expected only after having instead of the "two-sided" theorem (1.1)-(1.4) a "one-sided" one, assuring the existence of integers $v_1$ and $v_2$ in

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