ist ein erzeugendes Element der Automorphismengruppe von f. Es ist

$$A = \begin{pmatrix} \frac{1}{2}(u-bv) & -cv \\ av & \frac{1}{2}(u+bv) \end{pmatrix},$$

wenn $\varepsilon = \frac{1}{2}(u+v\sqrt{d})$ die Grundeinheit (positiver Norm) des Ringes $[1, \frac{1}{2}(1+\sqrt{d})]$ bzw. $[1, \frac{1}{2}\sqrt{d}]$ ist. Daraus ergibt sich (unter Benutzung der Reduktionsbedingungen)

$$\mu(A) = \tfrac{1}{4}\big(u^2+2(b^2-2a^2-2c^2)v^2\big) < \tfrac{3}{4}\varepsilon^2.$$

Folglich ist die Periodenlänge nach Satz 3

(32) $$l \leqslant c_8 \log \tfrac{3}{4}\varepsilon^2 \approx 0.69 \log \tfrac{3}{4}\varepsilon^2.$$

Übrigens ist die Periodenlänge l von der Klasse der Form unabhängig und daher ihr Produkt mit der Klassenzahl gleich der Anzahl der reduzierten Formen.

Literaturverzeichnis

[1] Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie,* 4. Aufl. Braunschweig 1894.

[2] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale,* Math. Zeitschr. 67 (1957), S. 267 - 298.

[3] — *Quadratische Formen und Modulfunktionen,* Acta Arith. 4 (1958), S. 217 - 239.

[4] C. L. Siegel, *Ausgewählte Fragen der Funktionentheorie II,* Göttingen 1954, S. 29 ff.

[5] I. V. Uspensky and M. A Heaslet, *Elementary Number Theory,* New York-London. 1939, S. 43,44.

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Irrational power series II*

by

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§ 1. Let a be a real irrational number and denote by $\{a\} = a-[a]$ the fractional part of a and so $0 \leqslant \{a\} < 1$. Some forty years ago, Hecke discussed the behaviour of the function

(1) $$g(x) = \sum_{n=1}^{\infty} \{na\} x^n$$

on its circle of convergence $|x| = 1$. He showed that this is a line of essential singularities of $g(x)$.

Similar questions have been discussed by Salem ([1]), M. Newman ([2]) and myself ([3], [4]).

In a very recent paper ([5]), Wolfgang Schwarz discusses, *inter alia*, a similar question for the function

(2) $$h(x) = \sum_{n=0}^{\infty} \varphi(\{na\}) x^n,$$

where $\varphi(y)$ is a function of y.

The proof is based on Hecke's method and uses the theory of uniform distribution. Some of his results are included in my subsequent ([6])

THEOREM. *Let*

(3) $$f(x) = \sum_{n=0}^{\infty} \varphi(\{an+\beta\}) x^n, \qquad |x| < 1,$$

where a is irrational and β is real, and $\varphi(y)$ is continuous for $0 \leqslant y \leqslant 1$. Then $f(x)$ is a rational function of x if and only if $\varphi(y)$ is a finite Fourier series $\varphi(y) = \sum_{r=-L}^{L} a_r e^{2r\pi i x}$, and so

(4) $$f(x) = \sum_{r=-L}^{L} \frac{a_r e^{2r\pi i \beta}}{1-xe^{2r\pi i a}}.$$

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If $f(x)$ is not a rational function of x, then $f(x)$ cannot be continued outside of the unit circle.

The series (4) makes obvious the singularities of $f(x)$. We shall now find an expansion for $f(x)$ in (3) of the form

$$f(x) = \sum_{r=-\infty}^{\infty} \frac{c_r}{1-xe^{2r\pi i a}}, \tag{5}$$

where here and throughout this paper, $\sum_{-\infty}^{\infty}$ means $\lim_{N\to\infty} \sum_{-N}^{N}$ and $\sum'$ means the omission of $n = 0$. The series (3) arises formally from (5) by expanding each of its terms in ascending powers of x, and rearranging the double series. Some instances were given in a former paper ([4]). We need two results from this paper.

LEMMA 1. *Let*

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{a_n}{1-xe^{2n\pi i a}}, \tag{6}$$

where a is a real irrational number and $\sum|a_n|$ converges. Then the series for $F(x)$ converges if $|x|<1$, and as $xe^{2k\pi i a}\to 1-0$ by radial approach, where k is an integer,

$$F(x) = \frac{a_k}{1-xe^{2k\pi i a}} + o\left(\frac{1}{1-xe^{2k\pi i a}}\right). \tag{7}$$

If $n = 0$ is excluded in (6), then as $x \to 1-0$, $F(x) = o\big(1/(1-x)\big)$. It seems very difficult to deal with the case when $\sum a_n$ converges conditionally. We have now, however,

LEMMA 2. *Let $a_n = e(2n\pi i\beta)/n$ where β is real and a is a real irrational number and $an+\beta$ is not an integer for any integer n. Write*

$$H(x) = \sum_{n=-\infty}^{\infty}{}' \frac{e^{2n\pi i\beta}}{n(1-xe^{2n\pi i a})}. \tag{8}$$

Then the series for $H(x)$ converges if $|x|<1$, and as $xe^{2k\pi i a}\to 1-0$ by radial approach, where k is a non zero integer,

$$H(x) = \frac{1}{k}\cdot\frac{e^{2k\pi i\beta}}{1-xe^{2k\pi i a}} + o\left(\frac{1}{1-xe^{2k\pi i a}}\right). \tag{9}$$

If $k = 0$, the first term on the right must be omitted.

The power series in x for both $F(x)$ and $H(x)$ are given by expanding each term in powers of x, and rearranging the terms of the double series. This is easily justified for $F(x)$ since the double series is absolutely convergent. The justification for $H(x)$ is more difficult and required Littlewood's-Tauberian theorem, as explained in the paper [4]. This led to

$$H(x)/2\pi i = \sum_{n=0}^{\infty} (\tfrac{1}{2} - \{an+\beta\})x^n, \tag{10}$$

if $an+\beta$ is not an integer for any integer $n \geqslant 0$. If $an+\beta$ is an integer for $n = l$, the term in (10) with x^l must be omitted.

In my paper [6], I showed that the behaviour of the power series for $F(x)$ and $H(x)$, as indicated by (7), (9), could be found more simply. If, however, we are given $F(x)$ defined by (6) where $\sum a_n$ is conditionally convergent, it is not an easy matter to determine if (6) is convergent or if (4) holds.

We now find the meromorphic expansion for

$$f(x) = \sum_{n=0}^{\infty} \varphi(\{na+\beta\})x^n, \tag{11}$$

where a, β are real numbers, a is irrational and $an+\beta$ is not an integer for any integer $n \geqslant 0$. We suppose that $\varphi(y)$ is defined and continuous for $0 \leqslant y \leqslant 1$ (one sided continuity at $y = 0, 1$) and has for $0 < y < 1$, a Fourier expansion,

$$\varphi(y) = \sum_{r=-\infty}^{\infty} a_r e^{2r\pi i y}, \tag{12}$$

where

$$a_r = \int_0^1 \varphi(y)\, e^{-2r\pi i y} dy.$$

Results follow on making assumptions about the convergence of the series $\sum a_n$.

Suppose first that $\sum|a_r|$ converges. Then from (12) since $\{na+\beta\} \neq 0$,

$$\varphi(\{na+\beta\}) = \sum_{r=-\infty}^{\infty} a_r e^{2r\pi i(na+\beta)}. \tag{13}$$

Then (11) becomes

$$f(x) = \sum_{n=0}^{\infty}\sum_{r=-\infty}^{\infty} a_r e^{2r\pi i(na+\beta)}x^n. \tag{14}$$

Consideration of the absolutely convergent double series

$$S = \sum_{n=0}^{\infty}\sum_{r=-\infty}^{\infty} a_r e^{2r\pi i(na+\beta)}x^n,$$

on summing first for n, gives

$$f(x) = \sum_{r=-\infty}^{\infty} \frac{a_r e^{2r\pi i\beta}}{1-xe^{2r\pi i\alpha}}. \tag{15}$$

It is not easy to find results when $\sum a_n$ is conditionally convergent. This, however, we can do when

$$a_r = \frac{c}{r} + b_r \quad (r \neq 0), \tag{16}$$

where c is independent of r and $\sum|b_r|$ converges. This will be the case if $\varphi''(y) = \dfrac{d^2\varphi(y)}{dy^2}$ is continuous for $0 \leqslant y \leqslant 1$ (one sided continuity at $y = 0, 1$), or more generally if $\varphi''(y)$ is integrable. From (10), integration by parts, when $r \neq 0$, gives

$$a_r = \frac{\varphi(1)-\varphi(0)}{2\pi ir} - \frac{\varphi'(1)-\varphi'(0)}{(2\pi ir)^2} + \int_0^1 \frac{\varphi''(y)e^{2r\pi iy}}{(2\pi ir)^2}dy. \tag{17}$$

This comes under (16) with $2\pi ic = \varphi(1)-\varphi(0)$. Then (14) becomes

$$f(x) = \frac{\varphi(1)-\varphi(0)}{2\pi i}\sum_{n=0}^{\infty}\Big(\sum_{r=-\infty}^{\infty}{}' \frac{e^{2r\pi i(n\alpha+\beta)}}{r}\Big)x^n + \sum_{n=0}^{\infty}\Big(\sum_{r=-\infty}^{\infty} b_r e^{2r\pi i(n\alpha+\beta)}\Big)x^n.$$

From Lemma 2, we have since $b_r = O(1/r^2)$,

$$f(x) = \frac{\varphi(1)-\varphi(0)}{2\pi i}\Big(\sum_{r=-\infty}^{\infty}{}' \frac{e^{2r\pi i\beta}}{r(1-xe^{2r\pi i\alpha})}\Big) + \sum_{r=-\infty}^{\infty} \frac{b_r e^{2r\pi i\beta}}{1-x^{2r\pi i\alpha}}. \tag{18}$$

§ 2. A new generalization is to the function defined for $|x| < 1$ by

$$f(x) = \sum_{n=0}^{\infty} \chi(\{\alpha n+\gamma\}, \{\beta n+\delta\})x^n, \tag{19}$$

where $\chi(y, z)$ is a function of two variables y, z; and α, β are irrational, γ, δ are real, and neither $\alpha n+\gamma$ nor $\beta n+\delta$ is an integer for any integer $n > 0$. Results similar to the theorem hold for $f(x)$ as shown in my paper [6]. A meromorphic expansion for (19) can be found when $\chi(y, z)$ has a Fourier expansion

$$\chi(y, z) = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} c_{rs}e^{2\pi i(ry+sz)}$$

for $0 < y < 1$, $0 < z < 1$. For simplicity, we consider only the case when $\chi(y, z) = \varphi(y)\psi(z)$, and we have for $0 < y < 1$, $0 < z < 1$, the Fourier expansions

$$\varphi(y) = \sum_{r=-\infty}^{\infty} a_r e^{2r\pi iy}, \qquad \psi(z) = \sum_{s=-\infty}^{\infty} b_s e^{2s\pi iz}. \tag{20}$$

If $\sum|a_r|$, $\sum|b_s|$ both converge, then consideration of an obvious absolutely convergent triple series shows that

$$f(x) = \sum_{r,s=-\infty}^{\infty} \frac{a_r b_s e^{2\pi i(\gamma r+\delta s)}}{1-xe^{2\pi i(\alpha r+\beta s)}}.$$

The case when $\sum a_r$, $\sum b_s$ converge conditionally is rather difficult. Expansions can be found when

$$a_r = \frac{A}{r} + a_r', \qquad b_s = \frac{B}{s} + b_s', \tag{22}$$

where A and B are constants and $\sum|a_r'|$, $\sum|b_s'|$ converge.

Then from (19), (20), (22),

$$f(x) = \sum_{n=0}^{\infty}\Big(\sum_{r,s=-\infty}^{\infty}{}' C_{rs}\Big)x^n, \tag{23}$$

where $r = 0$ and $s = 0$ are omitted in the summation and

$$C_{rs} = \Big(\frac{AB}{rs} + \frac{A}{r}b_s' + \frac{B}{s}a_r' + a_r'b_s'\Big)e^{2n\pi i(\alpha r+\gamma s)+2\pi i(\beta r+\delta s)}. \tag{24}$$

The series for $f(x)$ splits into four obvious series, say S_1, S_2, S_3, S_4.

The question now arises, can we invert the order of summation for n in (23), e.g., is

$$\frac{S_1}{AB} = \sum_{r,s=-\infty}^{\infty}{}' \frac{e^{2\pi i(\gamma r+\delta s)}}{rs(1-xe^{2\pi i(\alpha r+\beta s)})}? \tag{25}$$

The justification of the change in the order of summation does not seem easy for general α, β. We can do this if

$$\sum x^n/\|\alpha n+\gamma\|, \qquad \sum x^n/\|\beta n+\delta\|,$$

where $\|k\|$ is the positive difference between k and its nearest integer, both converge for $|x| < 1$. This is so if the constants α, β, γ, δ are such that for constants $\lambda \geqslant 1$, $\mu \geqslant 1$,

$$\|\alpha n+\gamma\| > n^{-\lambda}, \qquad \|\beta n+\delta\| > n^{-\mu},$$

and so for all integers, $n > 0$ and m,

$$|\alpha n+\gamma-m| > n^{-\lambda}, \qquad |\beta n+\delta-m| > n^{-\mu}.$$

We show by Liouville's method that this is so if α, β, γ, δ are algebraic numbers.

Suppose that it is not true that $|\alpha n+\gamma-m| > n^{-\lambda}$ for all large integers n. Then for an infinity of n, $|\alpha n+\gamma-m| \leqslant n^{-\lambda}$. Denote by α', α'', ..., γ', γ'', ... the conjugates of α, γ respectively. Then

$$\prod |\alpha n+\gamma-m| > k,$$

where the product is extended over all the conjugates of α, γ, and k is a rational number independent of n, m, provided that γ is linearly independent of α, 1 over the rational integers. Now

$$|\alpha' n+\gamma'-m| = |\alpha n+\gamma-m+(\alpha'-\alpha)n| \leqslant n^{-\lambda}+|\alpha'-\alpha| n \leqslant k'n,$$

say.

Since $|\alpha n+\gamma-m| \prod |\alpha' n+\gamma'-m'| > k$,

$$|\alpha n+\gamma-m| > k/\prod(k'n) > n^{-\nu},$$

say. Hence we have a contradiction if $\lambda > \nu$.

For the justification of (25), we write

$$(26)\qquad \sum_{r,s=-N}^{N}{}' \frac{e^{2\pi i(\gamma r+\delta s)}}{rs(1-xe^{2\pi i(\alpha r+\beta s)})} = \sum_{n=0}^{\infty} \sum_{r,s=-N}^{N}{}' \frac{e^{2\pi i(\gamma r+\delta s)+2\pi i n(\alpha r+\beta s)}}{rs} x^n.$$

It is known, and as can be easily proved by partial summation, that if z is not an integer,

$$\frac{1}{2}-\{z\} = \frac{1}{2\pi i}\sum_{t=-N}^{N}{}' \frac{e^{2t\pi i z}}{t} + O\left(\frac{1}{N\|z\|}\right).$$

From this, the right-hand side of (26) becomes

$$(27)\qquad (2\pi i)^2 \sum_{n=0}^{\infty}\left(\frac{1}{2}-\{\alpha n+\gamma\}+O\left(\frac{1}{N\|\alpha n+\gamma\|}\right)\right)\left(\frac{1}{2}-\{\beta n+\delta\}+ +O\left(\frac{1}{N\|\beta n+\delta\|}\right)\right)x^n.$$

Hence if α, β, γ, δ are algebraic numbers, the series $\sum x^n/\|\alpha n+\gamma\|$ etc. converge. Then the limit of (27) when $N\to\infty$ is given by omitting the O terms and so

(28)

$$\sum_{r,s=-\infty}^{\infty}{}' \frac{e^{2\pi i(\gamma r+\delta s)}}{rs(1-xe^{2\pi i(\alpha r+\beta s)})} = (2\pi i)^2\sum_{n=0}^{\infty}\left(\frac{1}{2}-\{\alpha n+\gamma\}\right)\left(\frac{1}{2}-\{\beta n+\delta\}\right)x^n.$$

Take next

$$(29)\qquad \frac{S_2}{A} = \sum_{n=0}^{\infty}\left(\sum_{r,s=-\infty}^{\infty}{}' \frac{b'_s}{r} e^{2\pi i(\alpha r+\beta s)n+2\pi i(\gamma r+\delta s)}\right)x^n.$$

The same type of argument applies as for S_1. Write

$$h(n) = \sum_{s=-\infty}^{\infty} b'_s e^{2\pi i(n\beta+\delta)s}.$$

Since $\sum|b'_s|$ converges,

$$h(n) = \sum_{s=-N}^{N} b'_s e^{2\pi i(n\beta+\delta)s} + o(1),$$

where the error term is independent of n and tends to zero as $N\to\infty$. Then

$$\sum_{n=0}^{\infty}\sum_{r,s=-N}^{N}{}' \frac{1}{r} b'_s e^{2\pi i(\alpha r+\beta s)n+2\pi i(\gamma r+\delta s)} x^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}-\{\alpha n+\gamma\}+O\left(\frac{1}{N\|\alpha n+\gamma\|}\right)\right)\left(h(n)-o(1)\right)x^n.$$

Since $h(n) = o(1)$ uniformly in n, the limit as $N\to\infty$ is given by omitting the error terms. Hence

$$(30)\qquad S_2/A = \sum_{n=0}^{\infty}\left(\tfrac{1}{2}-\{\alpha n+\gamma\}\right)h(n)x^n.$$

Similarly if $g(n) = \sum_{r=-\infty}^{\infty} a'_r e^{2\pi i(n\alpha+\gamma)r}$,

$$(31)\qquad S_3/B = \sum_{n=0}^{\infty}\left(\tfrac{1}{2}-\{\beta n+\delta\}\right)g(n)x^n.$$

Finally, it is clear that

$$(32)\qquad S_4 = \sum_{n=0}^{\infty} g(n)h(n)x^n.$$

References

[1] R. Salem, *Power series with integral coefficients*, Duke Math. Journ. 12 (1945), pp. 153 - 172.

[2] M. Newman, Proc. Amer. Math. Soc. 11 (1960), pp. 699 - 702.

[3] L. J. Mordell, *Irrational power series*, Proc. Amer. Math. Soc. 12 (1961), pp. 522 - 526.

[4] L. J. Mordell, *The series* $\sum a_n/(1-xe^{2n\pi ia})$, J. London Math. Soc. 38 (1963), pp. 111 - 116.

[5] Wolfgang Schwarz, *Irrationale Potenzreihen,* Arch. Math. 13 (1962), pp. 220 - 240.

[6] L. J. Mordell, *Irrational power series III,* Proc. Amer. Math. Soc.

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On a problem of Sierpiński

(Extract from a letter to W. Sierpiński)

by

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Denote by μ_s the least integer so that every integer $> u_s$ is the sum of exactly s integers >1 which are pairwise relatively prime. Sierpiński ([3]) proved that $u_2 = 6$, $u_3 = 17$ and $u_4 = 30$ and he asks for a determination or estimation of u_s. Denote by $f_1(s)$ the smallest integer so that every $l > f_1(s)$ is the sum of s distinct primes; $f_2(s)$ is the smallest integer so that every $l > f_2(s)$ is the sum of s distinct primes or squares of primes where a prime and its square are not both used and $f_3(s)$ is the least integer so that every $l > f_3(s)$ is the sum of s distinct integers >1 which are pairwise relatively prime. By definition $f_3(s) = u_s$. Clearly

$$f_3(s) \leqslant f_2(s) \leqslant f_1(s).$$

Let $p_1 = 2$, $p_2 = 3, \ldots$ be the sequence of consecutive primes. Put

$$A(s) = \sum_{i=1}^{s} p_i, \quad B(s) = \sum_{i=2}^{s+1} p_i.$$

Theorem. *$f_2(s) < B(s)+C$ where C is an absolute constant independent of s.*

First we prove two lemmas.

Lemma 1. *Let C_1 be a sufficiently large absolute constant. Then*

(1) $$f_1(s) < A(s)+c_1 s\log s.$$

We shall first prove

(2) $$f_1(s) < A(s)+c_1 s\log s\log\log s$$

and then we will outline the proof of (1).

Denote by $r_k(N)$ the number of representations of N as the sum of k odd primes. It easily follows from the well-known theorem of Hardy-Littlewood-Vinogradoff ([2], p. 198), that

(3) $$r_3(N) > c_2 N^2/(\log N)^3.$$