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The discrepancy of random sequences $\{kx\}$

by

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1. Introduction. It was R. Bellman [2] who first suggested the investigation of the limit distribution of

$$\sum_{k=1}^{N} f(y+kx;\ a, b) - N(b-a) \tag{1.1}$$

if the pair x, y is a random variable, uniformly distributed in the unit square and if, for $0 \leqslant a \leqslant b \leqslant 1$,

$$f(\xi;\ a, b) = \begin{cases} 1 & \text{if} \quad a \leqslant \xi \leqslant b, \\ 0 & \text{if} \quad 0 \leqslant \xi < a \text{ or } b < \xi \leqslant 1, \end{cases}$$

$$f(\xi+1;\ a, b) = f(\xi;\ a, b).$$

If $\{\xi\} = \xi - [x]$ denotes the fractional part of ξ, then $\sum_{k=1}^{N} f(y+kx;\ a, b)$ is simply the number of $k, 1 \leqslant k \leqslant N$, with $\{y+kx\} \in [a, b]$ and (1.1) measures the deviation of this number from its average.

In [4] and [5] the author found the limiting distribution of (1.1). In this note those results are extended by studying the discrepancy

$$D_N(x) = \frac{1}{N} \sup_{0 \leqslant a \leqslant b \leqslant 1+a} \left| \sum_{k=1}^{N} f(kx;\ a, b) - N(b-a) \right|. \tag{1.2}$$

(If $1 < b \leqslant 1+a$ we define $f(\xi;\ a, b)$ in an obvious way, namely as $f(\xi;\ a, 1) + f(\xi;\ 0, b-1)$.) Our main result is Theorem 2 below for which we consider x as a point from the measure space $[0, 1]$ with Lebesgue measure.

THEOREM 2.

$$\frac{N \cdot D_N(x)}{\log N \cdot \log\log N} \to \frac{2}{\pi^2} \textit{ in measure on } [0, 1] \textit{ as } N \to \infty.$$

The first part of the proof (section 2) gives an asymptotic expression for $D_N(x)$ in terms of the continued fraction denominators of x, which may have some independent interest. It is a slight refinement of work of Ostrowski [10] and Behnke [1] and even though methods similar to [10] and [1] have been used by others, Theorem 1 does not seem to appear in the literature. By means of Theorem 1 the proof of Theorem 2 reduces to a metric problem for continued fractions which is solved in section 3 by means of known probabilistic results on continued fractions. Some easy corollaries of our proof are given at the end of section 3.

2. A relation between $D_N(\xi)$ and the continued fraction of ξ. We recall that every irrational number $\xi \epsilon [0,1]$ has an infinite regular continued fraction which we always write as

(2.1) $$[a_1(\xi), a_2(\xi), \ldots] = \cfrac{1}{a_1(\xi)+\cfrac{1}{a_2(\xi)+\ldots}} = \xi.$$

The convergents $\frac{p_n(\xi)}{q_n(\xi)}$ satisfy (cf. [3])

(2.2a) $$p_0 = 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1}+p_{n-2}, \quad n \geqslant 2,$$

and

(2.2b) $$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1}+q_{n-1}, \quad n \geqslant 2.$$

Since $q_n < q_{n+1} < (a_{n+1}+1)q_n$ we can expand any positive number z in a unique way as

(2.3) $$z = \sum_{n=0}^{\infty} c_n(z, \xi) q_n(\xi)+\{z\}$$

where c_i is an integer satisfying

(2.4) $$0 \leqslant c_n(z, \xi) \leqslant a_{n+1}(\xi)$$

and

(2.5) $$r_n(z, \xi) = z-\sum_{i=n}^{\infty} c_i(z, \xi) q_i(\xi) < q_n(\xi)$$

(compare [10]). Actually the sum in (2.3) only runs as far as $m = m(z, \xi)$, which is determined by

(2.6) $$q_m \leqslant z < q_{m+1}.$$

If z is an integer the term $\{z\}$ in (2.3) vanishes.

If not explicitly stated otherwise, an expansion of the form $z = \sum c_i q_i+\{z\}$ will always stand for this unique expansion.

LEMMA 1. *For each $\varepsilon > 0$ there exists a $u = u(\varepsilon)$ such that*

$$cq_n(\xi) D_{cq_n(\xi)}(\xi) \leqslant a_{n+1}(\xi)\left(\frac{c}{a_{n+1}(\xi)}\left(1-\frac{c}{a_{n+1}(\xi)}\right)+\varepsilon\right)$$

whenever

(2.7) $$a_{n+1}(\xi) \geqslant u \quad \textit{and} \quad 0 \leqslant c \leqslant a_{n+1}(\xi) \quad (c \textit{ integer}).$$

For any integer $0 \leqslant c \leqslant a_{n+1}(\xi)$

(2.8) $$cq_n(\xi) D_{cq_n(\xi)}(\xi) \leqslant 2c.$$

This lemma will be proved together with the next lemma which gives a lower bound for D.

For any a and b we can find integers j_1, j_2 such that

(2.9) $$a = \frac{j_1-s}{q_n}, \quad b = \frac{j_2+t}{q_n}$$

with $0 \leqslant s, t < 1$. Of special interest to us are those a, b which for some large u satisfy the conditions

(2.10u) $$0 \leqslant t \leqslant 1-s \leqslant 1, \quad 0 \leqslant s \leqslant \frac{1}{u}, \quad \left|t-\frac{c}{a_{n+1}(\xi)}\right| \leqslant \frac{1}{u}$$

or

(2.11u) $$0 \leqslant s < 1-t \leqslant 1, \quad \left|s-\frac{c}{a_{n+1}(\xi)}\right| \leqslant \frac{1}{u}, \quad 0 \leqslant t \leqslant \frac{1}{u},$$

as well as

(2.12) $$0 \leqslant j_1 \leqslant j_2 \leqslant q_n(\xi)+j_1 \quad \text{and} \quad j_2-j_1+s+t \leqslant q_n(\xi).$$

LEMMA 2. *For each $\varepsilon > 0$ there exists a $u = u(\varepsilon)$ such that if* (2.7) *and* (2.12) *hold and either* (2.10u), *in case n is even, or* (2.11u) *in case n is odd, then*

(2.13) $$R_{cq_n(\xi)}(\xi; a, b) = \sum_{k=1}^{cq_n(\xi)} f(k\xi; a, b)-cq_n(\xi)(b-a)$$
$$\geqslant a_{n+1}(\xi)\left(\frac{c}{a_{n+1}(\xi)}\left(1-\frac{c}{a_{n+1}(\xi)}\right)-\varepsilon\right).$$

Proof. For convenience we drop the argument ξ in most functions. Consider now $R_{cq_n}(\xi; a, b)$ and write a, b in the form (2.9), with $0 \leqslant s, t < 1$. If $j_2 \geqslant q_k$ $[a, b]$ will mean $[a, 1] \cup [0, b]$ with the corresponding meaning for $f(\xi; a, b)$ (cf. comment to (1.2)).

It is well known (cf. [3]) that

$$\xi-\frac{p_n}{q_n}=\frac{(-1)^n}{q_n(a'_{n+1}q_n+q_{n-1})}=\frac{\delta_n}{q_n}, \tag{2.14}$$

where

$$a'_{n+1}=a_{n+1}+[a_{n+2},a_{n+3},\ldots]$$

and

$$\delta_n=\frac{(-1)^n}{a'_{n+1}q_n+q_{n-1}}. \tag{2.15}$$

Moreover, by Theorem 150 in [3],

$$(p_n,q_n)=(q_{n-1},q_n)=1. \tag{2.16}$$

For the remainder of the proof we shall only consider the case n even and thus $\delta_n\geqslant 0$. One only has to change the role of s and t to treat odd values of n.

By (2.16), the numbers kp_n, $k=v, v+1,\ldots,v+q_n-1$ form a complete residue system $\operatorname{mod} q_n$ and thus among the numbers $\left\{\frac{kp_n}{q_n}\right\}$, $v\leqslant k\leqslant v+q_n-1$, each value j/q_n, $0\leqslant j<q_n$, occurs exactly once. Consequently of the numbers $\left\{\frac{kp_n}{q_n}\right\}$, $1\leqslant k\leqslant cq_n$, exactly $c(j_2-j_1+1)$ belong to $[a,b]$. The situation is slightly different for the numbers $\{k\xi\}$. Namely, if

$$\left\{\frac{kp_n}{q_n}\right\}=\frac{\lambda_k}{q_n},\qquad 0\leqslant\lambda_k\leqslant q_n-1,$$

then

$$\{k\xi\}=\frac{\lambda_k+k\delta_n}{q_n}$$

since (for even n)

$$0\leqslant k\delta_n\leqslant cq_n\delta_n\leqslant\frac{a_{n+1}q_n}{a'_{n+1}q_n+q_{n-1}}<1. \tag{2.17}$$

Therefore $\left\{k\frac{p_n}{q_n}\right\}\epsilon[a,b]$ if and only if $[k\xi]\epsilon[a,b]$ as long as $\lambda_k\neq j_1-1$ (¹) and $\lambda_k\neq j_2$ (resp. j_2-q_n if $j_2\geqslant q_n$). If $\lambda_k=j_1-1$(¹) then $\{k\xi\}\epsilon[a,b]$ if and only if $k\delta_n\geqslant 1-s$. Since $\lambda_k=j_1-1$ for some $k=k_1$, $1\leqslant k_1\leqslant q_n$

(¹) Replace j_1-1 by q_n-1 if $j_1=0$.

and then for all k of the form k_1+wq_n we have

$$\{(k_1+wq_n)\xi\}\notin[a,b]\quad\text{for}\quad w\leqslant\left[\frac{1-s}{q_n\delta_n}\right]-2$$

but

$$\{(k_1+wq_n)\xi\}\epsilon[a,b]\quad\text{for}\quad \left[\frac{1-s}{q_n\delta_n}\right]+1\leqslant w\leqslant a_{n+1}-1.$$

It will not be necessary to investigate precisely what happens for $\left|w-\left[\frac{1-s}{q_n\delta_n}\right]\right|\leqslant 1$. Similarly $\lambda_k=j_2$ (j_2-q_n if $j_2\geqslant q_n$) for $k=k_2+wq_n$ and

$$\{(k_2+wq_n)\xi\}\epsilon[a,b]\quad\text{for}\quad w\leqslant\left[\frac{t}{q_n\delta_n}\right]-2,$$

$$\{(k_2+wq_n)\xi\}\notin[a,b]\quad\text{for}\quad w\geqslant\left[\frac{t}{q_n\delta_n}\right]+1.$$

These arguments show that exactly (j_2+j_1-1) of the values $\{(k+wq_n)\xi\}$, $1\leqslant k\leqslant q_n$, belong to $[a,b]$ if

$$w\leqslant\min\left(\left[\frac{1-s}{q_n\delta_n}\right],\left[\frac{t}{q_n\delta_n}\right]\right)-2\quad\text{or}\quad w>\max\left(\left[\frac{1-s}{q_n\delta_n}\right],\left[\frac{t}{q_n\delta_n}\right]\right)+1.$$

If

$$\min\left(\left[\frac{1-s}{q_n\delta_n}\right],\left[\frac{t}{q_n\delta_n}\right]\right)+1\leqslant w\leqslant\max\left(\left[\frac{1-s}{q_n\delta_n}\right],\left[\frac{t}{q_n\delta_n}\right]\right)-2$$

there will be j_2-j_1+2 values $\{(k+wq_n)\xi\}\epsilon[a,b]$, $1\leqslant k\leqslant q_n$, in case $1-s<t$ but only j_2-j_1 in case $t<1-s$. For all other values of $w\leqslant a_{n+1}-1$ the number will be between j_2-j_1 and j_2-j_1+2. This already proves (2.8) because

$$j_2-j_1\leqslant q_n(b-a)=j_2-j_1+s+t\leqslant j_2-j_1+2 \tag{2.18}$$

and therefore

$$\left|\sum_{wq_n+1}^{(w+1)q_n}f(k\xi;\,a,b)-q_n(b-a)\right|\leqslant 2$$

uniformly in (a,b). For the other parts of Lemmas 1 and 2 we need to refine this argument. It is necessary to distinguish between the cases $1-s\leqslant t$ and $t<1-s$ and in addition one has to take into account whether or not c exceeds one or both the numbers $\left[\frac{1-s}{q_n\delta_n}\right]$ and $\left[\frac{t}{q_n\delta_n}\right]$.

As a generic example we consider the case where

$$\left[\frac{t}{q_n\delta_n}\right]\leqslant c\leqslant\left[\frac{1-s}{q_n\delta_n}\right]. \tag{2.19}$$

In this case the number of $\{k\xi\}\in[a,b]$, $1\leqslant k\leqslant cq_n$, is

$$\left[\frac{t}{q_n\delta_n}\right](j_2-j_1+1)+\left(c-\left[\frac{t}{q_n\delta_n}\right]\right)(j_2-j_1)+6\theta_1.$$

Here and in the sequel θ_i will denote a constant of absolute value at most one. The term $6\theta_1$ comes in for the ambiguity in the number of $\{(k_i+wq_n)\xi\}\in[a,b]$, when $i=1$ and $\left|w-\left[\frac{1-s}{q_n\delta_n}\right]\right|\leqslant 1$ respectively $i=2$ and $\left|w-\left[\frac{t}{q_n\delta_n}\right]\right|\leqslant 1$. Using (2.18) we obtain for this case

$$R_{cq_n}(\xi;\ a,b)=\left[\frac{t}{q_n\delta_n}\right]-c(s+t)+6\theta_1.$$

By the definition (2.15) of δ_n

$$a_{n+1}t-1\leqslant\left[\left(a'_{n+1}+\frac{q_{n-1}}{q_n}\right)t\right]=\left[\frac{t}{q_n\delta_n}\right]\leqslant a_{n+1}t+2 \tag{2.20}$$

and therefore, in case of (2.19),

$$\begin{aligned} R_{cq_n}(\xi;\ a,b)&=a_{n+1}t-c(t+s)+8\theta_2 \\ &=a_{n+1}\left(t\left(1-\frac{c}{a_{n+1}}\right)-s\frac{c}{a_{n+1}}\right)+8\theta_2. \end{aligned} \tag{2.21}$$

From (2.21) we see immediately that (2.7), (2.10u) and (2.12) (still in the case (2.19)) imply (2.13) when u is sufficiently large. Similar computations show that also for the cases

$$c\leqslant\left[\frac{t}{q_n\delta_n}\right]\leqslant\left[\frac{1-s}{q_n\delta_n}\right]\quad\text{and}\quad\left[\frac{t}{q_n\delta_n}\right]\leqslant\left[\frac{1-s}{q_n\delta_n}\right]\leqslant c$$

(2.7), (2.10u) and (2.12) imply (2.13) as soon as u is sufficiently large. This proves Lemma 2.

In order to complete the proof of Lemma 1 we maximize in (2.21) with respect to s and t with the restrictions (2.19) while c is kept fixed. One trivially obtains from (2.21) and (2.15), that under the conditions (2.19)

$$\begin{aligned} R_{cq_n}(\xi;\ a,b)&\leqslant a_{n+1}q_n\delta_n(c+1)\left(1-\frac{c}{a_{n+1}}\right)+8 \\ &\leqslant a_{n+1}\frac{c+1}{a_{n+1}}\left(1-\frac{c}{a_{n+1}}\right)+8\leqslant a_{n+1}\left(\frac{c}{a_{n+1}}\left(1-\frac{c}{a_{n+1}}\right)+\varepsilon\right) \end{aligned}$$

for all $a_{n+1}\geqslant\dfrac{9}{\varepsilon}$.

Again similar computations for the other cases show that for all $a=\dfrac{j_1-s}{q_n}$, $b=\dfrac{j_2+t}{q_n}$ with $0\leqslant s,t<1$, $0\leqslant a\leqslant b\leqslant a+1$

$$R_{cq_n}(\xi;\ a,b)\leqslant a_{n+1}\left(\frac{c}{a_{n+1}}\left(1-\frac{c}{a_{n+1}}\right)+\varepsilon\right)$$

as soon as $a_{n+1}\geqslant\dfrac{14}{\varepsilon}$. Since this holds uniformly in a,b, also

$$\sup_{0\leqslant a\leqslant b\leqslant a+1}R_{cq_n}(\xi;\ a,b)\leqslant a_{n+1}\left(\frac{c}{a_{n+1}}\left(1-\frac{c}{a_{n+1}}\right)+\varepsilon\right). \tag{2.22}$$

This completes the proof of Lemma 1 since the left-hand side of (2.22) actually is $cq_nD_{cq_n}(\xi)$. After all, $R_{cq_n}(\xi;\ 0,1)=0$ so that for $(a',b')=[0,1]-[a,b]$,

$$R_{cq_n}(\xi;\ a,b)=-R_{cq_n}(\xi;\ a',b').$$

THEOREM 1. *Let*

$$h(z)=z(1-z),\qquad 0\leqslant z\leqslant 1, \tag{2.23}$$

and let N be an integer with expansion

$$N=\sum_{n=0}^{m(N,\xi)}c_n(N,\xi)q_n(\xi)$$

as in (2.3)–(2.6). Then there exists for each $\varepsilon>0$ a $v=v(\varepsilon)$ such that

$$\left|ND_N(\xi)-\sum_{n=0}^{m(N,\xi)}a_{n+1}(\xi)h\left(\frac{c_n(N,\xi)}{a_{n+1}(\xi)}\right)\right|\leqslant\varepsilon\sum_{n=0}^{m(N,\xi)}a_{n+1}(\xi) \tag{2.24}$$

whenever

$$\sum_{n=0}^{m(N,\xi)}a_{n+1}(\xi)\geqslant v\big(m(N,\xi)+1\big). \tag{2.25}$$

Proof. Without risk of confusion we shall drop the arguments N and ξ in most functions. Introducing

$$\varrho_n=\varrho_n(N,\xi)=\sum_{i=n+1}^{m}c_iq_i\qquad(\varrho_m=0)$$

we have

$$\begin{aligned} ND_N(\xi)&\leqslant\sum_{n=0}^{m}\sup_{a,b}\left|\sum_{k=1}^{c_nq_n}f\big((\varrho_n+k)\xi;\ a,b\big)-c_nq_n(b-a)\right| \\ &=\sum_{n=0}^{m}\sup_{a,b}\left|\sum_{k=1}^{c_nq_n}f(k\xi;\ a-\varrho_n\xi,b-\varrho_n\xi)-c_nq_n(b-a)\right|=\sum_{n=0}^{m}c_nq_nD_{c_nq_n}(\xi). \end{aligned} \tag{2.26}$$

By Lemma 1, however,

$$\sum_{n=0}^{m} c_n q_n D_{c_n q_n}(\xi) \leqslant \sum_{\substack{0\leqslant n\leqslant m\\ a_{n+1}\leqslant u(\frac12\varepsilon)}}{}' 2c_n + \sum_{\substack{0\leqslant n\leqslant m\\ a_{n+1}>u(\frac12\varepsilon)}} a_{n+1}\left(h\left(\frac{c_n}{a_{n+1}}\right)+\frac{\varepsilon}{2}\right)$$

$$\leqslant 2(m+1)u\left(\frac{\varepsilon}{2}\right)+\sum_{n=0}^{m} a_{n+1}h\left(\frac{c_n}{a_{n+1}}\right)+\frac{\varepsilon}{2}\sum_{n=0}^{m} a_{n+1}$$

$$\leqslant \sum_{n=0}^{m} a_{n+1}h\left(\frac{c_n}{a_{n+1}}\right)+\varepsilon\sum_{n=0}^{m} a_{n+1}$$

as soon as $\sum a_{n+1} \geqslant \frac{4u(\frac12\varepsilon)}{\varepsilon}(m+1)$. This proves the required upper bound for D_N. For the lower bound we use the following analogue of (2.26)

$$ND_N(\xi) \geqslant \sup_{a,b}\left|\sum_{n=0}^{m}\sum_{k=1}^{c_nq_n} f(k\xi;\, a-\varrho_n\xi,\, b-\varrho_n\xi)-c_nq_n(b-a)\right| \tag{2.27}$$

$$\geqslant \sup_{a,b}\left|\sum_{\substack{0\leqslant n\leqslant m\\ a_{n+1}>64u(\frac12\varepsilon)}}\sum_{k=1}^{c_nq_n} f(k\xi;\, a-\varrho_n\xi,\, b-\varrho_n\xi)-c_nq_n(b-a)\right|$$

$$-\sum_{\substack{0\leqslant n\leqslant m\\ a_{n+1}\leqslant 64u(\frac12\varepsilon)}} c_nq_nD_{c_nq_n}(\xi).$$

By (2.8) and (2.7)

$$\sum_{\substack{0\leqslant n\leqslant m\\ a_{n+1}\leqslant 64u(\frac12\varepsilon)}} c_nq_nD_{c_nq_n}(\xi) \leqslant 128(m+1)u(\tfrac12\varepsilon). \tag{2.28}$$

We now show that there exist a and b such that

$$R_{c_nq_n}(\xi,\, a-\varrho_n\xi,\, b-\varrho_n\xi) \tag{2.29}$$

$$=\sum_{k=1}^{c_nq_n} f(k\xi;\, a-\varrho_n\xi,\, b-\varrho_n\xi)-c_nq_n(b-a) \geqslant a_{n+1}\left(h\left(\frac{c_n}{a_{n+1}}\right)-\frac{\varepsilon}{2}\right),$$

simultaneously for all $0\leqslant n\leqslant m$ with

$$a_{n+1} > 64u(\tfrac12\varepsilon). \tag{2.30}$$

For the sake of argument we again assume that n is even and that (2.30) holds. Then by Lemma 2 (2.29) will hold if

$$a-\varrho_n\xi = L+\frac{j_1-s}{q_n},\qquad b-\varrho_n\xi = L+\frac{j_2+t}{q_n} \tag{2.31}$$

for some integers L, j_1, j_2 and numbers s and t satisfying (2.10u) and (2.12). But for each i

$$q_{i+2} = (a_{i+2}a_{i+1}+1)q_i+q_{i-1} \geqslant 2q_i$$

and thus

$$q_{i+j} \geqslant 2^{3/j}q_i \quad \text{for} \quad j\geqslant 2. \tag{2.32}$$

Consequently, if (2.30) holds

$$\{\varrho_n\xi\} = \left\{\sum_{i=n+1}^{m} c_iq_i\xi\right\} = \left\{\sum_{i=n+1}^{m} c_iq_i\left(\frac{p_i+\delta_i}{q_i}\right)\right\} = \left\{\sum_{i=n+1}^{m} c_i\delta_i\right\}$$

$$\leqslant \sum_{i=n+1}^{m}\frac{a_{i+1}}{a'_{i+1}q_i+q_{i-1}} \leqslant \sum_{i=n+1}^{\infty}\frac{1}{q_i} \leqslant \frac{2^{1/3}}{2^{1/3}-1}\cdot\frac{1}{q_{n+1}} \leqslant \frac{1}{8u(\frac12\varepsilon)}\cdot\frac{1}{q_n}.$$

Hence (2.31) is satisfied whenever

$$a = \frac{j_{1,n}-s_n}{q_n},\qquad b = \frac{j_{2,n}+t_n}{q_n} \tag{2.33}$$

with $j_{1,n}, j_{2,n}$ satisfying (2.12) and, abbreviating $u(\frac12\varepsilon)$ by u

$$\frac{1}{8u}\leqslant t\leqslant 1-\frac{1}{4u}-s\leqslant 1-\frac{3}{8u},\qquad \frac{1}{8u}\leqslant s\leqslant\frac{7}{8u},\quad \left|t-\frac{c_n}{a_{n+1}}\right|\leqslant\frac{7}{8u}. \tag{2.34}$$

Similarly, if n is odd and (2.30) holds, (2.29) will follow as soon as (2.33) holds for s_n, t_n satisfying

$$\frac{1}{8u}\leqslant s\leqslant 1-\frac{1}{4u}-t\leqslant 1-\frac{3}{8u},\qquad \left|s-\frac{c_n}{a_{n+1}}\right|\leqslant\frac{7}{8u},\quad \frac{1}{8u}\leqslant t\leqslant\frac{7}{8u}. \tag{2.35}$$

This reduces the problem to showing that there exist a and b which simultaneously satisfy (2.33) and (2.34) for all even $n\leqslant m$ for which (2.30) holds and (2.33) and (2.35) for all odd $n\leqslant m$ for which (2.30) holds. This really is not hard. In fact each of the conditions (2.34) and (2.35) allow s_n and t_n to vary independently over intervals of length at least $1/4u$. In view of (2.33) and (2.30) this means that a and b can vary independently over intervals of length

$$\frac{1}{4uq_n} \geqslant \frac{16}{a_{n+1}q_n} \geqslant \frac{16}{q_{n+1}}.$$

This allows us to choose s_n and t_n inductively. Assume that we have found intervals I_1 and I_2 each of length at least $16/q_{k+1}$ such that for any $a \in I_1$, $b \in I_2$ (2.33) and (2.34) (resp. (2.35)) are satisfied for all $n \leqslant k$ with (2.30). If then $a_{k+2}, \ldots, a_{k+i} < 64u$ but $a_{k+i+1} > 64u$ no restrictions are required on $j_{k+l,1}, j_{k+l,2}, s_{k+l}, t_{k+l}$ for $1 \leqslant l \leqslant i-1$ whereas we can find $j_{k+i,1}$ and $j_{k+i,2}$ such that

$$I_1' = \left[\frac{j_{k+i,1}-1}{q_{k+i}}, \frac{j_{k+i,1}}{q_{k+i}}\right] \subseteq I_1 \quad \text{and} \quad I_2' = \left[\frac{j_{k+i,2}}{q_{k+i}}, \frac{j_{k+i,2}+1}{q_{k+i}}\right] \subseteq I_2.$$

For these values of $j_{k+i,1}, j_{k+i,2}\ s_{k+i}, t_{k+i}$ can be any numbers between $[0,1]$ and still (cf. (2.33) for $n = k+i$)

$$a \in I_1, \qquad b \in I_2.$$

Therefore, we can find intervals $I_1'' \subseteq I_1'$ and $I_2'' \subseteq I_2'$, each of length at least $16/q_{k+i+1}$ such that for all $a \in I_1''$, $b \in I_2''$. (2.33) and (2.34) (resp. (2.35)) are satisfied for all $n \leqslant k+i$ with $a_{n+1} > 64u$. Continuing in this manner we find two intervals $\tilde{I}_1, \tilde{I}_2$ of length at least $16/q_{m+1}$ such that for any $a \in \tilde{I}_1$, $b \in \tilde{I}_2$ and any $n \leqslant m$ with $a_{n+1} > 64u$ (2.29) is satisfied. This, together with (2.27) and (2.28) shows

$$ND_N(\xi) \geqslant \sum_{\substack{0 \leqslant n \leqslant m \\ a_{n+1} > 64u}} a_{n+1}\left(h\left(\frac{c}{a_{n+1}}\right) - \frac{\varepsilon}{2}\right) - 128(m+1)u$$
$$\geqslant \sum_{n=0}^{m} a_{n+1}\left(h\left(\frac{c}{a_{n+1}}\right) - \varepsilon\right)$$

as soon as

$$\sum_{n=0}^{m} a_{n+1} \geqslant \frac{512u(\frac{1}{2}\varepsilon)}{\varepsilon}(m+1).$$

This proves Theorem 1.

3. The asymptotic behavior of $\dfrac{ND_N(x)}{\log N \cdot \log\log N}$. In this section we prove our main result, namely

THEOREM 2.

$$\frac{ND_N(x)}{\log N \cdot \log\log N} \to \frac{2}{\pi^2} \text{ in measure on } [0,1].$$

We use probabilistic terminology and consider x as a random variable uniformly distributed in $[0,1]$, i.e. if A is any event, $P\{A\}$ will denote the Lebesgue measure of the set of $x \in [0,1]$ for which A occurs. In particular $P\{g(x) \in B\} =$ Lebesgue measure of $\{x\colon g(x) \in B\}$. (Only in the proof of Lemma 3 will we use another probability measure.) Similarly one defines the conditional probability of A given B, $P\{A \mid B\} = \dfrac{P\{A \cap B\}}{P\{B\}}$.

The following facts can be found in the indicated references:

(i) $\lim\limits_{n\to\infty} \dfrac{1}{n} \log q_n(x) = \dfrac{1}{\tau}$ a.e., where $\tau = \dfrac{12\log 2}{\pi^2}$ ([7] and p. 320 of [8]).

(ii) For all $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{\left|\frac{\sum_{i=k}^{k+n} a_{i+1}(x)}{n\log n} - \frac{1}{\log 2}\right| > \varepsilon\right\} = 0$$

uniformly in k. In [6] Khintchine proves this for $k = 0$ only but all his estimates are uniform in k.

(iii) For all $\varepsilon > 0$

$$\lim_{n\to\infty} P\left\{\sum_{i=k}^{k+n} a_{i+1}^2(x) \geqslant \varepsilon n^2 \log n\right\} = 0$$

uniformly in k. A more precise result is indicated in p. 322 of [8] but an easy proof follows from Markov's inequality ([9], p. 158) and arguments similar to those in [6]. In fact

$$\sum_{i=k}^{k+n} P\{a_{i+1}(x) > n\log\log n\} = O\left(\frac{1}{\log\log n}\right)$$

(analogous to formula (28) on p. 378 in [6]) and

$$\sum_{i=k}^{k+n} \int_{a_{i+1}(\xi) \leqslant n\log\log n} a_{i+1}^2(\xi)\,d\xi = O(n^2 \log\log n)$$

(analogous to formula (21) of [6]).

(iv) If

$$y_n(x) = [a_{n+1}(x), a_{n+2}(x), \ldots] \tag{3.1}$$

then, for any measurable set $B \subseteq [0,1]$,

$\frac{1}{2}P\{x \in B\} = \frac{1}{2}$ Lebesgue measure of B

$\leqslant P\{y_n(x) \in B \mid a_i(x) = a_i,\ i = 1, \ldots, n\} \leqslant 2$ (Lebesgue measure of B),

(the conditional probability density of $y_n(x)$ is estimated by differentiation of formula 8, p. 292 of [8]).

Writing, for integral N,

$$N = \sum_{n=0}^{m(N,x)} c_n(N, x) q_n(x)$$

as in (2.3)–(2.6), we have

$$q_m(x) \leqslant N < q_{m+1}(x)$$

and hence, by (i),

$$\lim_{N\to\infty} \frac{m(N, x)}{\log N} = \tau \quad \text{a.e.} \tag{3.2}$$

If we take $0 < \eta < \dfrac{\varepsilon \log 2}{2}$ in the next few lines, we obtain from (ii)[1]

$$\begin{aligned}
\lim_{N\to\infty} P\left\{\left|\frac{\sum_{n=0}^{m(N,x)} a_{n+1}(x)}{\log N \log\log N} - \frac{\tau}{\log 2}\right| > \varepsilon\right\}
&\leqslant \lim_{N\to\infty} P\{|m(N, x) - \tau \log N| \geqslant \eta \log N\} + \\
&+ \lim_{N\to\infty} P\left\{\sum_{n=0}^{(\tau-\eta)\log N} a_{n+1}(x) \leqslant \left(\frac{\tau}{\log 2} - \varepsilon\right) \log N \log\log N\right\} + \\
&+ \lim_{N\to\infty} P\left\{\sum_{n=0}^{(\tau+\eta)\log N} a_{n+1}(x) \geqslant \left(\frac{\tau}{\log 2} + \varepsilon\right) \log N \log\log N\right\} = 0.
\end{aligned} \tag{3.3}$$

(3.2) and (3.3) together with Theorem 1 show that for every $\varepsilon > 0$

$$\lim_{N\to\infty} P\left\{\left|ND_N(x) - \sum_{n=0}^{m(N,x)} a_{n+1}(x) h\left(\frac{c_n(N, x)}{a_{n+1}(x)}\right)\right| > \varepsilon \log N \log\log N\right\} = 0. \tag{3.4}$$

We shall now prove in a sequence of lemmas that for every function $g(\xi)$ with bounded derivative on $[0,1]$ and for every $\varepsilon > 0$

$$\lim_{N\to\infty} P\left\{\left|\sum_{n=0}^{m(N,x)} a_{n+1}(x)\left(g\left(\frac{c_n(N, x)}{a_{n+1}(x)}\right) - \int_0^1 g(\xi)\,d\xi\right)\right| \geqslant \varepsilon \sum_{n=0}^{m(N,x)} a_{n+1}(x)\right\} = 0. \tag{3.5}$$

This will prove Theorem 2 since

$$\int_0^1 h(\xi)\,d\xi = \int_0^1 \xi(1-\xi)\,d\xi = \frac{1}{6}$$

[1] We freely use expressions such as $(\tau-\eta)\log N$ as bounds in summations. In reality these bounds should be integers. Both $[(\tau-\eta)\log N]$ and $[(\tau-\eta)\log N]+1$ are admissible in most of our formulae.

and (3.4), (3.5), (3.3) and the value of τ imply

$$\lim_{N\to\infty} P\left\{\left|ND_N(x) - \frac{2}{\pi^2}\log N \log\log N\right| \geqslant 4\varepsilon \log N \log\log N\right\} = 0$$

for every $\varepsilon > 0$.

In the following lemmas $g(\xi)$ is a differentiable function on $[0,1]$,

$$\tilde{g}(\xi) = g(\xi) - \int_0^1 g(\xi)\,d\xi$$

and we assume

$$|\tilde{g}(\xi)| \leqslant C_0, \qquad |\tilde{g}'(\xi)| \leqslant C_0. \tag{3.6}$$

In the Lemmas 3, 4 and the Corollary we expand any $z \geqslant 0$ as

$$z = \sum_{n=u}^{\infty} c_n(z) q_n + r_n(z)$$

where $0 \leqslant c_n(z) \leqslant a_{n+1}$ and $r_n(z) = z - \sum_{i=n}^{\infty} c_i(z) q_i < q_n$. The c's and a's are integers ($a_n \geqslant 1$) but we do not insist that the q_n are integers. We only require $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $n \geqslant u$, $0 \leqslant q_{u-1}$, $1 \leqslant q_u < q_{u+1} \ldots$ In particular one still has (cf. (2.32)) $q_{u+j} \geqslant q_u 2^{j/3}$ for $j \geqslant 2$.

LEMMA 3[2]. *Let $\lambda > 0$. Then the number of integers $k \in [0, q_{u+v})$ for which*

$$\left|\sum_{n=u}^{u+v-1} a_{n+1} \tilde{g}\left(\frac{c_n(k)}{a_{n+1}}\right)\right| \geqslant \lambda C_0 \Big(\sum_{n=u}^{u+v-1} a_{n+1}^2\Big)^{1/2} + C_0\Big(\sum_{n=u}^{u+2} a_{n+1} + 4\sum_{n=u+3}^{u+v-1} \frac{a_{n+1}}{q_n} + 8v\Big)$$

is at most

$$\frac{12([q_{u+v}]+1)}{\lambda^2}.$$

[2] We only apply the Lemmas 3 and 4 when g equals h. The greater generality may be useful when studying $\sum_{k=0}^{N} (\{kx\} - \frac{1}{2})$. For this reason we also want to point out that Lemmas 3 and 4 and their proofs remain valid when the sums over n are restricted to odd n's or to even n's.

If $\mu \geqslant 0$, $0 \leqslant s \leqslant v$ and $q_{u+v-s} \geqslant 1$ then the number of integers $k \in [0, q_{u+v})$ for which

$$(3.7)\qquad 0 \leqslant r_{u+v-s}(k) \leqslant \mu \quad \text{or} \quad q_{u+v-s}-\mu \leqslant r_{u+v-s}(k) \leqslant q_{u+v-s}$$

is at most $\dfrac{4(\mu+1)}{q_{u+v-s}-1}\cdot(q_{u+v}+1)$.

Proof. Consider the probability space $\{0, 1, \ldots, [q_{n+v}]\}$ in which each point has probability $1/([q_{u+v}]+1)$. Just for this proof $P\{\cdot\}$ will refer to this probability measure. (If $[q_{u+v}] \notin [0, q_{u+v})$ we replace this by $P\{k\} = 1/q_{u+v}$, $0 \leqslant k < q_{u+v}$.) The first part of the lemma can then be restated as

$$(3.8)\qquad P\Bigg\{\Bigg|\sum_{n=u}^{u+v-1} a_{n+1}\tilde{g}\left(\frac{c_n(k)}{a_{n+1}}\right)\Bigg| \geqslant \lambda C_0\Big(\sum_{n=u}^{u+v-1} a_{n+1}^2\Big)^{1/2} + C_0\Big(\sum_{n=u}^{u+2} a_{n+1} + 4\sum_{n=u+3}^{u+v-1}\frac{a_{n+1}}{q_n} + 8v\Big)\Bigg\} \leqslant \frac{12}{\lambda^2}$$

which we prove by an application of Tchebychev's inequality. We introduce the intervals $J_{b_1,\ldots,b_s}$ $(1 \leqslant s \leqslant v,\ 0 \leqslant b_i \leqslant a_{u+v-i+1})$ as the set of $z \in [0, q_{u+v})$ with $c_{u+v-i}(z) = b_i$, $i = 1, \ldots, s$. By $|J|$ we denote the length of J. The number of integers in J is $|J|+\theta$ for some $|\theta| \leqslant 1$. In general θ_i will stand for a number of absolute value at most one.

Since $q_{n+1} = a_{n+1}q_n + q_{n-1}$,

$$J_{b_1} = [b_1 q_{u+v-1},\ (b_1+1)q_{u+v-1}) \quad \text{for} \quad 0 \leqslant b_1 < a_{u+v},$$

and

$$J_{a_{u+v}} = [a_{u+v}q_{u+v-1},\ q_{u+v}) = [a_{u+v}q_{u+v-1},\ a_{u+v}q_{u+v-1}+q_{u+v-2}).$$

At the next step, each interval J_{b_1} with $b_1 < a_{u+v}$ is the union of adjacent intervals $J_{b_1,0}, J_{b_1,1}, \ldots, J_{b_1,a_{u+v-1}}$ where

$$|J_{b_1,b_2}| = q_{u+v-2} \quad \text{if} \quad b_2 < a_{u+v-1} \quad \text{and} \quad |J_{b_1,a_{u+v-1}}| = q_{u+v-3}$$

whereas

$$J_{a_{u+v}} = J_{a_{u+v},0} \quad \text{and} \quad |J_{a_{u+v},0}| = q_{u+v-2}.$$

In general, one shows that $J_{b_1,\ldots,b_s}$ is non empty only if $b_i < a_{u+v-i+1}$ or $b_i = a_{u+v-i+1}$, $b_{i+1} = 0$ $(i = 1, \ldots, s)$. If this is the case and $b_s < a_{u+v-s+1}$, then

$$(3.9a)\qquad |J_{b_1,\ldots,b_s}| = q_{u+v-s},$$

$$(3.9b)\qquad J_{b_1,\ldots,b_s} = \bigcup_{0\leqslant b_{s+1}\leqslant a_{u+v-s}} J_{b_1,\ldots,b_{s+1}},$$

$$(3.9c)\qquad |J_{b_1,\ldots,b_{s+1}}| = q_{u+v-s-1} \quad \text{for} \quad 0 \leqslant b_{s+1} < a_{u+v-s},$$

$$(3.9d)\qquad |J_{b_1,\ldots,b_s,a_{u+v-s}}| = q_{u+v-s-2}.$$

Moreover, if $b_s = a_{u+v-s+1}$ and $J_{b_1,\ldots,b_s}$ is not empty, then

$$(3.10a)\qquad J_{b_1,\ldots,b_{s-1},a_{u+v-s+1}} = J_{b_1,\ldots,b_{s-1},a_{u+v-s+1},0},$$

$$(3.10b)\qquad |J_{b_1,\ldots,b_{s-1},a_{u+v-s+1}}| = q_{u+v-s-1}.$$

The conditional distribution of $c_{u+v-s-1}(k)$, given $c_{u+v-1}(k) = b_1, \ldots, c_{u+v-s+1}(k) = b_{s-1}$, or equivalently given $k \in J_{b_1,\ldots,b_{s-1}}$, is now easily determined. First assume $b_{s-1} < a_{u+v-s+2}$ and $0 < b_{s+1} < a_{u+v-s}$. Then, by (3.9), (3.10) for $k \in J_{b_1,\ldots,b_{s-1}}$, $c_{s+1}(k)$ will equal b_{s+1} if and only if

$$(3.11)\qquad k \in \bigcup_{0\leqslant b<a_{u+v-s+1}} J_{b_1,\ldots,b_{s-1},b,b_{s+1}}$$

which contains $a_{u+v-s+1}$ intervals of length $q_{u+v-s+1}$. We cannot allow $b = a_{u+v-s+1}$ in (3.11) because then b_{s+1} must be zero (cf. (3.10a)). Thus if $b_{s-1} < a_{u+v-s+2}$ ([3])

$$(3.12a)\qquad P\{c_{u+v-s-1}(k) = b_{s+1} \mid c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\} = \frac{a_{u+v-s+1}(q_{u+v-s-1}+\theta_1)}{|J_{b_1,\ldots,b_{s-1}}|+\theta_2} = \frac{a_{u+v-s+1}(q_{u+v-s-1}+\theta_1)}{q_{u+v-s+1}+\theta_2},$$

$0 < b_{s+1} < a_{u+v-s}$.

If $b_{s+1} = 0$ we also can take $b = a_{u+v-s+1}$ in (3.11) so that

$$(3.12b)\qquad P\{c_{u+v-s-1}(k) = 0 \mid c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\} = \frac{(a_{u+v-s+1}+1)(q_{u+v-s-1}+\theta_3)}{q_{u+v-s+1}+\theta_2}.$$

Finally,

$$(3.12c)\qquad P\{c_{u+v-s-1}(k) = a_{u+v-s} \mid c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\} = \frac{a_{u+v-s+1}(q_{u+v-s-2}+\theta_4)}{q_{u+v-s+1}+\theta_2}$$

since $c_{u+v-s-1}(k) = a_{u+v-s}$ only occurs if

$$k \in \bigcup_{0\leqslant b<a_{u+v-s+1}} J_{b_1,\ldots,b_{s-1},b,a_{u+v-s}}$$

([3]) Strictly speaking this conditional probability is defined only if $P\{c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\} > 0$ but as usual we may take any value for the conditional probability if $P\{c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\} = 0$ without affecting the argument.

which consists of $a_{u+v-s+2}$ intervals of length $q_{u+v-s-2}$ (cf. (3.9d)). The formulae (3.12) show that, as long as $q_{u+v-s-1} \geqslant 2$ (and thus for all $u+v-s-1 \geqslant u+3$) (4)

$$E\left\{a_{u+v-s}\tilde{g}\left(\frac{c_{u+v-s-1}(k)}{a_{u+v-s}}\right)\middle| c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\right\}$$

$$= a_{u+v-s}\left(\sum_{b=0}^{a_{u+v-s-1}} \frac{a_{u+v-s+1}q_{u+v-s-1}}{q_{u+v-s+1}+\theta_2}\tilde{g}\left(\frac{b}{a_{u+v-s}}\right)+\right.$$

$$\left.+\frac{q_{u+v-s-1}}{q_{u+v-s+1}+\theta_2}\tilde{g}(0)+\frac{a_{u+v-s+1}q_{u+v-s-2}}{q_{u+v-s+1}+\theta_2}\tilde{g}(1)\right)+4\theta_5 C_0\frac{a_{u+v-s}}{q_{u+v-s-1}}$$

$$= \frac{a_{u+v-s+1}a_{u+v-s}q_{u+v-s-1}}{(a_{u+v-s+1}a_{u+v-s}+1)q_{u+v-s-1}+q_{u+v-s-2}+\theta_2}\sum_{b=0}^{a_{u+v-1}-1}\tilde{g}\left(\frac{b}{a_{u+v-s}}\right)+$$

$$+4\theta_6 C_0\left(\frac{a_{u+v-s}}{q_{u+v-s-1}}+1\right),$$

cf. (3.6). Since $|\tilde{g}'(\xi)| \leqslant C_0$ and $\int_0^1 \tilde{g}(\xi)d\xi = 0$, $b_{s-1} < a_{u+v-s+2}$ implies

$$(3.13)\quad E\left\{a_{u+v-s}\tilde{g}\left(\frac{c_{u+v-s-1}(k)}{a_{u+v-s}}\right)\middle| c_{u+v-i}(k) = b_i,\ 1 \leqslant i \leqslant s-1\right\}$$

$$= \frac{a_{u+v-s+1}a_{u+v-s}q_{u+v-s-1}}{(a_{u+v-s+1}a_{u+v-s}+1)q_{u+v-s-1}+q_{u+v-s-2}+\theta_2}a_{u+v-s}\int_0^1\tilde{g}(\xi)d\xi+$$

$$+4\theta_7 C_0\left(\frac{a_{u+v-s}}{q_{u+v-s-1}}+2\right) = 4\theta_7 C_0\left(\frac{a_{u+v-s}}{q_{u+v-s-1}}+2\right).$$

Actually (3.13) remains valid even if $b_{s-1} = a_{u+v-s+2}$. This easily follows from

$$J_{b_1,\ldots,b_{s-2},a_{u+v-s+2}} = \bigcup_{0\leqslant b\leqslant a_{u+v-s}} J_{b_1,\ldots,b_{s-2},a_{u+v-s+2},0,b}$$

(4) For our simple, finite probability space the definition of the expectation $E\{Y\}$ of a random variable $Y(\cdot)$ defined on the probability space $\{0, 1, \ldots, [q_{u+v}]\}$ becomes $E\{Y\} = \sum_k Y(k)P\{k\}$ where $P\{k\} = ([q_{u+v}]+1)^{-1}$, the probability assigned to the point k. The conditional expectation, given the event B, becomes $E\{Y|B\} = \sum_{k\in B}\frac{Y(k)P\{k\}}{P\{B\}}$ where $P\{B\} = \sum_{k\in B}P\{k\}$. This of course is only well defined if $P\{B\} \neq 0$. For our particular simple case all properties of conditional expectations reduce to trivial properties of finite sums, and as far as we use it, we may take $E\{Y|B\}$ any finite number if $P\{B\} = 0$.

by an argument similar to the above. Putting

$$Z_s(k) = a_{u+v-s}\tilde{g}\left(\frac{c_{u+v-s-1}(k)}{a_{u+v-s}}\right) - E\left\{a_{u+v-s}\tilde{g}\left(\frac{c_{u+v-s-1}(k)}{a_{u+v-s}}\right)\middle| c_{u+v-i}(k),\ 1 \leqslant i \leqslant s-1\right\}$$

one has trivially

$$E\{Z_s^2(k)\} \leqslant 4C_0^2 a_{u+v-s}^2.$$

We can now write (by 3.13)

$$(3.14)\quad \sum_{n=u}^{u+v-1} a_{n+1}\tilde{g}\left(\frac{c_n(k)}{a_{n+1}}\right) = \sum_{s=0}^{v-4} Z_s + \theta_8 C_0\left(\sum_{n=u}^{u+2} a_{n+1} + 4\sum_{n=u+3}^{u+v-1}\frac{a_{n+1}}{q_n} + 8v\right)$$

where, by elementary properties of conditional expectations (cf. [9], p. 386)

$$E\left(\sum_{s=0}^{v-4} Z_s\right)^2 = \sum_{s=0}^{v-4} EZ_s^2 + 2\sum_{s_1<s_2} EZ_{s_1}Z_{s_2} = \sum_{s=0}^{v-4} EZ_s^2 + 2\sum_{s=0}^{v-5} EZ_sZ_{s+1}$$

$$\leqslant 3\sum_{s=0}^{v-4} EZ_s^2 \leqslant 12C_0^2\sum_{n=u}^{u+v-1} a_{n+1}^2.$$

By Tchebychev's inequality ([9], p. 158), therefore

$$P\left\{\left|\sum_{s=0}^{v-4} Z_s\right| \geqslant \lambda C_0\left(\sum_{n=u}^{u+v-1} a_{n+1}^2\right)^{1/2}\right\} \leqslant \frac{12}{\lambda^2}$$

which together with (3.14) implies (3.8) and hence the first part of the lemma. The second part is easy now since, for any non-empty $J_{b_1,\ldots,b_{s-1}}$

$$|J_{b_1,\ldots,b_{s-1}}| = q_{u+v-s+1} \quad\text{or}\quad q_{u+v-s}.$$

In the first case there are at most $(a_{u+v-s+1}+1)2(\mu+1)$ integers $k \in J_{b_1,\ldots,b_{s-1}}$ which satisfy (3.7) and in the second case at most $2(\mu+1)$. In each case

$$P\{(3.7)\text{ holds}\,|\,k \in J_{b_1,\ldots,b_{s-1}}\} \leqslant \frac{4(\mu+1)}{q_{u+v-s}+\theta_9}$$

which immediately implies the last statement of the lemma.

LEMMA 4. *Let $q_{u+v} > z_1 > z_2 > \ldots > z_t \geqslant 0$ be real numbers such that $z_i - z_{i+1} \geqslant 1/d > 0$. Then for every s with $u+2 \leqslant u+v-s-1 \leqslant \leqslant u+v-1$, the number of indices $1 \leqslant j \leqslant t$ with*

$$\left|\sum_{n=u}^{u+v-1} a_{n+1}\tilde{g}\left(\frac{c_n(z_j)}{a_{n+1}}\right)\right|$$

$$\geqslant \lambda C_0\left(\sum_{n=u}^{u+v-1} a_{n+1}^2\right)^{1/2} + C_0\left(4\sum_{n=u+3}^{u+v-1}\frac{a_{n+1}}{q_n} + 8v + 3\sum_{n=u}^{u+v-s-1} a_{n+1}\right)$$

is at most

$$d+\frac{24d(q_{u+v}+1)}{\lambda^2}+\frac{16d(q_{u+v}+1)}{q_{u+v-s}}.$$

If $q_{u+v-s}\geqslant 1$, *then the number of indices* $1\leqslant j\leqslant t$ *with*

$$0\leqslant r_{u+v-s}(z_j)\leqslant\mu \quad \textit{or} \quad q_{u+v-s}-\mu\leqslant r_{u+v-s}(z_j)\leqslant q_{u+v-s}$$

is at most

$$d+\frac{4d(\mu+3)}{q_{u+v-s}-1}(q_{u+v}+1).$$

Proof. Let k_j be the unique integer with $k_j\leqslant z_j<k_j+1$. There are at most d points z_j with $k_j+1\geqslant q_{u+v}$. We disregard those in the argument below. It is easy to see that if

$$c_n(k_j)=c_n(k_j+1) \quad \text{for} \quad u+v-s\leqslant n\leqslant u+v-1 \tag{3.15}$$

then

$$c_n(k_j)=c_n(z_j) \quad \text{for} \quad u+v-s\leqslant n\leqslant u+v-1.$$

Consequently, whenever (3.15) holds, as well as

$$\Big|\sum_{n=u}^{u+v-1}a_{n+1}\tilde g\left(\frac{c_n(k)}{a_{n+1}}\right)\Big| \leqslant \lambda C_0\Big(\sum_{n=u}^{u+v-1}a_{n+1}^2\Big)^{1/2}+C_0\Big(\sum_{n=u}^{u+2}a_{n+1}+4\sum_{n=u+3}^{u+v-1}\frac{a_{n+1}}{q_n}+8v\Big) \tag{3.16}$$

for $k=k_j$ and $k=k_{j+1}$ then

$$\Big|\sum_{n=u}^{u+v-1}a_{n+1}\tilde g\left(\frac{c_n(z_j)}{a_{n+1}}\right)\Big| \leqslant \lambda C_0\Big(\sum_{n=u}^{u+v-1}a_{n+1}^2\Big)^{1/2}+C_0\Big(4\sum_{n=u+3}^{u+v-1}\frac{a_{n+1}}{q_n}+8v+3\sum_{n=u}^{u+v-s-1}a_{n+1}\Big). \tag{3.17}$$

But (3.15) holds unless $r_{u+v-s}(k_j+1)\leqslant 1$, which occurs for at most $8(q_{u+v}+1)/(q_{u-v-s}-1)$ values of k_j (by Lemma 3). Also by Lemma 3 (3.16) holds with the exception of at most $12(q_{u+v}+1)/\lambda^2$ values of k, so that both (3.15) and (3.16) for $k=k_j$ and $k=k_{j+1}$ hold for all but $\frac{16(q_{u+v}+1)}{q_{u+v-s}}+\frac{24(q_{u+v}+1)}{\lambda^2}$ values of j. But a fixed value k_j can correspond to at most d different z's since $z_i-z_{i+1}\geqslant 1/d$. The first part of the lemma follows and the second part is proved in a similar manner.

COROLLARY. *Let* $q_{u+v}>z_1>z_2>\ldots>z_t\geqslant 0$ *be real numbers such that* $z_i-z_{i+1}\geqslant\frac{1}{d}>0$. *If*

$$\sum_{n=u}^{u+v-1}a_{n+1}^2\leqslant\frac{v^2\log v}{C_0^2}, \tag{3.18}$$

$$\sum_{n=u}^{u+\log v-1}a_{n+1}\leqslant v, \tag{3.19}$$

$$\sum_{n=u}^{u+v-1}a_{n+1}\geqslant\frac{v\log v}{2\log 2}+\frac{4C_0}{\varepsilon v^{1/6}}\sum_{n=u}^{u+v-1}a_{n+1}+\frac{15vC_0}{\varepsilon}, \tag{3.20}$$

then for $v\geqslant v_0(d,\varepsilon)$ *the number of indices* $1\leqslant j\leqslant t$ *with*

$$\Big|\sum_{n=u}^{u+v-1}a_{n+1}\tilde g\left(\frac{c_n(z_j)}{a_{n+1}}\right)\Big|\geqslant\varepsilon\sum_{n=u}^{u+v-1}a_{n+1} \tag{3.21}$$

is at most $q_{u+v}/(\log v)^{1/2}$.

Proof. Under the conditions (3.18)-(3.20)

$$\varepsilon\sum_{n=u}^{u+v-1}a_{n+1}\geqslant$$

$$\frac{\varepsilon C_0(\log v)^{1/2}}{2\log 2}\Big(\sum_{n=u}^{u+v-1}a_{n+1}^2\Big)^{1/2}+4v^{-1/6}C_0\sum_{n=u+\log v}^{u+v-1}a_{n+1}+8vC_0+7C_0\sum_{n=u}^{u+\log v-1}a_{n+1}.$$

Since $q_{u+\log v}\geqslant q_u 2^{\log v/3}\geqslant v^{1/6}$ (cf. (2.32)) we can take $\lambda=\frac{\varepsilon(\log v)^{1/2}}{2\log 2}$ and $u+v-s=u+\log v$ in Lemma 4.

In the proof of the next two lemmas we need the following simple formulae for continued fractions

$$\frac{q_{n-1}(\xi)}{q_n(\xi)}=[a_n(\xi),a_{n-1}(\xi),\ldots,a_1(\xi)] \tag{3.22}$$

and consequently the "reversed" fraction $[a_n(\xi),\ldots,a_1(\xi)]$ has the same nth denominator as $[a_1(\xi),a_2(\xi),\ldots,a_n(\xi)]$, namely $q_n(\xi)$ (cf. [11], p. 27).

If $\frac{p_{n,k}(\xi)}{q_{n,k}(\xi)}$ is the $(k-n)$-th convergent of $y_n(\xi)=[a_{n+1}(\xi),a_{n+2}(\xi),\ldots]$ then

$$q_k(\xi)=q_{n,k}(\xi)q_n(\xi)+p_{n,k}(\xi)q_{n-1}(\xi)\cdot(k\geqslant n). \tag{3.23}$$

This is obvious for $k=n$, $n+1$ since $p_{n,n}=0$, $p_{n,n+1}=1$, $q_{n,n}=1$, $q_{n,n+1}=a_{n+1}$ in accordance with (2.2). For general k, it follows by induction, since again by (2.2)

$$q_{n,k}q_n+p_{n,k}q_{n-1}=(a_kq_{n,k-1}+q_{n,k-2})q_n+(a_kp_{n,k-1}+p_{n,k-2})q_{n-1}$$
$$=a_k(q_{n,k-1}q_n+p_{n,k-1}q_{n-1})+(q_{n,k-2}q_n+p_{n,k-2}q_{n-1})=a_kq_{k-1}+q_{k-2}=q_k.$$

LEMMA 5. *For all* $\varepsilon,\eta>0$

$$\lim_{N\to\infty}P\left\{\left|\sum_{n=0}^{(\frac{1}{2}\tau-2\eta)\log N-1}a_{n+1}(x)\tilde{g}\left(\frac{c_n(N,x)}{a_{n+1}(x)}\right)\right|\geqslant\varepsilon\sum_{n=0}^{(\frac{1}{2}\tau-2\eta)\log N-1}a_{n+1}\right\}=0. \tag{3.24}$$

Proof. For shortness we abbreviate the event between braces in (3.24) by $E(N,\varepsilon,\eta)$ and write M for $(\frac{1}{2}\tau-2\eta)\log N$. Then

$$\begin{aligned}P\{E(N,\varepsilon,\eta)\}&\leqslant P\{q_M(x)\geqslant N^{\frac{1}{2}-\frac{\eta}{\tau}}\}+\\&+P\left\{\sum_{n=0}^{M-1}a_{n+1}^2>\frac{M^2\log M}{C_0^2}\right\}+P\left\{\sum_{n=0}^{\log M-1}a_{n+1}>M\right\}+\\&+P\left\{\left(1-\frac{4C_0}{\varepsilon M^{1/6}}\right)\sum_{n=0}^{M-1}a_{n+1}<\frac{M\log M}{2\log 2}+\frac{15MC_0}{\varepsilon}\right\}+\\&+\sum{}'P\{a_n(x)=b_n,\ 1\leqslant n\leqslant M\}\cdot P\{E(N,\varepsilon,\eta)\mid a_n(x)=b_n,\ 1\leqslant n\leqslant M\}\end{aligned}\tag{3.25}$$

where $\sum'$ is a sum over all M-tuples $b_1,\dots,b_M$ $(b_i\geqslant 1)$ for which

$$\sum_{n=0}^{M-1}b_{n+1}^2\leqslant\frac{M^2\log M}{C_0^2},\qquad\sum_{n=0}^{\log M-1}b_{n+1}\leqslant M,$$

$$\left(1-\frac{4C_0}{\varepsilon M^{1/6}}\right)\sum_{n=0}^{M-1}b_{n+1}\geqslant\frac{M\log M}{2\log 2}+\frac{15MC_0}{\varepsilon}\quad\text{and}\quad q_M<N^{\frac{1}{2}-\frac{\eta}{\tau}}$$

where q_M is defined by (2.2) with a_n replaced by b_n. By (i)-(iii) the first four terms in the right-hand side of (3.25) tend to zero as $N\to\infty$. Since

$$\sum{}'P\{a_n(x)=b_n,\ 1\leqslant n\leqslant M\}\leqslant 1$$

it suffices to prove that

$$\lim_{N\to\infty}P\{E(N,\varepsilon,\eta)\mid a_n(x)=b_n,\ 1\leqslant n\leqslant M\}=0 \tag{3.26}$$

uniformly over all $b_1,\dots,b_M$ in $\sum'$.

Once $a_1(x),\dots,a_M(x)$ and hence $q_0(x),\dots,q_M(x)$ are fixed, the coefficients in the expansion

$$k=\sum_{n=0}^{M-1}c_n(k,x)q_n(x)$$

are determined for each $k<q_M(x)$ and therefore, so are all the integers $k_1,\dots,k_\varrho<q_M(x)$ for which

$$\left|\sum_{n=0}^{M-1}a_{n+1}(x)\tilde{g}\left(\frac{c_n(k,x)}{a_{n+1}(x)}\right)\right|>\varepsilon\sum_{n=0}^{M-1}a_{n+1}(x). \tag{3.27}$$

If $a_n(x)=b_n$, $1\leqslant n\leqslant M$, where $b_1,\dots,b_M$ is included in $\sum'$ then, by the Corollary to Lemma 4, for sufficiently large N

$$\varrho\leqslant q_M/(\log M)^{1/2}=o(q_M). \tag{3.28}$$

If the expansion of N is

$$N=\sum_{n=M}^{m(N,x)}c_n(N,x)q_n(x)+r_M(N,x)$$

then clearly $c_n(N,x)=c_n\big(r_M(N,x),x\big)$ for $n<M$ so that $E(N,\varepsilon,\eta)$ can occur only if $r_M(N,x)$ is one of the k's for which (3.27) occurs, i.e. if $r_M(N,x)$ equals some k_i, $1\leqslant i\leqslant\varrho$.

One concludes that

$$\begin{aligned}&P\{E(N,\varepsilon,\eta)\mid a_n(x)=b_n,\ 1\leqslant n\leqslant M\}\\&\leqslant\sum_{\nu=1}^{\varrho}P\Big\{N-k_\nu=\sum_{i=M}^{m(x,N)}c_iq_i(x)\ \text{for some}\ 0\leqslant c_i\leqslant a_{i+1}(x)\mid\\&\qquad a_n(x)=b_n,\ 1\leqslant n\leqslant M\Big\}.\end{aligned}\tag{3.29}$$

We finally prove

$$\begin{aligned}&P\Big\{N'=\sum_{i=M}^{m(x,N')}c_iq_i(x)\ \text{for some}\ 0\leqslant c_i\leqslant a_{i+1}(x)\mid\\&\qquad a_n(x)=b_n,\ 1\leqslant n\leqslant M\Big\}=O(q_M^{-1})\end{aligned}\tag{3.30}$$

uniformly for all $N-q_M\leqslant N'\leqslant N$ and $q_M\leqslant N^{\frac{1}{2}-\frac{\eta}{\tau}}$. Since ϱ in (3.29) is $o(q_M)$ this will prove (3.26) and hence the lemma. In order to prove (3.30) we use (3.23).

$$\sum_{i=M}^{m}c_iq_i=\sum_{i=M}^{m}c_i(q_{M,i}q_M+p_{M,i}q_{M-1})$$

and therefore

$$(3.31)\quad P\Big\{N' = \sum_{i=M}^{m} c_i q_i \text{ for some } 0 \leqslant c_i \leqslant a_{i+1} \mid a_n(x) = b_n,\ 1 \leqslant n \leqslant M\Big\}$$

$$\leqslant \sum_{\substack{0\leqslant\lambda_1\leqslant\lambda_2\\ \lambda_1 q_{M-1}+\lambda_2 q_M = N'}} P\Big\{\sum_{i=M}^{m} c_i p_{M,i}(x) = \lambda_1,\ \sum_{i=M}^{m} c_i q_{M,i}(x) = \lambda_2 \text{ for some } 0 \leqslant c_i \leqslant a_{i+1}(x) \mid a_n(x) = b_n,\ 1 \leqslant n \leqslant M\Big\}.$$

Recalling that $p_{M,i}(x) = p_{i-M}(y_M(x))$ and similarly for q, one immediately finds

$$\Big|\sum_{i=M}^{m} c_i p_{M,i}(x) - y_M(x)\sum_{i=M}^{m} c_i q_{M,i}(x)\Big| \leqslant \sum_{i=M}^{m} a_{i+1}(x) q_{M,i}(x)\Big|y_M(x) - \frac{p_{M,i}(x)}{q_{M,i}(x)}\Big|$$

$$\leqslant \sum_{i=M}^{m} \frac{1}{q_{M,i}(x)} \leqslant 2 + \sum_{j=0}^{m} 2^{-j/3} \text{ (by (2.32))} \leqslant 10,$$

and consequently the right hand side of (3.31) does not exceed

$$(3.32)\quad \sum_{\substack{0\leqslant\lambda_1\leqslant\lambda_2\\ \lambda_1 q_{M-1}+\lambda_2 q_M = N'}} P\{|\lambda_1 - y_M(x)\lambda_2| \leqslant 10 \mid a_n(x) = b_n,\ 1 \leqslant n \leqslant M\}$$

$$\leqslant \sum_{\substack{0\leqslant\lambda_1\leqslant\lambda_2\\ \lambda_1 q_{M-1}+\lambda_2 q_M = N'}} \frac{40}{\lambda_2} \text{ (by (iv)).}$$

Clearly $\lambda_1 q_{M-1} + \lambda_2 q_M = N'$ allows at most one value of λ_1 for a given λ_2, namely $\lambda_1 = (N' - \lambda_2 q_M)/q_{M-1}$ if this last expression is an integer. Otherwise no term occurs in (3.32) corresponding to this λ_2. Since $(q_{M-1}, q_M) = 1$ (cf. (2.16)) $q_{M-1} \mid N' - \lambda_2 q_M$ only if λ_2 is of the form

$$(3.33)\quad \lambda_2 = \lambda_0 + j q_{M-1},\quad j = 0, \pm 1, \pm 2, \ldots$$

for some fixed $0 \leqslant \lambda_0 < q_{M-1}$. Moreover $0 \leqslant \lambda_1 \leqslant \lambda_2$ means $0 \leqslant \dfrac{N' - \lambda_2 q_M}{q_{M-1}} \leqslant \lambda_2$ or

$$(3.34)\quad \frac{N'}{q_M + q_{M-1}} \leqslant \lambda_2 \leqslant \frac{N'}{q_M}.$$

Together with (3.33) this implies that there occur at most

$$1 + \frac{1}{q_{M-1}}\Big(\frac{N'}{q_M} - \frac{N'}{q_M + q_{M-1}}\Big) = 1 + \frac{N'}{q_M(q_M + q_{M-1})}$$

summands in the right-hand side of (3.32), each one not exceeding $40(q_M + q_{M-1})/N'$. It follows that

$$P\Big\{N' = \sum_{i=M}^{m} c_i q_i \text{ for some } 0 \leqslant c_i \leqslant a_{i+n} \mid a_n(x) = b_n,\ 1 \leqslant n \leqslant M\Big\} \leqslant 40\Big(\frac{2q_M}{N'} + \frac{1}{q_M}\Big) \leqslant \frac{120}{q_M}$$

whenever $N' \geqslant q_M^2$. This proves (3.30) and hence the lemma.

LEMMA 6. *For all $\varepsilon, \eta > 0$*

$$(3.35)\quad \lim_{N\to\infty} P\Big\{\Big|\sum_{n=K(N,x)}^{m(N,x)-1} a_{n+1}(x)\tilde{g}\Big(\frac{c_n(N,x)}{a_{n+1}(x)}\Big)\Big| \geqslant \varepsilon \sum_{n=K(N,x)}^{m(N,x)-1} a_{n+1}\Big\} = 0$$

where $K = K(N, x)$ stands for $m(N, x) - (\frac{1}{2}\tau - 2\eta)\log N$.

Proof. This proof is quite similar to the one of the last lemma. The main difficulty will turn out to be that the $q_n(x)$ for $m - (\frac{1}{2}\tau - \eta)\log N \leqslant n \leqslant m$ do not determine $q_m(x)/q_k(x)$ uniquely ($K \leqslant k < m$). We shall use the abbreviation $F(N, \varepsilon, \eta)$ for the event between braces in (3.35), M for $(\frac{1}{2}\tau - 2\eta)\log N$, M' for $(\frac{1}{2}\tau - \eta)\log N$ and L for $m - (\frac{1}{2}\tau - \eta)\log N$. Obviously,

$$(3.36)\quad P\{F(N, \varepsilon, \eta)\} \leqslant \sum_{w=1}^{w_0-1} P\Big\{F(N, \varepsilon, \eta),\ \frac{N}{w+1} < q_m(x) \leqslant \frac{N}{w}\Big\} + P\Big\{q_m(x) \leqslant \frac{N}{w_0}\Big\}$$

and

$$P\Big\{q_m(x) \leqslant \frac{N}{w_0}\Big\} \leqslant \sum_{k=0}^{N/w_0}\sum_{n=0}^{\infty} P\{q_n(x) = k,\ q_{n+1}(x) > N\}$$

$$\leqslant \sum_{k=0}^{N/w_0}\sum_{n=0}^{\infty} P\{q_n(x) = k\} P\Big\{a_{n+1}(x) \geqslant \frac{N}{k} - 1 \mid q_n(x) = k\Big\}$$

$$\leqslant 2\sum_{k=0}^{N/w_0}\sum_{n=0}^{\infty} P\{q_n(x) = k\} P\Big\{a_1(x) \geqslant \frac{N}{k} - 1\Big\} \text{ (by (iv))}$$

$$\leqslant 2\sum_{k=0}^{N/w_0} \frac{k}{N-k}\sum_{n=0}^{\infty} P\{q_n(x) = k\} \leqslant \frac{4}{w_0 - 1}.$$

The last inequality results from

$$\sum_{n=0}^{\infty} P\{q_n(x) = k\} = P\{\text{some convergent of } x \text{ has } k \text{ as denominator}\} \leqslant \sum_{j=0}^{k} P\Big\{\Big|x - \frac{j}{k}\Big| \leqslant \frac{1}{k^2}\Big\} \leqslant \frac{2}{k}.$$

The last term in (3.36) can be made small by choosing w_0 large and it therefore suffices to prove, for each fixed w

$$(3.37)\qquad \lim_{N\to\infty} P\left\{F(N, \varepsilon, \eta),\ \frac{N}{w+1} < q_m(x) \leqslant \frac{N}{w}\right\} = 0.$$

Quite analogous to (3.25) we have as $N\to\infty$

$$(3.38)\qquad P\left\{F(N, \varepsilon, \eta),\ \frac{N}{w+1} < q_m(x) \leqslant \frac{N}{w}\right\}$$

$$= o(1) + \sum{}'' P\left\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ F(N, \varepsilon, \eta),\ \frac{N}{w+1} < q_m(x) \leqslant \frac{N}{w}\right\}$$

where $\sum''$ runs over all M' tuples $b_1, \ldots, b_{M'}$ $(b_i \geqslant 1)$ which satisfy (3.39) -(3.42) below.

$$(3.39)\qquad \bar{q}_{M'} \leqslant N^{\frac{1}{2}-\frac{\eta}{2\tau}}$$

where we define $\bar{p}_k/\bar{q}_k = \bar{p}_k(b)/\bar{q}_k(b)$ as the kth convergent of $[b_1, b_2, \ldots, b_{M'}]$.

$$(3.40)\qquad \sum_{n=\eta\log N}^{M'-1} b_{n+1}^2 \leqslant \frac{M^2\log M}{C_0^2}.$$

$$(3.41)\qquad \sum_{n=\eta\log N}^{\eta\log N+\log M-1} b_{n+1} \leqslant M.$$

$$(3.42)\qquad \left(1-\frac{4C_0}{\varepsilon M^{1/6}}\right)\sum_{n=\eta\log N}^{M'-1} b_{n+1} \geqslant \frac{M\log M}{2\log 2} + \frac{15MC_0}{\varepsilon}.$$

The proof of (3.38) is slightly more complicated than its analogue in Lemma 4 because m and hence K, L are random variables. This difficulty is easily overcome by means of (3.2). E.g. by (3.2) and (iii)

$$\lim_{N\to\infty} P\left\{\sum_{n=K}^{m-1} a_{n+1}^2(x) > \frac{M^2\log M}{C_0^2}\right\}$$

$$\leqslant \lim_{N\to\infty} P\left\{\left|\frac{m(N, x)}{\log N} - \tau\right| > \eta\right\} + \lim_{N\to\infty} P\left\{\sum_{(\frac{1}{2}\tau+\eta)\log N}^{(\tau+\eta)\log N} a_{n+1}^2(x) > \frac{M^2\log M}{C_0^2}\right\} = 0.$$

We now imitate the proof of (3.26). To start let us estimate, for some fixed $b_1, \ldots, b_{M'}$ in $\sum''$ and $N/(w+1) < N' \leqslant N/w$,

$$(3.43)\qquad P\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ q_m(x) = N'\}.$$

By (3.23) $a_{L+n}(x) = b_n$ for $1 \leqslant n \leqslant M'$, $q_m = N'$ can occur only if

$$(3.44)\qquad N' = q_m = \bar{q}_{M'}q_L(x) + \bar{p}_{M'}q_{L-1}(x)$$

and hence the probability in (3.43) is bounded by

$$(3.45)\qquad \sum_{\substack{0\leqslant\lambda_1\leqslant\lambda_2\\ \lambda_1\bar{p}_{M'}+\lambda_2\bar{q}_{M'}=N'}} \sum_{k=0}^{\infty} P\{q_{k-1}(x) = \lambda_1,\ q_k(x) = \lambda_2,$$

$$a_{k+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\} \leqslant 2P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}\times$$

$$\times \sum_{\substack{0\leqslant\lambda_1\leqslant\lambda_2\\ \lambda_1\bar{p}_{M'}+\lambda_2\bar{q}_{M'}=N'}} \sum_{k=0}^{\infty} P\{q_{k-1}(x) = \lambda_1,\ q_k(x) = \lambda_2\} \quad \text{(by (iv))}.$$

Just as in (3.32) the sum over λ_1, λ_2 contains at most $1+\dfrac{N'}{\bar{q}_M(\bar{q}_{M'}+\bar{p}_{M'})}$ terms, each with $\lambda_2 \geqslant \dfrac{N'}{\bar{q}_{M'}+\bar{p}_{M'}}$. Moreover, it was proved in [5] (pp. 367, 368, near formula (3.18) and (3.19)) that

$$\sum_{k=0}^{\infty} P\{q_{k-1}(x) = \lambda_1,\ q_k(x) = \lambda_2\} \leqslant \frac{4}{\lambda_2^2}.$$

Combining these estimates with (3.39) we obtain

$$(3.46)\qquad P\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ q_m(x) = N'\}$$

$$\leqslant 8P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}\left(1+\frac{N'}{\bar{q}_{M'}(\bar{q}_{M'}+\bar{p}_{M'})}\right)\left(\frac{\bar{q}_{M'}+\bar{p}_{M'}}{N'}\right)^2$$

$$\leqslant \frac{20}{N'}P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}$$

as soon as $8(\bar{q}_{M'}+\bar{p}_{M'})^2 \leqslant 32N'^{1-\frac{\eta}{\tau}} < 4N'$.

For $N/(w+1) < q_m = N' \leqslant N/w$, $c_m(N, x) = w$ and thus

$$(3.47)\qquad \frac{N}{wN'}(N-wN') = \frac{N}{wq_m(x)}\big(N - c_m(N, x)q_m(x)\big)$$

$$= \sum_{n=u}^{m-1} c_n(N, x)\frac{N}{wq_m(x)}q_n(x) + \frac{N}{wq_m(x)}r_u(N, x).$$

We assume $a_{L+n}(x) = b_n$, $1 \leqslant n \leqslant M'$, fixed for the moment and define

$$z_j = \frac{N}{wj}(N-wj),$$

and

$$\frac{s_n}{t_n} = \frac{s_n(b)}{t_n(b)} = [b_n, b_{n-1}, \ldots, b_1], \quad (s_n, t_n) = 1, \quad q'_m = \frac{N}{w},$$

$$q'_n = \frac{N}{w} \prod_{i=n+1-L}^{m-L} \frac{s_i}{t_i} = \frac{N}{w} \prod_{i=n+1-L}^{M'} \frac{s_i}{t_i}, \quad L \leqslant n < m.$$

Finally the integral coefficients $c'_n(N, x)$, $L \leqslant n < m$, are defined by the expansion

$$(3.48) \qquad z_{q_m}(x) = \frac{N}{wq_m(x)}\big(N - c_m(N, x) q_m(x)\big) = \sum_{n=u}^{m-1} c'_n(N, x) q'_n + r'_u(N, x)$$

where

$$0 \leqslant c'_n \leqslant b_{n+1-L}, \quad 0 \leqslant r'_u(N, x) < q'_u, \quad L \leqslant u < m.$$

We claim that except for a set of small probability $c_n = c'_n$ for $K \leqslant n < m$. This is of course a consequence of the special definition of q'_n. In fact if $a_{L+n}(x) = b_n$, $1 \leqslant n \leqslant M'$, and $\frac{N}{w+1} < q_m \leqslant \frac{N}{w}$ then (cf. (3.22))

$$\frac{q_n(x)}{q_{n+1}(x)} = [a_{n+1}(x), a_n(x), \ldots, a_1(x)]$$

$$= [b_{n+1-L}, b_{n-L}, \ldots, b_1, a_L(x), \ldots, a_1(x)]$$

$$= \frac{s_{n+1-L}}{t_{n+1-L}} + \frac{\theta_{10}}{t^2_{n+1-L}} = \frac{s_{n+1-L}}{t_{n+1-L}}\left(1 + \frac{\theta_{11}}{t_{n+1-L}}\right) \quad (|\theta_i| \leqslant 1).$$

But also

$$\frac{q'_n}{q'_{n+1}} = \frac{s_{n+1-L}}{t_{n+1-L}}$$

so that

$$(3.49) \qquad \left|\frac{Nq_n}{wq_m} - q'_n\right| = \frac{N}{w}\left|\prod_{i=n+1-L}^{M'} \frac{s_i}{t_i} - \prod_{i=n+1-L}^{M'} \frac{s_i}{t_i}\left(1 + \frac{\theta_{11,i}}{t_i}\right)\right| \leqslant \frac{w+1}{w} q_n \left| e^{\sum_{i=n+1-L}^{M'} \frac{2}{t_i}} - 1\right|.$$

Finally, by the remark immediately after (3.22) t_i equals $\bar{q}_i$, the denominator of the continued fraction $[b_1, b_2, \ldots, b_i]$. In particular (cf. (2.32))

$t_{i+j} \geqslant 2^{j/3} t_i$ $(j \geqslant 2)$ and

$$t_i = \bar{q}_i \geqslant 2^{\eta \log N/3} \quad \text{for} \quad i \geqslant \eta \log N = K - L.$$

Combining (3.49) with (3.23) we obtain for $n \geqslant K$

$$(3.50) \qquad \left|\frac{Nq_n}{wq_m} - q'_n\right| \leqslant 8q_n \sum_{i=n+1-L}^{M'} \frac{1}{t_i} \leqslant \frac{2^{13/3}}{2^{1/3}-1} \cdot \frac{q_n}{\bar{q}_{n+1-L}} = \frac{2^{13/3}}{2^{1/3}-1} \cdot \frac{1}{\bar{q}_{n+1-L}} (\bar{q}_{n-L} q_L + \bar{p}_{n-L} q_{L-1}) \leqslant 2^8 \frac{\bar{q}_{n-L}}{\bar{q}_{n+1-L}} q_L.$$

Therefore if $c'_n(N, x) = c_n(N, x)$ for $n > u \geqslant K$ but $c'_u(N, x) \neq c_u(N, x)$ we must have (cf. (3.47), (3.48))

$$r'_u(N, x) - \frac{Nr_u(N, x)}{wq_m(x)} = \big(c_u(N, x) - c'_u(N, x)\big) \frac{Nq_u(x)}{wq_m(x)} + c'_u(N, x)\left(\frac{Nq_u(x)}{wq_m(x)} - q'_u(x)\right) + \sum_{n=u+1}^{m-1} c_n(N, x)\left(\frac{Nq_n(x)}{wq_m(x)} - q'_n\right)$$

$$= (c_u - c'_u)\frac{Nq_u}{wq_m} + 2^8 \theta_{12} \sum_{n=u}^{m-1} b_{n+1-L} \frac{\bar{q}_{n-L}}{\bar{q}_{n+1-L}} q_L$$

$$= (c_u - c'_u)\frac{Nq_u}{wq_m} + 2^8 \theta_{13} (m-u) q_L$$

$$= (c_u - c'_u)\frac{Nq_u}{wq_m} + 2^8 \theta_{14} M \cdot 2^{-\eta \log N/3} q_u$$

or, for sufficiently large N,

$$(3.51) \qquad \frac{Nr_u(N, x)}{wq_m(x)} = r'_u(N, x) + (c'_u - c_u)\frac{Nq_u(x)}{wq_m} + \theta_{15} N^{-\eta/6} q'_u.$$

In view of $0 \leqslant r_u < q_u$, $0 \leqslant r'_u < q'_u$ (3.51) can occur only if

$$c'_u - c_u = 1 \quad \text{and} \quad 0 \leqslant r'_u \leqslant N^{-\eta/6} q'_u$$

or

$$c'_u - c_u = -1 \quad \text{and} \quad q'_u - 2N^{-\eta/6} q'_u \leqslant \frac{Nq_u}{wq_m} - N^{-\eta/6} q'_u \leqslant r'_u < q'_u.$$

We can now apply Lemma 4 because

$$q'_{n+1} = \frac{t_{n+1-L}}{s_{n+1-L}} q'_n = \left(b_{n+1-L} + \frac{s_{n-L}}{t_{n-L}}\right) q'_n = b_{n+1-L} q'_n + q'_{n-1}$$

and for $N/(w+1) < j \leqslant N/w$, $N \geqslant w \geqslant 1$

$$(3.52) \qquad z_j - z_{j+1} = \frac{N}{wj}(N - wj) - \frac{N}{w(j+1)}\big(N - w(j+1)\big) \geqslant \frac{Nw}{N+w} \geqslant \frac{1}{2}.$$

Thus, by Lemma 4, if $j_1, j_2, \ldots, j_\varrho$ are all the values q_m can take such that z_{q_m} in (3.48) gives rise to

$$0 \leqslant r'_u \leqslant N^{-\eta/6} q'_u \quad \text{or} \quad q'_u - 2N^{-\eta/6} q'_u \leqslant r'_u < q'_u$$

for some $K \leqslant u \leqslant m-1$ then

$$\varrho \leqslant \sum_{u=K}^{m-1} \left(2 + \frac{8(2N^{-\eta/6} q'_u + 3)(q'_m + 1)}{q'_u - 1}\right)$$
$$\leqslant 2M + 64\left(\frac{N}{w} + 1\right)\left(MN^{-\eta/6} + \sum_{u=K}^{m-1} \frac{1}{q'_u}\right)$$
$$\leqslant 1000\left(MN^{1-\eta/6} + \frac{N}{q'_K}\right) \leqslant 1000\left(MN^{1-\eta/6} + \frac{N}{q_L}\right) \leqslant N^{1-\eta/12}$$

(cf. (3.44) and (3.39)). This proves our claim about the c'_n, since by (3.46)

$$(3.53) \quad P\Big\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ \frac{N}{w+1} < q_m \leqslant \frac{N}{w} \text{ and }$$
$$c'_u(N, x) \neq c_u(N, x) \text{ for some } K \leqslant u < m\Big\}$$
$$\leqslant \sum_{\nu=1}^{\varrho} P\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ q_m = j_\nu\}$$
$$\leqslant \frac{20\varrho(w+1)}{N} P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}$$
$$\leqslant 20(w+1)N^{-\eta/12} P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}.$$

The advantage of the c'_n over the c_n is that they are uniquely determined by the a_{L+n}, $1 \leqslant n \leqslant M'$. The proof of the lemma is now completed by an application of the Corollary to Lemma 4, because

(3.54)

$$\sum{}'' P\Big\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ F(N, \varepsilon, \eta),\ \frac{N}{w+1} < q_m(x) \leqslant \frac{N}{w}\Big\}$$
$$\leqslant \sum{}'' P\Big\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ \frac{N}{w+1} < q_m \leqslant \frac{N}{w},$$
$$c'_u(N, x) \neq c_u(N, x) \text{ for some } K \leqslant u < m\Big\}$$
$$+ \sum{}'' \sum_{\lambda=1}^{\mu} P\{a_{L+n}(x) = b_n \text{ for } 1 \leqslant n \leqslant M',\ q_m = k_\lambda\}$$

where $k_1, k_2, \ldots, k_\mu$ are all the possible values, q_m can take between $\frac{N}{w+1}$ and $\frac{N}{w}$ such that

$$\left|\sum_{n=K}^{m-1} b_{n+1-L}\tilde{g}\left(\frac{c'_n}{b_{n+1-L}}\right)\right| \geqslant \varepsilon \sum_{n=K}^{m-1} b_{n+1-L}.$$

The values of the k_i as well as their number μ will depend on $b_1, \ldots, b_{M'}$ but for each choice of $b_1, \ldots, b_{M'} \in \sum''$ one concludes from (3.40)-(3.42), (3.52) and the Corollary of Lemma 4 (applied to the q'_n) that

$$\mu \leqslant \frac{q'_m}{(\log M)^{1/2}} = \frac{N}{w(\log M)^{1/2}}.$$

Thus, by (3.46), the second sum in the right-hand side of (3.54) is at most

$$\sum{}'' \frac{N}{w(\log M)^{1/2}} \cdot \frac{20(w+1)}{N} P\{a_n(x) = b_n \text{ for } 1 \leqslant n \leqslant M'\}$$
$$\leqslant \frac{20(w+1)}{w(\log M)^{1/2}} = o(1) \quad (N \to \infty).$$

The first sum in the right-hand side of (3.54) tends to zero as $N \to \infty$ because of (3.53). This proves (3.37) and the lemma.

To complete the proof of Theorem 2 from Lemmas 5 and 6 is rather trivial because

$$\left|\sum_{n=M}^{K-1} a_{n+1}(x)\tilde{g}\left(\frac{c_n(N, x)}{a_{n+1}(x)}\right)\right| \leqslant C_0 \sum_{n=M}^{K-1} a_{n+1}(x)$$

and

$$P\Big\{C_0 \sum_{n=M}^{K-1} a_{n+1}(x) \geqslant \varepsilon \sum_{n=0}^{m-1} a_{n+1}(x)\Big\}$$
$$\leqslant P\left\{\left|\frac{m(N, x)}{\log N} - \tau\right| \geqslant \eta\right\} + P\left\{\sum_{n=0}^{m} a_{n+1}(x) \leqslant \frac{\tau}{2\log 2} \log N \log\log N\right\} +$$
$$+ P\left\{C_0 \sum_{n=(\tau/2-2\eta)\log N}^{(\tau/2+3\eta)\log N} a_{n+1}(x) > \frac{\varepsilon\tau}{2\log 2} \log N \log\log N\right\}.$$

These three terms in the right hand side tend to zero as $N \to \infty$ if $\eta < \varepsilon\tau/10C_0$ by (3.2), (3.3) and (ii). In much the same way [5]

$$\lim_{N\to\infty} P\left\{\left|a_{m+1}(x)\tilde{g}\left(\frac{c_m(N, x)}{a_{m+1}(x)}\right)\right| \geqslant \varepsilon \sum_{n=0}^{m} a_{n+1}(x)\right\} = 0.$$

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[5] By an argument similar to the one following (3.36) we can even show $\lim_{w\to\infty} P\{a_{m(N,x)+1} > w\} = 0$ uniformly in N.

Combining these estimates with Lemmas 5 and 6 one obtains (3.5) with ε replaced by 4ε.

As pointed out before this proves Theorem 2.

For comparison we point out another theorem whose proof is almost immediate from Theorem 1.

THEOREM 3.

$$\max_{1\leqslant k\leqslant N} \frac{kD_k(x)}{\log N \log\log N} \to \frac{3}{\pi^2} \text{ in measure on } [0,1]. \tag{3.55}$$

For all $\varepsilon > 0$

$$\limsup_{N\to\infty} ND_N(x)\Big(\sum_{n=0}^{(\tau+\varepsilon)\log N} a_{n+1}(x)\Big)^{-1} \leqslant \frac{1}{4} \text{ a.e. in } [0,1] \tag{3.56a}$$

and

$$\limsup_{N\to\infty} ND_N(x)\Big(\sum_{n=0}^{(\tau-\varepsilon)\log N} a_{n+1}(x)\Big)^{-1} \geqslant \frac{1}{4} \text{ a.e. in } [0,1]. \tag{3.56b}$$

We shall not say more about the proof than that $h(\xi)$ takes its maximum value $\frac{1}{4}$ at $\xi = \frac{1}{2}$ and if $a_1(x), \ldots, a_n(x)$ are fixed then one will maximize $ND_N(x)$ over $N \leqslant q_n(x)$ roughly by taking $c_k(N, x) = [\frac{1}{2}a_{n+1}(x)]$, $k = 0, \ldots, n-1$. This argument can also be used to prove almost everywhere statements about

$$\limsup_{N\to\infty} \frac{D_N(x)}{\Phi(N)}$$

for suitable functions Φ. E.g. by pp. 295, 296 of [8]

$$\limsup_{N\to\infty} \frac{ND_N(x)}{\log N \log\log N \log\log\log N} = \infty \text{ a.e. in } [0,1].$$

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