where $J(i)$ is an index set depending on i. If we define a function f by specifying that on R_g

$$f(\gamma_j) = \delta_i \quad (j \in J(i),\ i = 1, \ldots, r)$$

and by allowing f to be arbitrary elsewhere, then f will satisfy (9.12).

References

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On the abstract theory of primes I

by

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Introduction. 1. For a semi-group $\mathfrak{G}$ (with respect to multiplication) of real numbers $a \geqslant 1$ satisfying some given asymptotical laws of distribution Beurling [2] investigated the asymptotical distribution of the generators b of $\mathfrak{G}$. He proved a general theorem which, applied to the semi-group of natural integers, gives the prime number theorem of Hadamard and Vallée-Poussin (¹). Forman and Shapiro [11] divided the numbers a into classes H_i $(1 \leqslant i \leqslant h)$ forming a group K (for any $a \in H_i$ and $a' \in H_j$ we have $aa' \in H_k$ where k depends only on i and j) and satisfying

$$(1) \qquad \sum_{x \geqslant a \in H_i} 1 = a_i x + O(x^{1-\vartheta})$$

with some positive constants a_i, ϑ $(\vartheta \leqslant 1)$. They proved that under those circumstances the numbers $\pi(x, H_i)$ of the generators $b \leqslant x$, $b \in H_i$ are asymptotically the same for all the classes H_i forming a sub-group K_0 of K, whereas the number of the remaining generators $\leqslant x$ (if $K_0 \neq K$) has a smaller order of magnitude as $x \to \infty$ (²). As special cases of this abstract theorem we may deduce the asymptotical laws for primes in arithmetical progressions or prime ideals in ideal classes.

The aim of all the work in the abstract theory of primes up to now has been the proof of the asymptotical law for $\pi(x, H)$. In a short note [9] I have mentioned that in the abstract scheme used by Forman and Shapiro one can treat the smallest prime problem for different progressions simultaneously. For this purpose it is necessary to change the

(¹) Other writers after Beurling (as Nyman [16], Erdös [3]) either start from different conditions or use, instead of the analytical method of the zeta function, the elementary method of A. Selberg.

(²) This is an intentionally simplified description. Actually Forman and Shapiro start from a free Abelian group G on a countable number of generators and use a homomorphism into positive rationals such that the images of the generators are all integral. The distribution of the generators in the classes H of a semi-group is also the subject of a recent work of Amitsur [1], who replaces the remaining term in (1) by $O(x/\log^c x)$.

scheme as follows. Firstly we introduce a parameter $D > D_0 > 2$ which corresponds to the difference in the case of arithmetical progressions and which may increase indefinitely. We suppose that

$$(2)\qquad 1 \leqslant h \leqslant D^{c_0},$$

where c_0 is any positive constant. Secondly we take for granted that the coefficients a_i in (1) do not depend on i. Otherwise in some of the classes there might be no generators [3] and thus we could not prove the existence of a generator $b = D^{O(1)}$, $b \epsilon H_i$ for every H_i, which is our aim. A proof of the existence of a generator $b \epsilon H_i$ is usually based on the preponderance of the residue at $s = 1$ in the sum of all residues of a certain expression containing the logarithmic derivatives of all L-functions $\zeta(s, \chi)$ with characters χ of the group K (cf. [17], p. 134). If there are in (1) at least two different coefficients a_i, then, as will be proved in the Appendix (§ 28), $s = 1$ is a pole for at least two functions ζ'/ζ and in some cases the dominating term in the residue sum disappears, giving no evidence of the existence of a generator in the corresponding class H_i. In all the applications of any importance (they will be given in a continuation of this paper) the equality condition $a_1 = \ldots = a_h$ for the coefficients a_i in (1) will be satisfied.

According to these changes we replace (1) by

$$(3)\qquad \sum_{x \geqslant a \epsilon H_i} 1 = ax + O(D^{c_1} x^{1-\vartheta}), \qquad a = D^l,$$

where the constants l, c_1, ϑ do not depend on i $(0 \gtreqless l \ll 1,\ 0 \leqslant c_1 \ll 1,\ 0 < \vartheta \leqslant 1)$. Next we introduce another parameter q such that [4]

$$(4)\qquad 1 \leqslant q \leqslant D^{\vartheta}$$

and we suppose that if $q > 1$, then the numbers $a \epsilon \mathfrak{G}$ and x in (3) are exclusively terms of the progression

$$(5)\qquad 1,\ q,\ q^2,\ q^3,\ \ldots$$

It is understood that all further constants (including those of the symbols O and $\ll$) may depend on c_0, c_1, ϑ, l.

Denoting by $R(x)$ the remainder term in (3) we have for $q > 1$ and for any number x of the progression (5):

$$(6)\qquad \sum_{a=x} 1 = a_1 x + R(x) - R(x/q), \qquad a_1 = a(1-q^{-1}), \qquad R(x) \ll D^{c_1} x^{1-\vartheta},$$

(3) See the example of § 28.

(4) The parameter q was introduced following professor Turán's suggestion of extending the theorem of [9] in such a manner as to make it applicable to irreducible polynomials over a finite field of constants.

whence

$$\sum_{q^k > x} \frac{R(q^k) - R(q^{k-1})}{q^k}$$

$$= -\frac{R(x)}{qx} + \sum_{q^k > x} R(q^k)\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) = -\frac{R(x)}{qx} + \left(1 - \frac{1}{q}\right) \sum_{q^k > x} \frac{R(q^k)}{q^k}$$

$$\ll D^{c_1} x^{-\vartheta} + (1-q^{-1}) D^{c_1} \sum_{q^k > x} q^{-k\vartheta} \ll D^{c_1}\left(x^{-\vartheta} + (qx)^{-\vartheta} \frac{1-q^{-1}}{1-q^{-\vartheta}}\right) \ll D^{c_1} x^{-\vartheta},$$

$$(7)\qquad \sum_{x \geqslant a \epsilon H} \frac{1}{a} = a_1 \sum_{q^k \leqslant x} 1 + \sum_{q^k \leqslant x} \frac{R(q^k) - R(q^{k-1})}{q^k}$$

$$= a_1\left(1 + \frac{\log x}{\log q}\right) + \left(\sum_{k=0}^{\infty} - \sum_{q^k > x}\right) \frac{R(q^k) - R(q^{k-1})}{q^k}$$

$$= \frac{1-q^{-1}}{\log q} a \log x + c_H + O(D^{c_1} x^{-\vartheta}).$$

The last estimate for the left-hand side of (7) holds as well if $q = 1$ (now for any $x \geqslant 1$), but then the factor $(1-q^{-1})/\log q$ becomes unity. In this case (7) could be proved more conveniently by the use of Abel's identity ([17], p. 371).

For even h let K_j denote any sub-group of the group K with the index 2. Then we have, by (7),

$$(8)\qquad \lim_{x \to \infty} \left(\sum_{x \geqslant a \epsilon K} \frac{1}{a} - \sum_{x \geqslant a \notin K_j} \frac{1}{a}\right) = C_j = C_j(D, q).$$

With the use of this notation the principal result of the present paper may be formulated as the following

THEOREM. (i) *If $\vartheta > \frac{1}{2}$, then there is a positive constant c such that for any $x \geqslant 1$ and any H_i in the interval (x, xD^c) there is at least one generator $b \epsilon H_i$. For an odd class-number h the conclusion holds as well in the case of $\vartheta \leqslant \frac{1}{2}$.*

(ii) *Let h be even and $\vartheta \leqslant \frac{1}{2}$. If there is a constant $c_2 > 0$ such that in (8) for any j*

$$(9)\qquad C_j > D^{-c_2},$$

then the conclusion of (i) holds (with the constant c depending also on c_2).

Corollary. *For appropriate constants $c_3, c_4 > 0$ and any $x > D^{c_3}$ we have*

$$\pi(x, H_i) > x/D^{c_4}\log x. \tag{10}$$

If $x > \exp(D^{c_5})$ (with a suitable $c_5 > 0$), then by known methods one can prove the asymptotical formula for $\pi(x, H_i)$, which is a stronger result. Nevertheless from that formula one cannot deduce the estimate for the least generator $b \epsilon H_i$,

$$\min b_{H_i} < D^c \tag{11}$$

the proof of which is our present task. Since the proof of the existence of a generator $b \epsilon H_i$ in the interval (x, xD^c) is only a little more complicated, instead of (11) we shall prove the assertion of the theorem. It follows from part (i) of the theorem that Linnik's estimate (11) for the least prime of arithmetical progression $Du+d$ $(u = 0, 1, \ldots;\ (D, d) = 1)$ (see [15]) can be proved without using the functional equation of Dirichlet L-functions and their existence in the whole plane.

If $q \ll 1$, $x \to \infty$ and other restrictions are imposed, then the theorem holds for intervals (x, xD^{ε}), where ε denotes an arbitrary small positive constant. For a sketch of the proof see §§ 25-27.

In the present theorem we are concerned with a linear distribution problem in a semi-group. An analogous theorem on two-dimensional distribution will be proved in the continuation of this paper. From that theorem we will deduce an estimate for the smallest prime representable by a primitive binary quadratic form $\psi(u, v)$ if the point (u, v) is bound to a given angle in the plane of the variables. From the one-dimensional theorem of the present paper we can deduce among other things an upper bound for the height of a polynomial $\equiv r(x)\big(\mathrm{mod} f(x)\big)$ irreducible in a fixed algebraic field. These results have been announced in [9], [10].

2. In this paper we shall use the analytical method of the zeta function with a complex variable, since the elementary method at present does not give a satisfactory estimate for the least prime $\equiv d(\mathrm{mod}\, D)$. The method is in outline the same as that used by Linnik [15], its chief weapon being an estimate for the density of zeros of all L-functions in the neighbourhood of $s = 1$ (cf. the fundamental Lemmas 19 and 25 of the present paper). A variant of this method, due to Rodoskiĭ (see [18] or [17], Ch. X), was used in my previous papers [5]-[8], where the least prime problem was solved for classes of ideals in any algebraic field. In these papers for the proof of the main auxiliary theorems a general lemma ([6], p. 133) was used together with an upper bound for the number of ideals of any class having a special property ([7], Lemma 7, proved by Selberg's sieve method). In Linnik's, Rodoskiĭ's and my own

papers these theorems were proved by a method which essentially depends on the distribution of the small primes (see, for example, [6], p. 133 and [7], p. 238, where the sum over the primes $\sum_{p<Z} p^{-1}\log p$ has to be small enough for a sufficiently large Z). The same method cannot be used for the proof of the present theorem, since (3) does not give the necessary information about the distribution of the small numbers $a \epsilon \mathfrak{G}$. But the method would work if we added a new condition (a rather unnatural one):

$$\sum_{b \leqslant x} \log b \ll x \quad \text{for} \quad 1 \leqslant x \leqslant D^{c_3}.$$

To avoid the necessity of any new condition, for the proof of the main lemmas in this paper we shall follow Turán's method, which was used by Turán [22] and Knapowski [13] in proving the corresponding theorems for Dirichlet L-functions.

It seems probable that the truth of (11) does not depend on the distribution of the large numbers $a > D^{O(1)}$ of $\mathfrak{G}$; that is to say, one might hope to prove (11) using (3) merely for $x \leqslant D^A$ with some sufficiently large $A \ll 1$. However, this restriction excludes the use of the method of L-functions, the only known method by which (11) can be proved.

All the constants used in this paper (with at most one exception of the constant l) are positive. The values of $c_0, c_1, c_2, \vartheta, l$ remain as fixed in § 1, whereas other constants (generally denoted by $c, c', c_3, c_4, \ldots$) retain the meaning only throughout the same paragraph. More important constants and constants of some general lemmas are denoted by A, B, C.

The complex variable is generally denoted by $s = \sigma + it$ $(\sigma = \mathrm{re}\, s$, $t = \mathrm{im}\, s)$, but in a few cases we use w as well.

By $d \mid a$ ('d divides a') we mean that d, a are numbers of $\mathfrak{G}$ and there is a number $a' \epsilon \mathfrak{G}$ such that $a = da'$; the maximal d which divides a and a' will be denoted by (a, a').

Some details in the proofs may differ according to whether $q = 1$ or $q > 1$. In order to treat both cases simultaneously we shall use the following factors:

$$\theta = \begin{cases} (1-q^{-1})/\log q & \text{if} \quad q > 1, \\ 1 & \text{if} \quad q = 1; \end{cases} \qquad \vartheta = \begin{cases} 0 & \text{if} \quad q > 1 + D^{-1}, \\ 1 & \text{otherwise}; \end{cases}$$

$$\partial = \begin{cases} 1 & \text{if} \quad q > 1, \\ 0 & \text{if} \quad q = 1. \end{cases}$$

The functions $\zeta(s, \chi)$ and their zeros near the line $\sigma = 1$.

3. Lemma 1. *The function*

$$\zeta(s, H) = \sum_{a \epsilon H} a^{-s} \quad (\sigma > 1)$$

is regular in $\sigma > 1-\vartheta$ except for a simple pole at $s=1$ with residue a if $q=1$, or simple poles at $s=1+2k\pi i/\log q$ $(k=0, \pm 1, \ldots)$ with residues θa if $q>1$. In the latter case $\zeta(s,H)$ is a periodic function, its period being $2\pi i/\log q$.

We have uniformly in the strip $G\,(1-\vartheta+\eta \leqslant \sigma \leqslant 2)$ $(0<\eta<\frac{1}{8}\vartheta)$

$$\zeta(s,H) = \begin{cases} a(s-1)^{-1}+O(D^c\eta^{-1}|s|) & \text{if} \quad q=1, \\ a\dfrac{1-q^{-1}}{1-q^{1-s}}+O\big(D^c\eta^{-1}(1+\vartheta|s|)\big) & \text{if} \quad q>1. \end{cases} \tag{12}$$

Proof. First let $q=1$ and let $\zeta(s)$ be the zeta-function of Riemann. By (3) in $\sigma>1$

$$\zeta(s,H) = \sum_{a\epsilon H} a^{-s} = s\int_1^\infty \frac{ax+R(x)}{x^{s+1}}\,dx$$

$$= a\zeta(s)+s\int_1^\infty \frac{R_1(x)}{x^{s+1}}\,dx \qquad (R_1(x) \ll a+D^{c_1}x^{1-\vartheta}).$$

In G the last integral may be represented by the sum

$$\sum_{n=1}^\infty g_n(s) \quad \text{with} \quad g_n(s) = \int_n^{n+1} \frac{R_1(x)}{x^{s+1}}\,dx \ll \eta^{-1}D^c\big(n^{-\eta}-(n+1)^{-\eta}\big),$$

whence the regularity and (12) follows.

If $q>1$, then, by (6), in $\sigma>1$

$$\zeta(s,H) = a_1\sum_{k=0}^\infty \frac{1}{q^{k(s-1)}}+(1-q^{-s})\sum_{k=0}^\infty \frac{R(q^k)}{q^{ks}}.$$

Since the general term of the last series in G satisfies $\ll D^{c_1}q^{-\eta k}$, the function represented by that series is regular and $\ll D^{c_1}/(1-q^{-\eta})$. This proves (12) for $s \ll 1/\log q$. The periodicity of $\zeta(s,H)$ in $\sigma>1$ follows from the representation

$$\zeta(s,H) = d_0+d_1q^{-s}+d_2q^{-2s}+\ldots, \quad \text{where} \quad d_k = \sum_{q^k=a\epsilon H} 1,$$

and further it holds by analytic continuation.

4. Further let χ denote the characters of the group K, χ_0 being the principal character, and let $\chi(a)=\chi(H)$ for all $a \epsilon H$. Write

$$\zeta(s,\chi) = \sum_H \chi(H)\zeta(s,H) = \sum_a \chi(a)a^{-s} \qquad (\sigma>1). \tag{13}$$

Since

$$\sum_H \chi(H) = \begin{cases} h & \text{if} \quad \chi=\chi_0, \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

(see [12], § 10), by Lemma 1 the function $\zeta(s,\chi)$ is regular in $\sigma>1-\vartheta$, except for a simple pole of $\zeta(s,\chi_0)$ at $s=1$ with residue ha (if $q=1$) or simple poles at $s=1+2k\pi i/\log q$ $(k=0,\pm1,\ldots)$ with residues θah (if $q>1$). In the latter case all the functions $\zeta(s,\chi)$ are periodic, the period being $2\pi i/\log q$. By (13), (12) and (2) we have uniformly in $1-\vartheta+\eta \leqslant \sigma \leqslant 2$

$$\zeta(s,\chi) = \begin{cases} e_0ha(s-1)^{-1}+O(D^c\eta^{-1}|s|) & \text{if} \quad q=1, \\ e_0ha\dfrac{1-q^{-1}}{1-q^{1-s}}+O\big(D^c\eta^{-1}(1+\vartheta|s|)\big) & \text{if} \quad q>1, \end{cases} \tag{15}$$

where

$$e_0 = \begin{cases} 1 & \text{if} \quad \chi=\chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathfrak{G}$ being a semi-group with the generators b, in the half-plane E $(\sigma>1)$ we have

$$\zeta(s,\chi) = \prod_b \big(1-\chi(b)b^{-s}\big)^{-1},$$

whence there are no zeros of $\zeta(s,\chi)$ in E.

Let $\mu(a)=(-1)^\nu$ if a is a product of ν different generators, and $=0$ if $b^2 \mid a$. Further let

$$\Lambda(a) = \begin{cases} \log b & \text{if} \quad a=b^n \quad (n \geqslant 1), \\ 0 & \text{otherwise.} \end{cases}$$

From the product-form of $\zeta(s,\chi)$ we deduce

$$\begin{aligned} 1/\zeta(s,\chi) &= \sum_a \chi(a)\mu(a)a^{-s}, \\ \zeta'/\zeta(s,\chi) &= -\sum_a \chi(a)\Lambda(a)a^{-s} \qquad (\sigma>1). \end{aligned} \tag{16}$$

5. Lemma 2. *Let R be the rectangle whose vertices are $-1\pm\pi i/\log q$, $2\pm\pi i/\log q$ and let*

$$g(s) = \frac{\log q}{1-q^{1-s}}-\frac{1}{s-1}.$$

Then in R

$$g(s) \ll \log eq.$$

Proof. On the horizontal sides of R $|g(s)| = (1+q^{1-\sigma})^{-1}\log q + O(\log q) \ll \log q$, on the left vertical side $|g(s)| \leqslant (q^2-1)^{-1}\log q + O(1) = O(1) \ll \log eq$ and on the right vertical side $|g(s)| \leqslant (1-q^{-1})^{-1}\log q + O(1) \ll \log eq$. Hence the lemma follows by the maximum modulus principle.

COROLLARY. *In R*

$$\frac{\log q}{1-q^{1-s}} = \frac{1}{s-1} + O(\log eq). \tag{17}$$

Since, for any $q>1$, $1-q^{-1} \ll \log eq$, we have, by (16), (15), for all $\eta \in (0,1]$ and $q \geqslant 1$

$$1/\zeta(1+\eta+it, \chi) \ll \zeta(1+\eta, \chi_0) \ll \eta^{-1}D^c. \tag{18}$$

Let $f(s)$ be a regular function and let $|f(s)/f(s_0)| < e^M$ in $|s-s_0| \leqslant r$. Then by Landau's lemma (see [23], III, § 9)

$$f'/f(s) - \sum (s-\varrho)^{-1} \ll M/r \qquad (|s-s_0| \leqslant \tfrac{1}{4}r),$$

where ϱ runs through the zeros of $f(s)$ in $|s-s_0| \leqslant \frac{1}{2}r$. Taking

$$r = \vartheta, \qquad s_0 = 1+\eta+it_0, \qquad \eta = 1/D(1+\partial|t_0|)$$

and using (15), (18) we deduce that in $|s-s_0| \leqslant \frac{1}{4}\vartheta$

$$\zeta'/\zeta(s, \chi) + e_0\varphi(s) - \sum_{|\varrho-s_0|\leqslant\vartheta/2} (s-\varrho)^{-1} \ll \log D(1+\partial|t_0|), \tag{19}$$

where

$$\varphi(s) = \begin{cases} (s-1)^{-1} & \text{if} \quad q = 1, \\ \dfrac{\log q}{1-q^{1-s}} & \text{if} \quad q > 1. \end{cases}$$

Let $0 < \sigma - 1 \leqslant 1$. Then, by (17),

$$\frac{\log q}{1-q^{1-\sigma}} = \frac{1}{\sigma-1} + O(\log eq).$$

From this estimate and (19) by arguments used in [5], § 11 we can deduce the inequality

$$|\zeta'/\zeta(\sigma_0, \chi_0)| < \tfrac{5}{4}(\sigma_0-1)^{-1} \quad \text{where} \quad \sigma_0 = 1+c_3/\log D \tag{20}$$

and c_3 is small enough.

Taking $s = \sigma > 1$ we have in (19) re $\sum \geqslant 0$. Hence for all positive $r \leqslant 1$

$$|\zeta'/\zeta(1+r, \chi_0)| \leqslant 1/r + c_4\log D. \tag{21}$$

From (19), (21) we can deduce

$$\nu \leqslant r\log D(1+\partial|t_0|) \qquad (c_5/\log D(1+\partial|t_0|) \leqslant r \leqslant \tfrac{1}{4}\vartheta - 1/D(1+\partial|t_0|)) \tag{22}$$

where $\nu = \nu(r, \chi, t_0)$ denotes the number of zeros of $\zeta(s, \chi)$ in $|s-1-it_0| < r$ (cf. [5], § 10).

6. LEMMA 3. *The number of zeros of $\zeta(s, \chi)$ in the rectangle R $(1-\frac{1}{2}\vartheta \leqslant \sigma \leqslant 1,\ |t-t_0| \leqslant \frac{1}{2})$ does not exceed $\ll \log D(1+\partial|t_0|)$.*

Proof. Let C, C_1, C_x denote circles having a common centre at $s_0 = 1+\vartheta+\eta+it_0$ $(\eta = 1/D(1+\partial|t_0|))$, the radii being $2\vartheta, \frac{7}{4}\vartheta, x$, respectively. Further let $\nu(x)$ denote the number of zeros of $\zeta(s, \chi)$ in C_x. If $\chi \neq \chi_0$, then by (15) in C

$$|\zeta(s, \chi)| \leqslant \exp\{c\log D(1+\partial|t_0|)\}.$$

By Jensen's theorem ([24], § 3.61) and (18)

$$\int_0^{2\vartheta} \frac{\nu(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log|\zeta(s_0+2\vartheta e^{i\varphi}, \chi)|\,d\varphi - \log\zeta(s_0, \chi)$$
$$< c_3\log D(1+\partial|t_0|),$$

whence

$$c_3\log D(1+\partial|t_0|) > \int_0^{2\vartheta} \frac{\nu(x)}{x}\,dx \geqslant \int_{7\vartheta/4}^{2\vartheta} \frac{\nu(x)}{x}\,dx \geqslant \nu(7\vartheta/4)\int_{7\vartheta/4}^{2\vartheta} \frac{dx}{x}$$
$$= \nu(7\vartheta/4)\log 8/7$$

and thus

$$\nu(7\vartheta/4) \ll \log D(1+\partial|t_0|).$$

Observing that R can be covered by $\leqslant 1/\vartheta$ circles C' with the radii $7\vartheta/4$ (having their centres on the line $\sigma = 1+\vartheta+\eta$), we get the lemma for $\chi \neq \chi_0$.

If $\chi = \chi_0$, then we use the function $(s-1)\zeta(s, \chi_0)$ or $(1-q^{1-s})\zeta(s, \chi_0)$ (according as $q = 1$ or $q > 1$) and similar arguments.

7. LEMMA 4. *If $q = 1$, then for appropriate c no function $\zeta(s, \chi)$ has a zero in the region $\sigma > 1 - c/\log D(1+|t|)$, $|t| \geqslant 3$.*

The proof is based on the inequality

(23) $$-3\zeta'/\zeta(\sigma_0,\chi_0)-4\,\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi)-\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi^2)\geqslant 0\quad(\sigma_0>1)$$

(cf. [5], § 12). In $\sigma\geqslant 1-\frac{1}{4}\vartheta+\eta$ $(\eta=1/D(1+|t|))$ we have by (19)

$$\zeta'/\zeta(s,\chi)=\sum_{|\varrho-s_0|<\vartheta/2}(s-\varrho)^{-1}+O\big(\log D(1+|t|)\big)\qquad(s_0=1+\eta+it).$$

Let $\varrho=\beta+i\gamma$ $(|\gamma|\geqslant 3)$ be a zero of $\zeta(s,\chi)$. Since $\mathrm{re}\sum(s-\varrho)^{-1}\geqslant 0$ in $\sigma>1$, we have

$$\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi)\geqslant(\sigma_0-\beta)^{-1}-c_3\log D|\gamma|,$$

$$\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi^2)\geqslant -c_3\log D|\gamma|.$$

Take for σ_0 any number satisfying (20). Then, by (23),

$$\frac{15}{4}\cdot\frac{1}{\sigma_0-1}\geqslant 4\frac{1}{\sigma_0-1+1-\beta}-c_4\log D|\gamma|.$$

Putting

$$\sigma_0-1=1/A\log D|\gamma|,\qquad 1-\beta=1/A^2\log D|\gamma|$$

and dividing through by $A\log D|\gamma|$, we get the inequality

$$\frac{15}{4}\geqslant\frac{4}{1+1/A}-\frac{c_4}{A}$$

in which A cannot increase indefinitely. Since $\beta=1-1/A^2\log D|\gamma|$, the lemma follows.

LEMMA 5. *For appropriate c' no function $\zeta(s,\chi)$ with a complex χ has a zero in the region*

$$\sigma>1-c'/\log D,\quad |t|<5\quad\textit{or}\quad \sigma>1-c'/\log D(1+\vartheta|t|)$$

(*according as* $q=1$ *or* $q>1$).

χ^2 not being the principal character, the proof proceeds along the same lines as that of the previous lemma.

8. LEMMA 6. *For appropriate c no function $\zeta(s,\chi)$ with a real $\chi\neq\chi_0$ has a zero in the region*

$$\sigma>1-c/\log D,\qquad 0<|t|\leqslant 5$$

or

$$\sigma>1-c/\log D(1+\vartheta|t|),\quad 0<|t|<\pi/\log q$$

(*according as* $q=1$ *or* $q>1$).

Proof. First let $q=1$ and let $\zeta(s,\chi)$ have a zero $\varrho=\beta+i\gamma$ with

$$0<\gamma<1/B\log D,\qquad 0\leqslant 1-\beta<1/B\log D\qquad(B\lll 1).$$

Then it has another zero $\varrho'=\beta-i\gamma$. Writing

$$s_0=\sigma_0+i\gamma,\qquad \sigma_0=1+1/A\log D\qquad(A\lll 1)$$

we have, by (19),

$$\mathrm{re}\,\zeta'/\zeta(s_0,\chi)>\frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}-c_0\log D.$$

By (16), (20)

$$\mathrm{re}\,\zeta'/\zeta(s_0,\chi)\leqslant|\zeta'/\zeta(\sigma_0,\chi_0)|<\tfrac{5}{4}(\sigma_0-1)^{-1}$$

and thus

(24) $$\frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}<c_3\log D+\frac{\frac{5}{4}}{\sigma_0-1},$$

whence

$$\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}<c_3\log D+\frac{\frac{1}{4}}{\sigma_0-1}+\frac{1-\beta}{(\sigma_0-1)(\sigma_0-\beta)}$$

$$<c_3\log D+\frac{1}{4}A\log D+\frac{A^2}{B}\log D<\left(\frac{1}{2}A+\frac{A^2}{B}\right)\log D.$$

Since

$$\frac{1/A}{\log D}\leqslant\sigma_0-\beta\leqslant\frac{1/A+1/B}{\log D},$$

we have

$$\frac{1/A}{\log D}\leqslant\left\{\left(\frac{1/A+1/B}{\log D}\right)^2+\frac{4}{(B\log D)^2}\right\}\left(\frac{1}{2}A+\frac{A^2}{B}\right)\log D,$$

whence

$$\frac{1}{A}<\left\{\left(\frac{1}{A}+\frac{1}{B}\right)^2+\frac{4}{B^2}\right\}\left(\frac{1}{2}A+\frac{A^2}{B}\right).$$

This is impossible for any fixed A (with $\frac{1}{4}A>c_3$) and all large B, since the right-hand side tends to $1/2A$ as $B\to\infty$. This proves the lemma in the case we have been dealing with.

Now let us suppose that $\zeta(s,\chi)$ has a zero $\beta+i\gamma$ with $\gamma\geqslant 1/B\log D$. Since $\chi^2=\chi_0$, by (23)

(25) $$-3\zeta'/\zeta(\sigma_0,\chi_0)-4\,\mathrm{re}\,\zeta'/\zeta(\sigma_0+i\gamma,\chi)-\mathrm{re}\,\zeta'/\zeta(\sigma_0+2i\gamma,\chi_0)\geqslant 0$$

$$(\sigma_0>1).$$

The estimates for the first two terms of (25) may be acquired as in the proof of Lemma 5. By (19)

$$\operatorname{re}\zeta'/\zeta(\sigma_0+2i\gamma,\chi_0)$$

$$= -\operatorname{re}\frac{1}{\sigma_0+2i\gamma-1}+\operatorname{re}\sum_{|1+2i\gamma-\varrho|<\vartheta/2}\frac{1}{\sigma_0+2i\gamma-\varrho}+O(\log D) > -c_4\log D$$

(since $\operatorname{re}\sum \geqslant 0$ and $|\operatorname{re}(\sigma_0+2i\gamma-1)^{-1}| \leqslant |\sigma_0-1+2i\gamma|^{-1} < \frac{1}{2}B\log D$) and we may proceed as in Lemma 5.

Now let us consider the case of $q>1$. If $\pi/\log q > 2$, then the necessary inequalities for the zeros in $1 \leqslant |t| \leqslant \pi/\log q-1$ are obtainable as in the proof of Lemma 4. In the remaining parts $0<|t|\leqslant 1$ and $\pi/\log q-1 \leqslant |t| < \pi/\log q$ we use the function $Z(s) = \zeta(s-\pi i/\log q, \chi)$ and the same arguments as for $q=1$ of the present lemma but changing at most D to $D(1+\pi/\log q)$. Consider that the zeros of $Z(s)$ are conjugate, since the coefficients of the Dirichlet series of $Z(s)$ are real. The same proof holds if $\pi/\log q \leqslant 2$.

LEMMA 7. *Let χ be a real character $\neq \chi_0$. For appropriate c' there is at most one real zero $\beta' > 1-c'/\log D$ and (if $q>1$) at most one zero $\varrho'' = \beta''+i\pi/\log q$ with $\beta'' > 1-c'/\log D(1+\pi/\log q)$.*

Proof. Let β and β' be two real zeros of $\zeta(s,\chi)$ such that $\beta' \geqslant \beta$. By the arguments used in the proof of (24)

$$\frac{1}{\sigma_0-\beta}+\frac{1}{\sigma_0-\beta'} < c_5\log D+\frac{5}{4}(\sigma_0-1)^{-1},$$

whence for a sufficiently large $A \ll 1$

$$2/(\sigma_0-\beta) < c_5\log D+\tfrac{5}{4}A\log D < \tfrac{3}{2}A\log D,$$

$$\sigma_0-\beta > 4/3A\log D, \qquad \beta < \sigma_0-4/3A\log D = 1-c'/\log D.$$

This proves the statement concerning the real zeros. The same method of proof may be used for the zeros on the line $t=\pi/\log q$.

9. LEMMA 8. *For appropriate c the function $\zeta(s,\chi_0)$ has no zero in the region*

$$\sigma > 1-c/\log D, \qquad 0<|t|\leqslant 5$$

or

$$\sigma > 1-c/\log D(1+\vartheta|t|), \qquad 0<|t|<\pi/\log q$$

(according as $q=1$ or $q>1$).

Proof. Writing

$$G(s) = \begin{cases}(s-1)\zeta(s,\chi_0) & \text{if} \quad q=1,\\ (1-q^{1-s})\zeta(s,\chi_0) & \text{if} \quad q>1,\end{cases}$$

$$\sigma_0 = 1+1/A\log D, \qquad s_0 = \sigma_0+i\gamma$$

and using (19) and (22), by the arguments of [5], § 18 we can prove that for a sufficiently large $A \ll 1$ and a small $\gamma \ll 1/\log D$

(26) $$\operatorname{re}G'/G(s_0) < \tfrac{5}{4}(\sigma_0-1)^{-1}.$$

Let us suppose that $\zeta(s,\chi_0)$ has a zero $\varrho = \beta+i\gamma$ with $0<\gamma<1/B\log D$; then it has another zero $\varrho' = \beta-i\gamma$. By (19) and Lemma 3

$$G'/G(s_0) = \sum_{|s_0-\varrho|<\vartheta/2}(s_0-\varrho)^{-1}+O(\log D)$$

whence

$$\operatorname{re}G'/G(s_0) \geqslant \frac{1}{\sigma_0-\beta}+\frac{\sigma_0-\beta}{(\sigma_0-\beta)^2+4\gamma^2}-c_3\log D,$$

and using (26) we may go on as in Lemma 6. This proves the required result concerning zeros near the line $t=0$. If D is replaced by $D(1+ +\pi/\log q)$, the same method may be used for zeros near the line $t=\pi/\log q$.

Now suppose that $q=1$, $\gamma \geqslant 1/B\log D$ and σ_0 is any number for which (20) holds. Then by (19)

$$\operatorname{re}\zeta'/\zeta(\sigma_0+i\gamma,\chi_0) \geqslant (\sigma_0-\beta)^{-1}-c_4\log D,$$

$$\operatorname{re}\zeta'/\zeta(\sigma_0+2i\gamma,\chi_0) > -c_5\log D.$$

Using (20) and (25) (with $\chi=\chi_0$) and the arguments of Lemma 5 we prove the remaining part of the present lemma for $q=1$. The method of proof works as well for the zeros off the lines $t=0$, $t=\pi/\log q$ in the case of $q>1$.

LEMMA 9. *For appropriate c' the function $\zeta(s,\chi_0)$ has at most one real zero $\beta' > 1-c'/\log D$ and (if $q>1$) at most one zero $\beta''+i\pi/\log q$ with $\beta'' > 1-c'/\log D(1+\pi/\log q)$.*

This may be proved by the arguments of Lemma 7.

10. LEMMA 10. *For appropriate c' at most one of the $\leqslant h$ functions $\zeta(s,\chi)$ with a real character has a real zero $> 1-c'/\log D$ and, if $q>1$, at most one of them has a zero $\beta+i\pi/\log q$ with $\beta > 1-c'/\log D(1+\pi/\log q)$.*

Proof. First let $q=1$ and let the two functions $\zeta(s,\chi_1)$ and $\zeta(s,\chi_2)$ with two real and different characters χ_1,χ_2 have the real zeros

$\beta_1 > 1-\vartheta/6$ and $\beta_2 > 1-\vartheta/6$, respectively. Further let $\chi_1 \neq \chi_0$, $\chi_2 \neq \chi_0$. Taking $\sigma_0 = 1+1/A\log D$ with a sufficiently large $A \ll 1$ we have, by (19) and (20),

$$\zeta'/\zeta(\sigma_0, \chi_1) > (\sigma_0-\beta_1)^{-1} - c_3\log D,$$
$$\zeta'/\zeta(\sigma_0, \chi_2) > (\sigma_0-\beta_2)^{-1} - c_3\log D,$$
$$\zeta'/\zeta(\sigma_0, \chi_1\chi_2) > -c_3\log D,$$
$$\zeta'/\zeta(\sigma_0, \chi_0) > -\tfrac{5}{4}(\sigma_0-1)^{-1}.$$

Since the sum of the left-hand sides in this set of inequalities is $\leqslant 0$, by (16), we deduce

$$(\sigma_0-\beta_1)^{-1}+(\sigma_0-\beta_2)^{-1} < c_4\log D+\tfrac{5}{4}(\sigma_0-1)^{-1}$$

and may go on as in the proof of Lemma 7.

Now let $\chi_2 = \chi_0$, the other premises remaining unchanged. Then instead of $\zeta(s, \chi_2)$ we use the function $G(s)$ of § 9. By (19)

$$G'/G(\sigma_0) > (\sigma_0-\beta_2)^{-1} - c_5\log D$$

and

$$\zeta'/\zeta(\sigma_0, \chi_1) > (\sigma_0-\beta_1)^{-1} - c_3\log D.$$

Since, by (16),

$$\zeta'/\zeta(\sigma_0, \chi_1)+\zeta'/\zeta(\sigma_0, \chi_0) \leqslant 0,$$

we deduce

$$(\sigma_0-\beta_1)^{-1}+(\sigma_0-\beta_2)^{-1} < (\sigma_0-1)^{-1}+c_6\log D$$

and may go on as before.

The same arguments may be used in case of $q>1$ if, dealing with the zeros on $t = \pi/\log q$, we replace D by $D(1+\pi/\log q)$.

The results embodied in Lemmas 4-10 may be summarized by the following

FUNDAMENTAL LEMMA 11. *For appropriate c in the region*

(27) $$\sigma \geqslant 1-c/\log D(1+|t|) \quad (\geqslant 1-\vartheta/8)$$

there is no zero of $\zeta(s,\chi)$ in case of a complex χ. For at most one real χ there may be in (27) a simple real zero $\beta' \leqslant 1$ and, if $q>1$, the zeros $\beta'+2k\pi i/\log q$ $(k = 0, \pm 1, \ldots)$. Besides, if $q>1$, for at most one real χ there may be in (27) the set of simple zeros $\varrho''+2k\pi i/\log q$ $(k = 0, \pm 1, \ldots)$ with $\operatorname{re}\varrho'' \leqslant 1$, $\operatorname{im}\varrho'' = \pi/\log q$.

The zeros $\beta', \beta'+2k\pi i/\log q$ and $\varrho''+2k\pi i/\log q$ (if existing) will be called the *exceptional zeros* of $\zeta(s,\chi)$.

An upper bound for the number of generators.

11. LEMMA 12. *Let*

(28) $$a_m \quad (m = 1, 2, \ldots, N)$$

be a set of numbers $\epsilon\, \mathfrak{G}$ such that for any $d \,\epsilon\, \mathfrak{G}$

(29) $$\sum_{d|a_m} 1 = N/f(d)+R_d$$

where $f(d)$ is a multiplicative function $\neq 0$; that is to say, $f(a_i a_j) = f(a_i)f(a_j)$ whenever $(a_i, a_j) = 1$. Further let N_z denote the number of those numbers a_m of (28) which are not divisible in $\mathfrak{G}$ by any generator $b \leqslant z$. Write

(30) $$F(a) = \sum_{d|a}\mu(d)f\left(\frac{a}{d}\right), \qquad S_z = \sum_{d\leqslant z}\frac{\mu^2(d)}{F(d)}, \qquad S_z(a) = \sum_{\substack{d\leqslant z/a\\ (d,a)=1}}\frac{\mu^2(d)}{F(d)},$$

$$\lambda_a = \begin{cases}\mu(a)\prod_{b|a}\left(1-1/f(b)\right)^{-1}S_z(a)/S_z & \text{if} \quad a\leqslant z,\\ 0 \quad \text{otherwise.}\end{cases}$$

Then

(31) $$N_z \leqslant N/S_z + \sum_{a_1\leqslant z, a_2\leqslant z}|\lambda_{a_1}\lambda_{a_2}R_{a_1a_2/(a_1,a_2)}|.$$

This may be proved by the sieve method of A. Selberg [19]. See [7], § 3.

12. LEMMA 13. *In the previous lemma let (28) be all the numbers $a \leqslant x$ of any class H_i (as defined in § 1). If $z \geqslant x^{1/5}$, $x \geqslant D^{c_3}$ and c_3 is large enough, then the main term in (31) satisfies*

$$N/S_z \ll x/\theta h\log x$$

(with the θ defined in § 2).

Proof. By (3)

(32) $$N = ax+O(D^{c_1}x^{1-\vartheta}).$$

Considering that the classes H_i form a group, we deduce that for any $d \,\epsilon\, \mathfrak{G}$ the number of numbers $a\leqslant x$ such that $a \,\epsilon\, H_i$ and $d \mid a$, by (3) is

$$a\frac{x}{d}+O\left(D^{c_1}\left(\frac{x}{d}\right)^{1-\vartheta}\right) = \frac{N+O(D^{c_1}x^{1-\vartheta})}{d}+O\left(D^{c_1}\left(\frac{x}{d}\right)^{1-\vartheta}\right)$$
$$= \frac{N}{d}+O\left(D^{c_1}\left(\frac{x}{d}\right)^{1-\vartheta}\right).$$

Hence we have (29) with

(33) $$f(d) = d, \qquad R_d \ll D^{c_1}\left(\frac{x}{d}\right)^{1-\vartheta}.$$

Now by (30)

$$S_z = \sum_{a\leqslant z} \frac{\mu^2(a)}{a\prod_{b|a}(1-1/b)} = \sum_{a\leqslant z}\mu^2(a)\prod_{b|a}\left(\frac{1}{b}+\frac{1}{b^2}+\ldots\right) = \sum_{a\epsilon(z)}\frac{1}{a},$$

where (z) stands for the set of numbers $a'\epsilon\mathfrak{G}$ such that the product of all different generators [5] of any a' is $\leqslant z$. Hence

$$S_z > \sum_{a\leqslant z}\frac{1}{a} > \sum_{\sqrt{z}<a<z}\frac{1}{a}.$$

If c_3 is large enough, then by (7) the last sum is $>\frac{1}{2}h\theta\alpha\log\sqrt{z} > c_4\theta h\alpha\log x$. Since $N<2\alpha x$, by (32), we get the required result.

13. LEMMA 14. *Let W denote the remaining term in* (31) *and let in Lemma* 13

(34) $$z^{2\vartheta} = x^{\vartheta}/\alpha^2h^3D^{c_4}\log x \quad \textit{where} \quad c_4 = c_1+c_0+\max(c_1, l);$$

then

$$W \ll x/h\log x.$$

Proof. For any a we have $|\lambda_a|\leqslant 1$ (see [7], (38)), whence by (31), (33)

$$W \ll D^{c_1}x^{1-\vartheta}\sum_{a_1\leqslant z, a_2\leqslant z}\left(\frac{(a_1, a_2)}{a_1a_2}\right)^{1-\vartheta},$$

and the desired estimate would follow from

(35) $$\sum_{a_1\leqslant z, a_2\leqslant z}\left(\frac{(a_1, a_2)}{a_1a_2}\right)^{1-\vartheta} \ll \frac{x^{\vartheta}}{hD^{c_1}\log x}.$$

Using (3), we prove first the estimates

(36) $$\sum_{a\leqslant z}a^{-(1-\vartheta)} \ll h\alpha z^{\vartheta}, \qquad \sum_{a\leqslant z}a^{-1-\vartheta} \ll D^{c_4-c_1},$$

whence

(37) $$\sum_{\substack{a_1\leqslant z, a_2\leqslant z\\ (a_1,a_2)=1}}\left(\frac{(a_1, a_2)}{a_1a_2}\right)^{1-\vartheta} \leqslant \left(\sum_{a\leqslant z}a^{-1+\vartheta}\right)^2 \ll (h\alpha)^2z^{2\vartheta}.$$

---

[5] There may be different generators of $\mathfrak{G}$ having the same numerical b. We suppose that they are distinguished by different indices.

Write

$$S_d(z) = \sum_{\substack{a_1\leqslant z, a_2\leqslant z\\ (a_1,a_2)=d}}\left(\frac{(a_1, a_2)}{a_1a_2}\right)^{1-\vartheta}.$$

By (37)

$$S_d(z) = d^{-1+\vartheta}S_1\left(\frac{z}{d}\right) \ll (h\alpha)^2z^{2\vartheta}d^{-1-\vartheta}$$

whence by (36), (34)

$$\sum_{a_1\leqslant z, a_2\leqslant z}\left(\frac{(a_1, a_2)}{a_1a_2}\right)^{1-\vartheta} \leqslant \sum_{d\leqslant z}S_d(z) \ll (h\alpha)^2z^{2\vartheta}\sum_{d\leqslant z}d^{-1-\vartheta} \ll D^{c_4-c_1}(h\alpha)^2z^{2\vartheta}$$
$$= x^{\vartheta}/hD^{c_1}\log x,$$

which proves (35).

14. LEMMA 15. *If $x\geqslant D^{c_3}$ with a sufficiently large c_3, then*

(38) $$\pi(x, H) \ll x/\theta h\log x.$$

Proof. Since $z<x^{3/5}$, by (34), it follows from Lemma 13 that all the generators $b\epsilon[z, x]$ of the class H are in the set of the N_z numbers defined in Lemma 12. Hence, by Lemmas 13, 14 and by (3)

$$\pi(x, H) \leqslant N_z+\pi(z, H) \leqslant N_z+h\alpha z+O(hD^{c_1}z^{1-\vartheta}) \ll x/\theta h\log x.$$

COROLLARY. *If $x\geqslant D^{c_3}$, then*

(39) $$\sum_{x\geqslant b\epsilon H}\log b \ll x/\theta h,$$

(40) $$\sum_{x\geqslant a\epsilon H}\Lambda(a) \ll x/\theta h,$$

(41) $$\sum_{\substack{a\epsilon H\\ x<a<x^3}}\frac{\Lambda(a)}{a} \ll h^{-1}(\theta^{-1}+\log x),$$

(42) $$\sum_{\substack{a\epsilon H\\ D^{c_3}\leqslant a\leqslant x}}\frac{\Lambda(a)}{a^{\sigma}} \ll (\theta h)^{-1}x^{1-\sigma} \qquad (0<\sigma<1-\vartheta/8).$$

Proof. The left-hand side of (39) being $\leqslant\pi(x, H)\log x$, the estimate is evident by (38).

Let b_1 be the least generator of $\mathfrak{G}$. If $\log b_1<D^{-A}$ for any $A\ll 1$, then there are at least $D^{A}\log D$ numbers $a\leqslant D$ of $\mathfrak{G}$ (the powers of b_1) and for A large enough this contradicts (3). Hence for appropriate c_4 we have $\log b_1 > D^{-c_4}$.

By (39) and by the definition of $\Lambda(a)$, (40) would clearly follow from

$$(43)\qquad \sum_{x\geqslant b^2\epsilon H}\log b+\sum_{x\geqslant b^3\epsilon H}\log b+\ldots \ll x/0h.$$

Since the number of terms on the left is $\ll \log x/\log b_1 < D^{c_4}\log x$, none of them exceeding $\{a\sqrt{x}+O(D^{c_1}x^{(1-\vartheta)/2})\}\log x$, the truth of (43) is evident (if c_3 is large enough).

(41) and (42) may be deduced from (40) by partial summation (cf. (7)).

A general lemma. 15. In this section the main weapon of proof will be the following inequality of Turán:

Let z_j $(j=1,2,\ldots,n;\ n\leqslant N)$ be a set of complex numbers with $\max_j|z_j|\geqslant 1$ and let $m\geqslant 0$. Then there is a constant c $(0<c<24)$ such that for appropriate integer

$$(44)\qquad \nu\,\epsilon\,[m+1,\ m+N]$$

we have

$$(45)\qquad |z_1^\nu+z_2^\nu+\ldots+z_n^\nu|\geqslant\left(\frac{N}{c(m+N)}\right)^N.$$

By [21], Satz X, this is true for $c=96e^2$. An improvement of c and a simplification of the proof are given in [20].

LEMMA 16. *Let k, A, D, λ, τ_0 denote unbounded parameters, k being an integer $\geqslant 2$,*

$$A>2/\vartheta_0,\quad D>2,\quad Ak\geqslant\log D,\quad 0<c'\leqslant\lambda\leqslant(\vartheta_0/4B)\log D$$

where $B>1$ is a sufficiently large constant, $0<\vartheta_0\leqslant 1$, $-D\leqslant\tau_0\leqslant D$. Further let $F(s)$ be meromorphic in $\sigma>1-\vartheta_0$ with simple poles $\varrho=\beta+i\gamma$ which lie in $\sigma\leqslant 1$. Denoting by m_ϱ the residue of $F(s)$ at $s=\varrho$ we take for granted that for any real t_0

$$(46)\qquad \Big|F(s)-\sum_{|\varrho-1-it_0|<\vartheta_0}\frac{m_\varrho}{s-\varrho}\Big|<c''\log D(1+|t_0|)\qquad(|s-1-it_0|<\vartheta_0/2)$$

and

$$(47)\qquad \sum_{|\varrho-1-it_0|\leqslant r}(1+|m_\varrho|)\ll r\log D(1+|t_0|)\qquad(c'/\log D(1+|t_0|)\leqslant r\leqslant\vartheta_0),$$

supposing that the constant in $\ll$ (as other constants during the proof) may depend on c', c'', ϑ_0. For any real τ let

$$(48)\qquad J(\tau,k,A)=-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{e^{3As}-e^{As}}{2As}\right)^kF(1+s+i\tau)\,ds$$

and let Q_E (for any bounded $E\geqslant 1$) denote the square $(1-E\lambda/\log D\leqslant\sigma\leqslant 1,\ |t-\tau_0|\leqslant E\lambda/2\log D)$.

(i) *If for all $\varrho\,\epsilon\,Q_E$ the residues m_ϱ are positive integers and if there is at least one $\varrho\,\epsilon\,Q_1$, then for appropriate $C_1>1$ and for any $C\geqslant C_1^2$ there is a $k\geqslant\max(2, C\lambda)$, $k<(C+C_1)\lambda$ such that for any τ of the interval $I(|\tau-\tau_0|\leqslant\lambda/2\log D)$*

$$(49)\qquad |J(\tau,k,\lambda^{-1}\log D)|>\exp(-C_2\lambda),$$

where the constant C_2 may depend on C, C_1.

(ii) *For all $\varrho\,\epsilon\,Q_E$ let the residues m_ϱ be positive integers except for at most $\ll E\lambda$ poles $\varrho_j=1+it_j$ with $m_{\varrho_j}=-1$ such that for any one of them there is a corresponding pole $\varrho_j'=\varrho_j-\delta$ (where $0<\delta\leqslant e^{-B_1\lambda}/\log D$, B_1 arbitrary large $\ll 1$) with $m_{\varrho_j'}=1$. Further let there be a pole $\varrho_0=1-\lambda/\log D+i\tau_0$ and let the number of the points $t_j\,\epsilon\,I(|\tau-\tau_0|\leqslant\lambda/2\log D)$ be $\ll\lambda$. Then there is a number $\tau=\tau_1\,\epsilon\,I$ for which (49) holds.*

Proof. In this paragraph we shall deal with case (i).

By (47) there is an infinite broken line L in the strip $-\vartheta_0/3-1/C_3\log D(1+|t|)<\sigma<-\vartheta_0/3+1/C_3\log D(1+|t|)$ (with a suitable constant $C_3>2$) such that for any selected $\tau\ll D$ the distance between every point $s=\sigma+it\,\epsilon\,L$ and the nearest pole of $F(1+s+i\tau)$ exceeds $1/C_3^2\log^2D(1+|t|)$ (cf. [8], § 8) whence by (46), (47) on the line $F(1+\sigma+it+i\tau)\ll\log^3D(1+|t|)$. Besides the length of the piece of L between any two of its points $\sigma+it$, $\sigma'+i(t+1)$ does not exceed 2.

Writing

$$(50)\qquad g(s)=\frac{e^{3As}-e^{As}}{2As}$$

and moving in (48) the path of integration to the line L we deduce

$$(51)\qquad J(\tau,k,A)=-\sum_{\varrho\epsilon G}m_\varrho g(\varrho-1-i\tau)^k-\frac{1}{2\pi i}\int_L g(s)^kF(1+s+i\tau)\,ds,$$

where G denotes the region on the right of L.

Since $k\geqslant 2$, $Ak\geqslant\log D$, $A>2/\vartheta_0$, on L we have, by (50), $g(s)^k\ll D^{-\vartheta_0/3}(1+|t|)^{-2}$ whence the integral in (51) is $\ll D^{-\vartheta_0/3}\log^3D$.

Now let us suppose that there is a pole $\varrho_0\,\epsilon\,Q_1$ of $F(s)$ and let τ_1 be any selected $\tau\,\epsilon\,I$. Suppose that for a fixed $C\geqslant C_1^2$ and for all integers $k\,\epsilon\,[\max(2, C\lambda),\ C\lambda+C_1\lambda]$ we have

$$(52)\qquad |J(\tau_1,k,\lambda^{-1}\log D)|\leqslant\exp(-B\lambda).$$

From this (49) will be proved by arriving at a contradiction.

First let us estimate that part S_1 (say) of the sum in (51) which corresponds to the region $G_1 \subset G$ with $t \geqslant \tau_0 + E\lambda/2\log D$. The sum $\sum |m_\varrho|$ over all ϱ of the rectangle

$$R_\delta(1-\delta \leqslant \sigma \leqslant 1, \quad \tau_0 + t_0 \leqslant t \leqslant \tau_0 + t_0 + E\lambda/2\log D) \qquad (t_0 \geqslant E\lambda/2\log D)$$

by (47) does not exceed

$$\ll \left(\delta + \frac{E\lambda}{\log D}\right) \log D(1+t_0).$$

Let G' be the region in the strip $\tau_0 + t_0 \leqslant t \leqslant \tau_0 + t_0 + E\lambda/2\log D$ between $\sigma = 1$ and the displacement L' of the line L by 1 in the direction of the positive real axis. The part of the sum in (51) (with $\tau = \tau_1$, $A = \lambda^{-1}\log D$) corresponding to G' satisfies

$$\ll \sum_{\varrho \in G'} \frac{e^{-kA(1-\beta)}}{(t_0 A)^k}$$

$$\ll (At_0)^{-k} \left\{ \int_0^{\vartheta_0/2} kAe^{-kA\delta} \left(\delta + \frac{E\lambda}{\log D}\right) \log D(1+t_0)\, d\delta + e^{-kA\vartheta_0/2} \log D(1+t_0) \right\}$$

$$\ll \frac{\log D(1+t_0)}{(At_0)^k} \left\{ \frac{E\lambda}{\log D} \int_0^{E\lambda/\log D} kAe^{-kA\delta}\, d\delta + \int_{E\lambda/\log D}^{\vartheta_0/2} kA\delta e^{-kA\delta}\, d\delta + e^{-kA\vartheta_0/2} \right\}$$

$$\ll \frac{\log D(1+t_0)}{(At_0)^k} \cdot \frac{E\lambda}{\log D}.$$

Summing over all $t_0 = nE\lambda/2\log D$ $(n = 1, 2, \ldots)$ we get

$$(53) \qquad S_1 \ll \frac{E\lambda}{\log D} \sum_{n=1}^{\infty} \frac{\log D(1 + nE\lambda/2\log D)}{\left(\dfrac{\log D}{\lambda} \cdot \dfrac{E\lambda n}{2\log D}\right)^k}$$

$$\ll \frac{E\lambda}{(E/2)^k} \sum_{n=1}^{\infty} \frac{1}{n^{k-1/2}} \ll E\lambda \left(\frac{1}{2} E\right)^{-k}.$$

The same estimate holds for the part S_2 of the sum in (51) which corresponds to the region $G_2 \subset G$ with $t \leqslant \tau_0 - E\lambda/2\log D$.

Let G_3 denote the remaining part of G without Q_E. The corresponding part of the sum in (51) (with $\tau = \tau_1$, $A = \lambda^{-1}\log D$) satisfies

$$\ll \sum_{\varrho \in G_3} \frac{e^{-kA(1-\beta)}}{(AE\lambda/\log D)^k} \ll E^{-k} \left\{ \int_{E\lambda/\log D}^{\vartheta_0/2} kAe^{-kA\delta} \delta \log D\, d\delta + e^{-kA\vartheta_0/2} \log D \right\}$$

$$= E^{-k} \log D \left\{ \frac{1}{kA} \int_{kE}^{C'\log D} ue^{-u}\, du + e^{-kA\vartheta_0/2} \right\}$$

$$\ll E^{-k} \log D \frac{kEe^{-kE}}{kA} = E^{-k} E\lambda e^{-kE}.$$

This being smaller than the right-hand side of (53), we have by (51) and (52)

$$\sum_{\varrho \in Q_E} m_\varrho g(\varrho - 1 - i\tau_1)^k \ll e^{-B\lambda} + E\lambda \left(\tfrac{1}{2}E\right)^{-k}$$

whence, counting every $\varrho \in Q_E$ in the sequel m_ϱ times, we obtain

$$(54) \qquad g(\varrho_0 - 1 - i\tau_1)^k \sum_{\varrho \in Q_E} \left(\frac{g(\varrho - 1 - i\tau_1)}{g(\varrho_0 - 1 - i\tau_1)} \right)^k \ll e^{-B\lambda} + E\lambda \left(\tfrac{1}{2}E\right)^{-k}.$$

Let R' denote the rectangle $(-\lambda/\log D \leqslant \sigma \leqslant 0,\ |t| \leqslant \lambda/\log D)$ and let $A = \lambda^{-1}\log D$. Writing $s_1 = \varrho_0 - 1 - i\tau_1$ $(s_1 \in R')$ we have

$$|g(s_1)| \geqslant e^{-1} \min_{s \in R'} \left| \frac{1 - e^{2As}}{2As} \right| > e^{-3},$$

since on the boundary of R' the function $(1 - e^{2As})/2As$ is in modulus $> e^{-2}$ and has no zeros $\in R'$. Hence $|g(\varrho_0 - 1 - i\tau_1)|^k > e^{-3k}$ and denoting

$$(55) \qquad z_\varrho = \frac{g(\varrho - 1 - i\tau_1)}{g(\varrho_0 - 1 - i\tau_1)}$$

we have, by (54),

$$(56) \qquad \sum_{\varrho \in Q_E} z_\varrho^k \ll e^{-B\lambda + 3k} + e^{3k} E\lambda \left(\tfrac{1}{2}E\right)^{-k}.$$

By (47) there is a constant C_4 such that if

$$(57) \qquad N = C_4 E\lambda,$$

the number of the poles $\varrho \in Q_E$ does not exceed N. Take

$$(58) \qquad m = [C\lambda].$$

Let us apply Turán's inequality (45) to the numbers z_ϱ, m, N. By (44) there is an integer $\nu > C\lambda$, $\nu < C\lambda+N$ for which (45) holds and further we use $k=\nu$. Then by (57) and (58) the restrictions imposed on k in the present lemma are satisfied with $C_1 = C_4E$. Observing that

$$\frac{N}{m+N} \geqslant \frac{1}{1+C/C_1},$$

we have, by (56), (45)

$$(59)\qquad \left(\frac{1}{c(1+C/C_1)}\right)^{C_4E\lambda} \ll e^{-(B-3C-3C_4E)\lambda}+E\lambda\left(\frac{1}{2e^3}E\right)^{-C\lambda}.$$

The logarithms of the reciprocal values of these numbers are, respectively,

$$U = C_4E\lambda\log c(1+C/C_1),$$
$$U_1 = (B-3C-3C_4E)\lambda,$$
$$U_2 = C\lambda\log E/2e^3-\log E\lambda > \tfrac{1}{2}C\lambda\log E/2e^3$$

(supposing E large enough). Now we take $B = 4C$. Then the orders of magnitude for U_1 and U_2 are higher than that of U as $E\to\infty$. Hence for a sufficiently large $E \ll 1$ the right-hand side of (59) is smaller than the left-hand side, which provides the desired contradiction.

16. Let us now consider case (ii) of the present lemma. By the same arguments as those used before we get instead of (56)

$$(60)\qquad \sum_{\substack{\varrho\in Q_E\\ \varrho\neq\varrho_j,\neq\varrho'_j}} z_\varrho^k - \sum_j \left(z_{\varrho_j}^k - z_{\varrho'_j}^k\right) \ll e^{-B\lambda+3k}+e^{3k}E\lambda(\tfrac{1}{2}E)^{-k}.$$

The number of the points $t_j\in I$ being $\ll\lambda$, there is at least one $\tau_1\in I$ such that the distance between τ_1 and the nearest t_j exceeds $1/C_5\log D$ (for appropriate $C_5\ll 1$). Further let τ_1 be any fixed point having this property.

By (55) and (50)

$$z_{\varrho_j} = \frac{g\big(i(t_j-\tau_1)\big)}{g\big(-\lambda/\log D+i(\tau_0-\tau_1)\big)},\qquad z_{\varrho'_j} = \frac{g\big(-\delta+i(t_j-\tau_1)\big)}{g\big(-\lambda/\log D+i(\tau_0-\tau_1)\big)},$$
$$g(s) = \frac{e^{3As}-e^{As}}{2As}\qquad (A = \lambda^{-1}\log D).$$

From the estimate of $g(s_1)$ in the previous paragraph we can deduce

$$\left|\frac{1}{g\big(-\lambda/\log D+i(\tau_0-\tau_1)\big)}\right| < e^3.$$

For any selected t_j and arbitrary $\delta_1\in[0,\delta]$ let

$$s = -\delta_1+i(t_j-\tau_1),\qquad \varepsilon = 2A\delta_1,\qquad t = 2A(t_j-\tau_1).$$

Then

$$|g(s)| \leqslant \left|\frac{1-e^{-\varepsilon+it}}{-\varepsilon+it}\right| \leqslant \frac{\sqrt{(1-\varepsilon)4\sin^2 t/2+4\varepsilon^2}}{|\varepsilon+it|} \leqslant \frac{2\sin t/2}{|\varepsilon+it|}+\frac{2\varepsilon}{|\varepsilon+it|}\leqslant 3,$$
$$|g'(s)| = \left|\frac{1}{2}\cdot\frac{3e^{3As}-e^{As}}{s}-\frac{e^{3As}-e^{As}}{2As}\cdot\frac{1}{s}\right| \leqslant \frac{2}{|t_j-\tau_1|}+3\frac{1}{|t_j-\tau_1|} < 5C_5\log D,$$

whence

$$z_{\varrho_j}^k - z_{\varrho'_j}^k \ll \delta k e^{3k}3^{k-1}\log D \leqslant ke^{5k-B_1\lambda},$$
$$\sum_j (z_{\varrho_j}^k - z_{\varrho'_j}^k) \leqslant E\lambda ke^{5k-B_1\lambda} < 2CE\lambda^2e^{-(B-5C-5C_1)\lambda}.$$

Now taking $B_1 = 7C+5C_1$ and using (60), we deduce

$$\sum_{\substack{\varrho\in Q_E\\ \varrho\neq\varrho_j,\neq\varrho'_j}} z_\varrho^k \ll e^{-B\lambda+3k}+e^{3k}E\lambda(\tfrac{1}{2}E)^{-k}+2CE\lambda^2e^{-2C\lambda}$$

and may proceed as before: Taking the logarithms of the reciprocal numbers we obtain next to U_1, U_2 one more term $U_3 = 2C\lambda-\log 2CE\lambda^2 > C\lambda$ with a higher order of magnitude than that of U, whence a contradiction.

A density lemma. 17. The constants of this paragraph do not depend on any constants used previously.

LEMMA 17. *Let k and $A>0$ denote unbounded parameters, k being an integer $\geqslant 2$, and let*

$$R(a) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{e^{3As}-e^{-As}}{2As}\right)^k e^{-s\log a}\,ds\qquad (a\geqslant 1).$$

Then for appropriate constant $c>1$

$$(61)\qquad |R(a)|\begin{cases}\leqslant e^{ck}/A & \text{if}\quad e^{kA}<a<e^{3kA},\\ =0 & \text{otherwise.}\end{cases}$$

Proof ([6]). By the binomial expansion we get from the integrand

$$e^{2Aks}\left(\frac{e^{As}-e^{-As}}{2As}\right)^k e^{-s\log a}$$

a finite number of terms $e^{\varkappa s}s^{-k}$ (say). If $a\geqslant e^{3kA}$, then all the exponents $\varkappa$ are $\leqslant 0$, whence $R(a)=0$, since

$$\int_{2-i\infty}^{2+i\infty} e^{\varkappa s}s^{-k}\,ds = 0\qquad(\varkappa\leqslant 0).$$

([6]) Using more intricate arguments Turán ([22], Lemma I) gives a sharper estimate of $R(a)$.

If $a \leqslant e^{kA}$, then all the exponents $\varkappa$ are $\geqslant 0$ and we obtain the desired result by moving the path of integration to $\sigma = -1$ and using the identity

$$\int_{-1-i\infty}^{-1+i\infty} e^{\varkappa s} s^{-k} ds = 0 \qquad (\varkappa \geqslant 0).$$

Now let $e^{kA} < a < e^{3kA}$. Then there are $\leqslant (1+1)^k < e^k$ terms $e^{\varkappa s}s^{-k}$ with $\varkappa > 0$; for these we move the contour of integration to $\sigma = -1$. Since

$$\operatorname*{Res}_{s=0} \frac{e^{\varkappa s}}{(2As)^k} = \frac{\varkappa^{k-1}}{(k-1)!} \cdot \frac{1}{(2A)^k} \leqslant \frac{(3kA)^{k-1}}{(k-1)!(2A)^k} < \left(\frac{3}{2}\right)^k \frac{k^{k-1}}{(k-1)!} \cdot \frac{1}{A} < \frac{e^{(c-1)k}}{A},$$

(61) follows by the theorem on residues (the contribution of the terms with $\varkappa \leqslant 0$ being zero).

18. By (19) and (22) any of the functions $F(s) = \zeta'/\zeta(s, \chi)$, with $\chi \neq \chi_0$, satisfies the conditions of Lemma 16(i). For any real τ let

$$(62) \qquad J_\chi(\tau, k, A) = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{e^{3As} - e^{As}}{2As}\right)^k \frac{\zeta'}{\zeta}(s+1+i\tau, \chi)\, ds,$$

where $A > 1$ and k is an integer $\geqslant 2$.

Lemma 18. *If c_3 is the constant of Lemma 15 and $e^{kA} > D^{c_3}$, then*

$$(63) \qquad \sum_\chi |J_\chi(\tau, k, A)|^2 < e^{c_4 k}.$$

Proof. By the definition of $R(a)$ with the aid of (62), (16), (41), (61) and

$$(64) \qquad \sum_\chi \chi(H) = \begin{cases} h & \text{if } H \text{ is the principal class } H_1, \\ 0 & \text{otherwise} \end{cases}$$

(cf. [12], § 10), we deduce

$$J_\chi(\tau, k, A) = \sum_{e^{kA} < a < e^{3kA}} \frac{\Lambda(a) R(a)}{a^{1+i\tau}} \chi(a) = \sum_{e^{kA} < a < e^{3kA}} e_a \chi(a) \quad \text{(say)},$$

$$\sum_\chi |J_\chi(\tau, k, A)|^2 = \sum_\chi \sum_{a_1, a_2} e_{a_1} \bar{e}_{a_2} \chi(a_1) \bar{\chi}(a_2)$$

$$= \sum_{a_1, a_2} \sum_\chi = h \sum_H \Big| \sum_{\substack{a \in H \\ e^{kA} < a < e^{3kA}}} e_a \Big|^2 \leqslant h \sum_H \Big(\sum_{\substack{a \in H \\ e^{kA} < a < e^{3kA}}} \frac{\Lambda(a)}{a} |R(a)| \Big)^2$$

$$\leqslant e^{2ck} A^{-2} h \sum_H \Big(\sum_{\substack{a \in H \\ e^{kA} < a < e^{3kA}}} \frac{\Lambda(a)}{a} \Big)^2 \ll e^{2ck} A^{-2} h \sum_H h^{-2} (\theta^{-1} + kA)^2 \ll k^2 e^{2ck}.$$

Corollary. *For any selected numbers τ_0, λ satisfying the conditions of Lemma 16 (where c' is now equal to the constant c of Lemma 11) let $\nu = \nu(\tau_0, \lambda)$ be the number of the functions $\zeta(s, \chi)$ $(\chi \neq \chi_0)$ having at least one zero in the square Q $(1 - \lambda/\log D \leqslant \sigma \leqslant 1,\ |t - \tau_0| \leqslant \lambda/2\log D)$. Then for at least $\nu/c_5\lambda$ of them (49) holds with the same $k = k_1 < c_5\lambda$. Therefore, by (49) and (63)*

$$(\nu/c_5\lambda) e^{-2C_2\lambda} < \sum_\chi |J_\chi(\tau_0, k_1, \lambda^{-1}\log D)|^2 < e^{c_4 k_1},$$

whence

$$(65) \qquad \nu < e^{c_6 \lambda}.$$

From (65) we get the following lemma, analogous to the Linnik-Rodosskiĭ density lemma:

Fundamental Lemma 19. *Let N_λ denote the number of the zeros of the function $Z(s) = \prod_\chi \zeta(s, \chi)$ in the rectanlge R_λ $(1 - \lambda/\log D \leqslant \sigma \leqslant 1,$ $|t| \leqslant e^\lambda/\log D)$. Then for appropriate c_7, c_8*

$$(66) \qquad N_\lambda < e^{c_7\lambda} \qquad (c \leqslant \lambda \leqslant c_8 \vartheta \log D).$$

Proof. Considering that R_λ may be covered by $\ll e^\lambda$ squares Q (corresponding to $\tau_0 = n\lambda/\log D$, $n = 0, \pm 1, \ldots$), we deduce that the number of the functions $\zeta(s, \chi)$ $(\chi \neq \chi_0)$ having at least one zero in R_λ is $\ll e^{(1+c_6)\lambda}$. It increases at most by unity if we remove the restriction $\chi \neq \chi_0$. Hence we get (66) observing that the number of zeros ϵR_λ for any function $\zeta(s, \chi)$ is $\ll e^\lambda$ by (22).

The lemma and the proof hold as well for rectangles $(1 - \lambda/\log D \leqslant \sigma \leqslant 1,\ |t - t_0| \leqslant e^\lambda/\log D)$ with $|t_0| \leqslant D$.

On the real exceptional zero. 19. In this section we suppose that some of the functions $\zeta(s, \chi)$ have the real exceptional zero

$$(67) \qquad \beta' = 1 - \delta, \qquad 0 \leqslant \delta < c_3/\log D$$

with arbitrary small $c_3 < 1$. By Lemma 10 there is at most one such character. It is real, and in what follows it will be denoted by χ'.

Lemma 20. *If $\chi' = \chi_0$, then for appropriate c_4 we have $\delta > D^{-c_4}$.*

Proof. Writing

$$G(s) = \varphi(s)\zeta(s, \chi_0) \qquad \text{where} \qquad \varphi(s) = \begin{cases} s - 1 & \text{if } q = 1, \\ (1 - q^{1-s})/(1 - q^{-1}) & \text{if } q > 1, \end{cases}$$

we have, by (15), $G(s) \ll D^{c_5}$ in $|s-1| \leqslant r$ $(0 < r < \frac{1}{2}\vartheta)$. Hence in $|s-1| \leqslant \frac{1}{2}r$

(68) $$G'(s) = \frac{1}{2\pi i}\int\limits_{|w-s|=r/2} \frac{G(w)}{(w-s)^2}\,dw \ll r^{-1}D^{c_5}.$$

By § 4

$$\theta ha = \operatorname*{Res}_{s=1}\zeta(s,\chi_0) = \lim_{s\to 1}(s-1)G(s)/\varphi(s) = \theta G(1),$$

whence, by (3), $G(1) = ha > D^{-c_6}$ and thus $D^{-c_6} < G(1)-G(\beta') = \delta G'(\sigma_1)$ $(\beta' < \sigma_1 < 1)$. Now using (68) (with $r = \frac{1}{4}\vartheta$) we obtain the required result.

COROLLARY. *If h (the class number of $\mathfrak{G}$) is odd, then $\delta > D^{-c_4}$.*

Proof. In the case of an odd h the single real character being χ_0, we have $\chi' = \chi_0$.

LEMMA 21. *If $\vartheta > \frac{1}{2}$, then for appropriate c_7 we have $\delta > D^{-c_7}$.*

Proof. We may suppose that $\chi' \neq \chi_0$ (otherwise we have the desired result by Lemma 20). The function

$$g(a) = \sum_{d|a}\chi'(d)$$

has the multiplicative property $g(aa') = g(a)g(a')$ whenever $(a, a') = 1$, and for any generator b

$$g(b)^k \geqslant \begin{cases} 0 & \text{always,}\\ 1 & \text{if } k \text{ even}\end{cases}$$

(cf. [5], p. 104), whence

$$g(a^2) \geqslant 1.$$

Further we have

$$\zeta(s,\chi')\zeta(s,\chi_0) = \sum_a g(a)a^{-s} \quad (\sigma > 1)$$

and thus for any $y > 0$

(69) $$\sum_a g(a)e^{-ya} = \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} y^{-s}\Gamma(s)\zeta(s,\chi')\zeta(s,\chi_0)\,ds$$

(cf. [17], p. 380). If $y = D^{-B}$ (with a sufficiently large $B \ll 1$), by (3) the left-hand side of (69) is

(70) $$\geqslant e^{-1}\sum_{a^2\leqslant D^B} 1 = e^{-1}\sum_{a\leqslant D^{B/2}} 1 > \tfrac{1}{2}e^{-1}haD^{B/2}.$$

Moving the path of integration in (69) to the line $L(\sigma = \sigma_1 = 1-\vartheta+1/\log D)$ we pass over a pole or poles of the integrand with the sum of residues

(71) $$D^B\zeta(1,\chi')\operatorname*{Res}_{s=1}\zeta(s,\chi_0)\Big\{1+\vartheta\sum_{\pm k\geqslant 1}\Gamma\Big(1+\frac{2k\pi i}{\log q}\Big)D^{B.2k\pi i/\log q}\Big\}$$
$$\ll \zeta(1,\chi')D^B\theta ha\Big(1+\vartheta\sum_{k\geqslant 1}\Big|\Gamma\Big(1+\frac{2k\pi i}{\log q}\Big)\Big|\Big) \ll \zeta(1,\chi')D^{c_8},$$

since, by the asymptotic estimate of $|\Gamma(s)|$ (cf. [17], p. 395) and by (4)

$$\sum_{k\geqslant 1}\Big|\Gamma\Big(1+\frac{2k\pi i}{\log q}\Big)\Big| \ll \sum_{k\geqslant 1}\Big(\frac{2k\pi}{\log q}\Big)^{1/2}\exp\Big(-\frac{\pi}{2}\cdot\frac{2k\pi}{\log q}\Big) \ll \log eq \ll \log D.$$

(For $\log q \leqslant 1$ the estimate of the exponential sum is evident. If $\log q > 1$, then writing $2\pi/\log q = \varepsilon$ we prove the estimate $\ll \varepsilon^{-1}$ separately for each of the two parts of the sum corresponding to $\varepsilon k < e^2$ and $\varepsilon k \geqslant e^2$, respectively.)

By (15)

$$\frac{1}{2\pi i}\int\limits_L D^{Bs}\Gamma(s)\zeta(s,\chi')\zeta(s,\chi_0)\,ds \ll D^{B(1-\vartheta)+c_9}.$$

For a sufficiently large $B \ll 1$ this has a smaller order of magnitude than (70). Hence (71) is not less than a half of (70), whence

(72) $$\zeta(1,\chi') > D^{-c_{10}}.$$

By the arguments of (68) we can prove that $\zeta'(s,\chi) \ll D^{c_{11}}$ $(\chi \neq \chi_0)$ in $|s-1| < \frac{1}{4}\vartheta$. This combined with (72) gives the desired estimate of δ.

LEMMA 22. *If (9) holds, then $\delta > D^{-c_{12}}$.*

Proof. We may suppose (as in the previous proof) that $\chi' \neq \chi_0$. Writing $f(\sigma) = \sum_a \chi'(a)a^{-\sigma}$ we have

(73) $$f(\sigma) = \zeta(\sigma,\chi') \quad \text{for all } \sigma > 1.$$

Using (3), we can prove by partial summation that the series $f(\sigma)$ converges uniformly in $1-\vartheta+\varepsilon \leqslant \sigma \leqslant 2$ $(0 < \varepsilon < \vartheta)$, whence $f(\sigma)$ is a continuous function [7] in that interval. Let in (8) K_j be the set of all the classes H with $\chi'(H) = 1$; then

(74) $$f(1) = \sum_a \frac{\chi'(a)}{a} = \lim_{x\to\infty}\Big(\sum_{x\geqslant a\epsilon K_j}\frac{1}{a} - \sum_{x\geqslant a\notin K_j}\frac{1}{a}\Big) = C_j.$$

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[7] By the arguments of [14], p. 157, one can prove that for any $\chi \neq \chi_0$ the series $\Sigma\chi(a)a^{-s}$ converges uniformly in $1-\vartheta+\varepsilon \leqslant \sigma \leqslant 2$, its sum being the function (13). However, in the present case we can do as well with a weaker relation.

Considering that the regular function $\zeta(s,\chi')$ is continuous at $s=1$ and taking limits in (73) as $\sigma\to 1+0$ we deduce

$$\zeta(1,\chi')=f(1).$$

Hence by (74) and (9)(8), $\zeta(1,\chi')>D^{-c_2}$ and we may go on as in the previous lemma.

The results proved in the present paragraph may be summarized by the following

LEMMA 23. *Under such circumstances as stated in the theorem of § 1 for appropriate c we have $\delta>D^{-c}$.*

20. LEMMA 24. *There are constants c, c' such that, if $\delta<c/\log D$ and $\varrho_0=\beta_0+i\gamma_0$ is a zero in $(1-\vartheta/2\leqslant\sigma\leqslant 1,\ |t|\leqslant D)$ of any function $\zeta(s,\chi)$, different from the exceptional zeros of Lemma 11, then*

$$\beta_0<1-\frac{c'}{\log D}\log\frac{ec}{\delta\log D}. \tag{75}$$

Proof. Let

$$\beta_0=1-\lambda/\log D.$$

It will be sufficient to prove the lemma merely for $\lambda\leqslant c_3\vartheta\log D$ with some small constant $c_3<1/16$. If, in fact, $\lambda>c_3\vartheta\log D$, then, since by Lemma 23, $\delta>D^{-c_4}$, we have $\log(ec/\delta\log D)<c_5\log D$, whence (75) holds for any $c'<c_3\vartheta/c_5$.

In this paragraph we shall deal with the case $\chi_0\neq\chi'$. Let us consider that the function (9)

$$F(s)=\begin{cases}\zeta'/\zeta(s,x)+\zeta'/\zeta(s-\delta,\chi\chi') & \text{if}\quad \chi=\chi_0,\\ \zeta'/\zeta(s,\chi)+\zeta'/\zeta(s+\delta,\chi\chi') & \text{if}\quad \chi\neq\chi_0\end{cases} \tag{76}$$

(8) In order to show the necessity of (9) (or some equivalent condition) let us consider that $\zeta(s,\chi)$ has no zeros in $\sigma>1$ and $\zeta(\sigma,\chi)\to 1$ as $\sigma\to\infty$. Hence for any real χ we have $\zeta(1,\chi)\geqslant 0$ and thus, by (74), $C_j\geqslant 0$. If C_j or $\zeta(1,\chi)$ vanishes, then in some of the classes H there may be no generators, which does not suit our purpose.

The situation will be shown more clearly at the end of this paper. For the present let it suffice to quote the following example from [11], p. 617: If $\mathfrak{G}$ denotes the semi-group of natural integers $n=\prod_i p_i^{a_i}$ (p_i different primes) divided into the classes H_1, H_2 according as $\sum a_i$ is even or odd, then $\zeta(s,\chi')=\zeta(2s)/\zeta(s)$ and thus $\zeta(1,\chi')=0$; there are no generators in H_1.

(9) Using some additional arguments we can base the proof exclusively on the second form of $F(s)$ in (76). In that case ϱ_0 denotes any pole of $F(s)$ subject to the same restrictions (cf. [13], § 2 or [18], pp. 342-343).

is regular at $s=1$ and $s=\beta'$, but it has a pole $\varrho_0\in Q$ $(1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t-\gamma_0|\leqslant\lambda/2\log D)$. Therefore by Lemma 16(i)

$$|y_\chi(\gamma_0)|>e^{-c_6\lambda},$$

where

$$J_\chi(\gamma)=\sum_{D^B<a<D^{3B}}\frac{\Lambda_1(a)R(a)}{a^{1+i\gamma}}\chi(a)$$

(with B arbitrarily large $\ll 1$, $c_6=c_6(B)$), $\Lambda_1(a)$ denoting the coefficients in the Dirichlet expansion $F(s)=\sum_a\Lambda_1(a)a^{-s}$ $(\sigma>1)$; by (76), (16)

$$\Lambda_1(a)=\begin{cases}\Lambda(a)\big(1+\chi'(a)a^{\delta}\big) & \text{if}\quad \chi=\chi_0,\\ \Lambda(a)\big(1+\chi'(a)a^{-\delta}\big) & \text{if}\quad \chi\neq\chi_0.\end{cases}$$

Hence

$$\sum_{D^B<a<D^{3B}}\frac{\Lambda(a)\big(a^{\delta}+\chi'(a)\big)}{a}|R(a)|>e^{-c_6\lambda},$$

resp.,

$$\sum_{D^B<a<D^{3B}}\frac{\Lambda(a)\big(1+\chi'(a)a^{-\delta}\big)}{a}|R(a)|>e^{-c_6\lambda},$$

which implies, by (61),

$$\sum_{D^B<b<D^{3B}}\frac{\log b}{b}\big(b^{\delta}+\chi'(b)\big)>e^{-c_7\lambda}\log D, \tag{77}$$

resp.,

$$\sum_{D^B<b<D^{3B}}\frac{\log b}{b}\big(1+\chi'(b)b^{-\delta}\big)>e^{-c_7\lambda}\log D.$$

We may suppose that $b^{\delta}-1$ and $1-b^{-\delta}$ do not exceed $e^{-2c_7\lambda}\log D$ for any $b\leqslant D^{3B}$ (otherwise $\lambda>c_8\log(c_9/\delta\log D)$, whence (75) follows; cf. [6], p. 137). Then by (77) in both cases

$$\sum_{\substack{D^B<b<D^{3B}\\ \chi'(b)=1}}1/b>e^{-c_{10}\lambda}$$

and thus

$$\sum_{D^B<u<D^{3B}}1/u>e^{-c_{10}\lambda}, \tag{78}$$

where u runs through all the numbers of $\mathfrak{G}$ generated exclusively by the b's $>D^B$ with $\chi'(b)=1$.

The arithmetical function $g(a)$ defined by the expansion

$$\zeta(s,\chi')\zeta(s,\chi_0)=\sum_a g(a)a^{-s}\quad(\sigma>1) \tag{79}$$

being multiplicative and $\geqslant 0$, we have

$$(80)\qquad \sum_{a\leqslant D^B}\frac{g(a)}{a}\sum_{D^B<u<D^{3B}}\frac{1}{u}\leqslant\sum_{a\leqslant D^B}\frac{g(a)}{a}\sum_{D^B<u<D^{3B}}\frac{g(u)}{u}\leqslant\sum_{D^B<a<D^{4B}}\frac{g(a)}{a}.$$

Writing

$$U_1=\sum_{a>D^B}\frac{g(a)}{a}\{\exp(-D^{-4B}a)-\exp(-D^{-B}a)\},$$

$$U_2=\sum_{a\leqslant D^B}\frac{g(a)}{a}\{\exp(-D^{-4B}a)-\exp(-D^{-B}a)\}$$

we have

$$U_1=\sum_{a>D^B}\frac{g(a)}{a}e^{-aD^{-4B}}(1-e^{-a(D^{-B}-D^{-4B})})$$

$$(81)\qquad >\sum_{D^B<a<D^{4B}}\frac{g(a)}{a}e^{-1}(1-e^{-1/2})>\frac{1}{8}\sum_{D^B<a<D^{4B}}\frac{g(a)}{a},$$

$$U_2\geqslant 0.$$

Further we use the identity

$$\sum_a\frac{g(a)}{a}e^{-ya}=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}y^{1-s}\Gamma(s-1)\zeta(s,\chi')\zeta(s,\chi_0)\,ds$$

(cf. [17], p. 380) with $y=D^{-4B}$, $y=D^{-B}$ and move the path of integration to the line $\sigma=1-\vartheta+1/\log D$. The integrand has a double pole at $s=1$ with the residue

$$E+(c-\log y)\mu$$

where

$$\mu=\zeta(1,\chi')\operatorname*{Res}_{s=1}\zeta(s,\chi_0),\qquad c=\lim_{z\to 0}\{\Gamma(z)-z^{-1}\}\ll 1,$$

$$E=\lim_{s\to 1}\{\zeta(s,\chi')\zeta(s,\chi_0)-\mu(s-1)^{-1}\},$$

and, if $q>1$, there are simple poles at $s=1+2k\pi i/\log q$ $(k=\pm1,\pm2,\ldots)$ with residues

$$y^{-2k\pi i/\log q}\,\Gamma\left(\frac{2k\pi i}{\log q}\right)\mu.$$

We have

$$\sum_{\pm k\geqslant 1}\left|\Gamma\left(\frac{2k\pi i}{\log q}\right)\right|\ll\sum_{k=1}^{\infty}\left(\frac{2k\pi}{\log q}\right)^{-1/2}\exp\left(-\frac{\pi}{2}\cdot\frac{2k\pi}{\log q}\right)$$

$$\ll(\log q)^{1/2}\sum_{k\leqslant\log eq}k^{-1/2}+\sum_{k>\log eq}\exp\left(-\frac{\pi^2k}{\log q}\right)\ll\log eq.$$

From these esimates we deduce by subtraction

$$(82)\qquad U_1+U_2=\mu\{3B\log D+O(\log eq)\}+O(D^{-B\vartheta+c_{11}}).$$

By § 19, $\mu>D^{-c_{12}}$. Hence, if B is large enough, then the remaining term in (82) is in modulus less than a third of the principal term. And, since $U_2\geqslant 0$, we get the inequality

$$\mu(4B\log D+c_{13}\log eq)>U_1,$$

whence, by (81), (80) and (78),

$$(83)\qquad \mu>\frac{c_{14}}{\log D}\sum_{D^B<a<D^{4B}}\frac{g(a)}{a}$$

$$\geqslant\frac{c_{14}}{\log D}\sum_{a\leqslant D^B}\frac{g(a)}{a}\sum_{D^B<u<D^{3B}}\frac{1}{u}>\frac{e^{-c_{15}\lambda}}{\log D}\sum_{a\leqslant D^B}\frac{g(a)}{a}.$$

By (79)

$$(84)\qquad \sum_a g(a)a^{-\beta'}\exp(-aD^{-B/2})$$

$$=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}D^{\frac12 B(s-\beta')}\Gamma(s-\beta')\zeta(s,\chi')\zeta(s,\chi_0)\,ds$$

(cf. [17], p. 380). Moving the path of integration to the line $\sigma=1-\vartheta+{}$ $+1/\log D$ we infer that the right-hand side of (84) is

$$D^{\frac12 B\delta}\mu\left\{\Gamma(\delta)+\vartheta\sum_{k=\pm1,\ldots}D^{Bk\pi i/\log q}\,\Gamma\left(\delta+\frac{2k\pi i}{\log q}\right)\right\}+R,\qquad R\ll D^{-\frac12 B\vartheta+c_{16}}.$$

Observing that, by (4), (67), δ is of a lower order of magnitude than $1/\log q$, we have $\sum_k\ll\log eq$, whence the expression in brackets is equivalent to $\Gamma(\delta)$. For a sufficiently large $B\ll 1$ the remaining term R is in modulus $<\frac14$ and thus the principal term exceeds $\frac34$ (since the left-hand side of (84) is >1). Obviously

$$\sum_a g(a)a^{-\beta'}\exp(-aD^{-B/2})-R<c_{17}\sum_{a\leqslant D^B}g(a)a^{-1}$$

and thus

$$\mu/\delta<c_{18}\sum_{a\leqslant D^B}g(a)a^{-1}.$$

Hence, by (83),

$$c_{18}\sum_{a\leqslant D^B}\frac{g(a)}{a}>\frac{\mu}{\delta}>\frac{e^{-c_{15}\lambda}}{\delta\log D}\sum_{a\leqslant D^B}\frac{g(a)}{a},$$

whence $c_{18} > e^{-c_{15}\lambda}/\delta\log D$. This proves the lemma in the case of $\chi_0 \neq \chi'$.

21. Now let us suppose that $\chi_0 = \chi'$. Then instead of (76) we use the function $F(s) = \zeta'/\zeta(s, \chi)$. Arguing as before (but in the case of $\chi = \chi_0$ using Lemma 16 (ii)) we get the inequality

$$\sum_{D^B < b < D^{3B}} 1/b > e^{-c_3\lambda}.$$

Hence

$$\sum_{D^B < u < D^{3B}} 1/u > e^{-c_3\lambda}, \tag{85}$$

where u denotes those numbers of $\mathfrak{G}$ the generators of which are all $> D^B$. We have

$$\sum_{a \leqslant D^B} \frac{1}{a} \sum_{D^B < u < D^{3B}} \frac{1}{u} \leqslant \sum_{D^B < a < D^{4B}} \frac{1}{a}. \tag{86}$$

Write

$$U_1 = \sum_{a > D^B} a^{-1}\{\exp(-aD^{-4B}) - \exp(-aD^{-B})\},$$

$$U_2 = \sum_{a \leqslant D^B} a^{-1}\{\exp(-aD^{-4B}) - \exp(-aD^{-B})\}.$$

Then

$$U_1 > \tfrac{1}{8} \sum_{D^B < a < D^{4B}} a^{-1}, \qquad U_2 \geqslant 0 \tag{87}$$

(cf. (81)) and

$$U_1 + U_2 = \mu_0\{3B\log D + O(\log eq)\} + O(D^{-B\vartheta + c_4}),$$

where

$$\mu_0 = \operatorname*{Res}_{s=1} \zeta(s, \chi_0) > D^{-c_5}.$$

For a large $B \ll 1$ the remaining term being unimportant, we deduce

$$\mu_0(4B\log D + c_6 \log eq) > U_1,$$

whence, by (87), (86), (85)

$$\mu_0 > \frac{c_7}{\log D} \sum_{D^B < a < D^{4B}} \frac{1}{a} > \frac{c_7}{\log D} \sum_{a \leqslant D^B} \frac{1}{a} \sum_{D^B < u < D^{4B}} \frac{1}{u} > \frac{e^{-c_8\lambda}}{\log D} \sum_{a \leqslant D^B} \frac{1}{a}. \tag{88}$$

By the arguments of (84)

$$\sum_{a} a^{-\beta'} \exp(-aD^{-B/2}) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D^{\frac{1}{2}B(s-\beta')} \Gamma(s-\beta')\zeta(s, \chi_0)\, ds \tag{89}$$

$$= D^{\frac{1}{2}B\delta}\mu_0\{\Gamma(\delta) + O(\log eq)\} + R, \qquad R \ll D^{-\frac{1}{2}B\vartheta + c_9},$$

$$\sum_{a} a^{-\beta'} \exp(-aD^{-B/2}) - R < c_{10} \sum_{a \leqslant D^B} a^{-1}.$$

Hence, by (89), (88)

$$c_{11} \sum_{a \leqslant D^B} \frac{1}{a} > \frac{\mu_0}{\delta} > \frac{e^{-c_8\lambda}}{\delta\log D} \sum_{a \leqslant D^B} \frac{1}{a}$$

and we may go on as in § 20.

A simple consequence of Lemma 24 (cf. [6], pp. 146, 147) is the following

FUNDAMENTAL LEMMA 25. *Let δ be defined by (67). For appropriate constant A and*

$$\delta_0 = \begin{cases} \delta & \text{if} \quad \delta \leqslant A/\log D, \\ A/\log D & \text{otherwise}\ (^{10}), \end{cases}$$

$$\lambda_0 = A \log \frac{eA}{\delta_0 \log D} \in [A, \vartheta\log D]$$

there are in $(1 - \lambda_0/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant D)$ no other zeros of the function $\prod_{\chi} \zeta(s, \chi)$ than at most the exceptional ones of Lemma 11.

Proof of the theorem. 22. Let L_χ denote a broken line in the strip $1 - \frac{1}{6}\vartheta - 1/C\log D(1+\partial|t|) < \sigma < -\frac{1}{6}\vartheta$ $(C \ll 1)$ such that (i) for any $s = \sigma + it \in L_\chi$ we have $\zeta'/\zeta(s, \chi) \ll \log^3 D(1+\partial|t|)$ and (ii) that the length of the piece of L_χ between any two of its points $\sigma + it$, $\sigma' + i(t+1)$ is < 2 (cf. § 15). Write

$$g = \tfrac{1}{6}\vartheta \leqslant \tfrac{1}{6}, \qquad \sigma_1 = 1 - g.$$

From the identity

$$\sum_{a} \frac{\chi(a)\Lambda(a)}{a^{\sigma_1}} \exp\left(-\frac{1}{4y}\log^2 a/x\right) = i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, \chi) x^{s-\sigma_1} e^{(s-\sigma_1)^2}\, ds$$

$$(x > 1,\ y > 0)$$

(10) That is to say, when the exceptional zero β' does not exist.

(cf. [4], p. 299) and (64) we deduce

(90)
$$h\sum_{a\in H}\frac{\Lambda(a)}{a^{\sigma_1}}\exp\left(-\frac{1}{4y}\log^2 a/x\right)=2\sqrt{\pi y}\Big(x^g e^{g^2 y}+\vartheta\sum_{k=\pm1,\ldots}x^{g+2k\pi i/\log q}e^{y(g+2k\pi i/\log q)^2}\Big)-$$
$$-2\sqrt{\pi y}\sum_{\chi,\varrho_\chi}\overline{\chi}(H)\operatorname*{Res}_{s=\varrho_\chi}\frac{\zeta'}{\zeta}(s,\chi)x^{s-\sigma_1}e^{(s-\sigma_1)^2 y}+$$
$$+\sum_{\chi}\overline{\chi}(H)i\sqrt{\frac{y}{\pi}}\int_{L_\chi}\frac{\zeta'}{\zeta}(s,\chi)x^{s-\sigma_1}e^{(s-\sigma_1)^2 y}ds,$$

where ϱ_χ runs through the zeros of $\zeta(s,\chi)$ to the right of the line L_χ. If $1<y\ll\log^2 D$, then the last term does not exceed

$$\ll h\sqrt{y}\log^3 D\int_0^\infty e^{-t^2 y}\log^3(2+t)\,dt\ll h\log^3 D\ll D^{c_0}\log^3 D.$$

Let β' and $\beta''+i\pi/\log q=\varrho''$ be the exceptional zeros of the functions $\zeta(s,\chi')$, $\zeta(s,\chi'')$ with real characters χ', χ'' (see Lemma 11) and let ϱ be a typical zero of $\prod_\chi\zeta(s,\chi)$. Further on we use the following notation:

$$\delta'=1-\beta',\qquad \delta''=1-\beta'',\qquad \varrho=1-\delta+i\gamma,$$
$$E'=\begin{cases}1 & \text{if } \beta' \text{ exists},\\ 0 & \text{otherwise},\end{cases}\qquad E''=\begin{cases}1 & \text{if } \varrho'' \text{ exists},\\ 0 & \text{otherwise};\end{cases}$$

(91)
$$S=1+\vartheta\sum_{k=\pm1,\ldots}e^{-\left(\frac{2k\pi}{\log q}\right)^2 y+\frac{4gk\pi i}{\log q}y}$$
$$-E''\chi''(H)x^{-\delta''+\pi i/\log q}e^{-\delta''(2g-\delta'')y}\sum_{k=0,\pm1,\ldots}e^{-\left(\frac{(2k+1)\pi}{\log q}\right)^2 y+\frac{2(g-\delta'')(2k+1)\pi i y}{\log q}}-$$
$$-E'\chi'(H)x^{-\delta'}e^{-\delta'(2g-\delta')y}\Big\{1+\vartheta\sum_{k=\pm1,\ldots}e^{-\left(\frac{2k\pi}{\log q}\right)^2 y+\frac{4(g-\delta')k\pi i}{\log q}y}\Big\},$$

(92)
$$S'=\sum_\chi\overline{\chi}(H)\sum_{\varrho_\chi}x^{-\delta}e^{\{-\delta(2g-\delta)-\gamma^2+2i\gamma(g-\delta)\}y+i\gamma\log x}.$$

In the last sum ϱ_χ runs over the regular (or non-exceptional) zeros of $\zeta(s,\chi)$ in the region G_χ (say) to the right of the line L_χ.

Let $\Phi=\Phi(x,y,H)$ denote the left-hand side of (90). For $q>1$ using $x=q^n$ (n integer $\geqslant 0$) we have by (91), (92)

(93)
$$\Phi=2\sqrt{\pi y}x^g e^{g^2 y}(S-S')+O(D^{c_0}\log^3 D).$$

Write $x=D^\xi$, $\xi\geqslant 0$; $y=\eta\log D$ $(\eta\geqslant\eta_0>2)$.

Let I_B denote the integration B times repeated with respect to η, the range of integration being $(\eta,\eta+1)$. By (93), (92)

(94)
$$I_B\frac{\Phi}{2\sqrt{\pi y}x^g e^{g^2 y}}\geqslant|I_B S|-|I_B S'|-c_3 D^{c_0-g^2\eta_0}\log^3 D,$$

(95)
$$|I_B S'|\leqslant\left(\frac{2}{\log D}\right)^B\sum_\varrho x^{-\delta}\frac{\exp\{-\eta_0(g\delta+\gamma^2)\log D\}}{|g\delta+2i\gamma(g-\delta)|^B}=T_1+T_2+T_3,$$

where T_1, T_2, T_3 stand for the parts of the previous sum obtained by cutting the region $G=\sum_\chi G_\chi$ as follows.

Let G_1 be the part of G with $|t|>1$. By (95) and Lemma 3

$$T_1\ll\sum_{\varrho\in G_1}e^{-\eta_0\gamma^2\log D}\ll h\int_1^\infty e^{-\eta_0 t^2\log D}t^2\log D\cdot\log D(1+t)\,dt$$
$$<h\log^2 D\int_1^\infty e^{-\eta_0 t^2\log D}t^2\log(2+t)\,dt$$
$$\ll h\log^2 D\int_1^\infty e^{-\eta_0 t\log D}dt\ll hD^{-\eta_0}\log D.$$

Let G_2 be the set of points $s=1-\lambda/\log D+i\tau/\log D\in G$ with $\lambda>\lambda_0$ (defined by Lemma 25), $|\tau|\leqslant\tau_1=\tau_1(\lambda)=\min(e^\lambda,\log D)$. Now we write the zeros $\varrho\notin G_1$ as follows:

$$\varrho=1-\lambda/\log D+i\tau/\log D,\qquad\lambda=\lambda_\sigma,\qquad\tau=\tau_\varrho.$$

By (95) and Lemma 19 (with c_7, c_8 defined in it)

$$T_2\ll\sum_{\varrho\in G_2}e^{-\lambda\xi-g\eta_0\lambda}\lambda^{-B}\ll\sum_{\varrho\in G_2}e^{-(\xi+g\eta_0)\lambda}$$
$$\ll\int_{\lambda_0}^{c_8\vartheta\log D}(\xi+g\eta_0)e^{-(\xi+g\eta_0-c_7)\lambda}d\lambda+e^{-(\xi+g\eta_0-c_7)c_8\vartheta\log D}+$$
$$+(h\log D)e^{-(\xi+g\eta_0)c_8\vartheta\log D}\ll e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0},$$

provided that $\eta_0\ll 1$ is large enough.

Let G_3 denote the remaining part of G to the left of the line $\sigma=1-\lambda_0/\log D$. Supposing that $\lambda_0<\log\log D$ (otherwise there is no G_3) and taking $B\geqslant 1+c_7$ we have by (95)

$$T_3\ll\sum_{\varrho\in G_3}e^{-(\xi+g\eta_0)\lambda}|\tau|^{-B}\ll e^{-(\xi+g\eta_0)\lambda_0}\sum_{\varrho\in G_3}|\tau|^{-B}$$
$$\ll e^{-(\xi+g\eta_0)\lambda_0}\Big\{\int_{\lambda_0}^{\log\log D}e^{-(B-c_7)\lambda}d\lambda+e^{-(B-c_7)\log\log D}\Big\}$$
$$\ll e^{-(\xi+g\eta_0+B-c_7)\lambda_0}\ll e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0}.$$

23. First let us suppose that $E'' = 0$ and $E' = 1$. Then, by (91),

$$S = 1-\chi'(H)x^{-\delta'}e^{-\delta'(2g-\delta')y}+ \tag{96}$$

$$+e\sum_{k=\pm1,\dots}\{e^{-\left(\frac{2k\pi}{\log q}\right)^2 y+\frac{4gk\pi i}{\log q}y}-\chi'(H)x^{-\delta'}e^{-\delta'(2g-\delta')y}e^{-\left(\frac{2k\pi}{\log q}\right)^2 y+\frac{4(gk-\delta')\pi i}{\log q}y}\}.$$

Denoting by V_1, V_2 the parts of the last sum with $|k| \leqslant k_0 = \log q$ and $|k| > k_0$, respectively (11), we have for a sufficiently large $\eta_0 \ll 1$

$$|V_2| < 4\sum_{k>k_0} e^{-\frac{2k\pi}{\log q}\eta\log D} \leqslant \frac{4e^{-2\pi k_0\eta_0\log D/\log q}}{1-e^{-2\pi\eta_0\log D/\log q}}$$

$$< 8e^{-2\pi\eta_0\log D} < \tfrac{1}{4}(1-x^{-g'}e^{-g\delta'\eta_0\log D})$$

(since $\delta' > D^{-c}$, by Lemma 23), whence

$$|I_B V_2| < \tfrac{1}{4}(1-x^{-\delta'}e^{-g'\eta_0\log D}). \tag{97}$$

Performing the operation I_B in (96) for any selected k $(0 < |k| \leqslant k_0)$ we get 2^B differences, the typical first term being

$$\frac{e^{\left\{\frac{4gk\pi i}{\log q}-\left(\frac{2k\pi}{\log q}\right)^2\right\}\eta\log D}}{\left\{\frac{4gk\pi i}{\log q}\log D-\left(\frac{2k\pi}{\log q}\right)^2\log D\right\}^B}.$$

Subtracting the corresponding second term we get a quantity of the same order of magnitude multiplied by a factor

$$\ll \left(\frac{4\delta' k\pi}{\log q}+2g\delta'\right)\eta\log D+(1-x^{-\delta'}) \ll 1-x^{-\delta'}e^{-g\delta'\eta_0\log D} = d$$

(say). We may suppose that $B \geqslant 2$. Then, if $\chi'(H) = 1$,

$$I_B V_1 \ll 2^B d\left(\frac{\log q}{4g\pi\log D}\right)^B \sum_{1\leqslant k\leqslant k_0} k^{-B} \ll d\left(\frac{\log q}{2g\pi\log D}\right)^B. \tag{98}$$

The last quotient being < 1 (since $q \leqslant D^{\vartheta}$, $2g\pi > \vartheta$), for a sufficiently large $B \ll 1$ we have $|I_B V_1| < \frac{1}{4}d$. Hence, by (96) and (97)

$$|I_B S| > \tfrac{1}{2}(1-x^{-\delta'}e^{-g\delta'\eta_0\log D}). \tag{99}$$

(Another proof of (99) will be given at the end of § 27.) If $\chi'(H) = -1$, then instead of (97) we use $|I_B V_2| < \frac{1}{4}$ and we use (98) with $d = 1$. It follows that $|I_B S| > \frac{1}{2}$, whence (99).

(11) If $\log q < 1$, then $V_1 = 0$.

Now let us denote the right-hand side of (96) by $1+U_1$. If $E'' = 1$, then by (91)

$$S = 1+U_1+x^{\pi i/\log q}U_2,$$

where U_2 stands for the appropriate term. Consider that

$$x^{\pi i/\log q} = \begin{cases} 1 & \text{if} \quad x = q^{2m}, \\ -1 & \text{if} \quad x = q^{2m+1} \end{cases}$$

(m integer). By what has been proved

$$|\mathrm{re}\, I_B(1+U_1)| > \tfrac{1}{2}(1-x^{-\delta'}e^{-g\delta'\eta_0\log D}), \tag{100}$$

if η_0 and B are large enough. Now imposing the restriction that x runs merely over the powers $x = q^n$ with a properly fixed parity of n we find that the real parts of $I_B(1+U_1)$ and $I_B x^{\pi i/\log q}U_2$ agree in sign. Then, by (100),

$$|\mathrm{re}\, I_B S| > \tfrac{1}{2}(1-x^{-\delta'}e^{-g\delta'\eta_0\log D}),$$

whence (99) again.

If β' does not exist, then a stronger inequality holds. Using the number δ_0 of Lemma 25 we have in any case

$$|I_B S| > \tfrac{1}{2}(1-e^{-g\delta_0\eta_0\log D}) \geqslant \tfrac{1}{2}(1-e^{-2g(\delta_0/A)\log D}) \geqslant \frac{g\delta_0\log D}{2A}$$

(supposing $\eta_0 > 2/A$). If η_0 is large enough, then T_1 taken together with the remaining term of (94) is in modulus $< \frac{1}{6}g(\delta_0/A)\log D$ (since $\delta_0 > D^{-c}$, by Lemmas 25 and 23) and so is T_2+T_3:

$$T_2+T_3 < c_3 e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0} \leqslant c_3 e^{-\frac{1}{2}g\eta_0\lambda_0} = c_3\exp\left(-\frac{1}{2}g\eta_0 A\log\frac{eA}{\delta_0\log D}\right)$$

$$= \left(\frac{\delta_0\log D}{A}\right)^{\frac{1}{2}g\eta_0 A} c_3 e^{-\frac{1}{2}g\eta_0 A} < \frac{1}{6}g(\delta_0/A)\log D.$$

Hence, if U denotes the left-hand side of (94),

$$U > \tfrac{1}{6}g(\delta_0/A)\log D. \tag{101}$$

24. Now we introduce the number

$$z = xe^{4y} = D^{\xi+4\eta}$$

and divide the sum Φ on the left of (90) into four sums:

$$\Phi = S_0+h\sum_{\substack{b\in H\\ x<b<z}}\frac{\log b}{b^{\sigma_1}}\exp\left(-\frac{1}{4y}\log^2 b/x\right)+S_1+hS_2, \tag{102}$$

where

$$(103)\quad S_0 = h\sum_{x\geqslant b\epsilon H}\frac{\log b}{b^{\sigma_1}}\exp\Big(-\frac{\log^2 b/x}{4y}\Big),\qquad S_1 = h\sum_{z\leqslant a\epsilon H}\frac{\Lambda(a)}{a^{\sigma_1}}\exp\Big(-\frac{\log^2 a/x}{4y}\Big),$$

$$S_2 = \sum_{\substack{z\geqslant a=b^m\epsilon H\\ m\geqslant 2}}\frac{\Lambda(a)}{a^{\sigma_1}}\exp\Big(-\frac{\log^2 a/x}{4yt}\Big).$$

By (42)

$$(104)\qquad \theta S_1 \ll \int_z^\infty t^g\exp\Big(-\frac{\log^2 t/x}{4y}\Big)\cdot\frac{\log t/x}{2yt}\,dt$$

$$= \int_z^\infty \exp\Big(-\frac{\log^2 t/x}{4y}+g\log t\Big)\Big(\frac{\log t/x}{2yt}-\frac{g}{t}+\frac{g}{t}\Big)dt$$

$$< 2\int_z^\infty \exp\Big(-\frac{\log^2 t/x}{4y}+g\log t\Big)\Big(\frac{\log t/x}{2yt}-\frac{g}{t}\Big)dt$$

$$= 2\exp\Big(-\frac{\log^2 z/x}{4y}+g\log z\Big) = 2z^g e^{-4y} = 2x^g e^{-4(1-g)y}.$$

From (103), (3) and (42) we deduce

$$(105)\qquad S_0 \ll h\sum_{x\geqslant b\epsilon H} b^{-\sigma_1}\log b = h\sum_{D^{c_3}\geqslant b\epsilon H} + h\sum_{\substack{b\epsilon H\\ D^{c_3}<b\leqslant x}} \ll h\alpha D^{c_4}+\theta^{-1}x^g$$

$$< \theta^{-1}e^{\frac{1}{4}g^2y}x^g.$$

Observing that $\sigma_1 = 1-g$, $0<g\leqslant\frac{1}{6}$, we have for any selected $g'<1-2g$

$$\sum_{\substack{z\geqslant b^m\epsilon H\\ m\geqslant 2}}\frac{\log b}{b^{m\sigma_1}}\exp\Big(-\frac{\log^2 b^m/x}{4y}\Big) \ll \sum_{a\epsilon H}a^{-1-g'}+D^{c_5}$$

$$\ll \int_1^\infty\frac{\alpha t+D^{c_1}t^{1-\vartheta}}{t^{2+g'}}\,dt+D^{c_5} \ll D^{c_6},$$

whence, if $\eta_0 \ll 1$ is large enough,

$$(106)\qquad hS_2 < \theta^{-1}e^{\frac{1}{4}g^2y}x^g.$$

Now by (102), (104), (105), (106)

$$h\sum_{\substack{b\epsilon H\\ x<b<z}}\frac{\log b}{b^{\sigma_1}}\exp\Big(-\frac{\log^2 b/x}{4y}\Big) = \Phi-(S_0+S_1+hS_2)$$

$$\geqslant \Phi-\theta^{-1}(c_7x^g e^{\frac{1}{4}g^2y}+c_8x^g e^{-4(1-g)y})$$

$$\geqslant \Phi-2\sqrt{\pi y}\,x^g e^{g^2y}\{c_9y^{-\frac{1}{2}}e^{-\frac{3}{4}g^2y}+c_9y^{-\frac{1}{2}}e^{-(4-4g+g^2)y}\}\,\theta^{-1}.$$

Hence, by (101)

$$(107)\qquad I_B\frac{1}{2\sqrt{\pi y}x^g e^{g^2y}}\,h\sum_{\substack{b\epsilon H\\ x<b<z}}\frac{\log b}{b^{\sigma_1}}\exp\Big(-\frac{\log^2 b/x}{4y}\Big)$$

$$\geqslant U-2c_9\frac{\log q}{1-q^{-1}}I_By^{-\frac{1}{2}}e^{-\frac{3}{4}g^2y} > \frac{1}{2}U,$$

if $\eta_0 \ll 1$ is large enough. Since $z\leqslant xD^{4(\eta_0+B)}$, the theorem follows for $q=1$. If $q>1$, then from (107) we can deduce the existence of a generator $b\epsilon H$ in the interval (x, xD^c) $(c=4\eta_0+4B)$ merely for all $x=q^n$ with a probable restriction that n runs over the integers with a fixed parity (cf. § 23). But then increasing c at most by $\log q/\log D$ we get the theorem for all x.

Considering that for any $x>D^{c_{10}}$ (with a sufficiently large c_{10}) the left-hand side of (107) does not exceed

$$\pi(z, H)\frac{D^{c_0}}{x^g}\cdot\frac{\log x}{x^{1-g}},$$

we deduce (10).

Improvement of the theorem for $q\ll 1$, $x\to\infty$. 25. By the use of some additional conditions the theorem of § 1 may be improved as follows.

For arbitrarily small ε_1, ε' and all $D>D_0(\varepsilon')$, $x\geqslant 1$, let

$$(108)\qquad h\alpha > D^{-\varepsilon'},$$

$$(109)\qquad \sum_{a\leqslant x}1 \leqslant c_3(\varepsilon_1)D^{\varepsilon'}x^{1+\varepsilon_1}$$

and let (9) hold for any $c_2>0$, $D>D_0(c_2)$. If $q\ll 1$, then there are constants c, c' (depending on c_0, c_1, l, ϑ, q) such that for any positive $\varepsilon\leqslant c$ (if $q=1$), $\leqslant c/2$ (if $q>1$[12])) and all $x\geqslant x_0 = D^{c'\log(c/\varepsilon)}$ there is a generator $b\epsilon H$ in the interval (x, xD^ε). Then if $x>x_0D^\varepsilon$, we have $\pi(x, H)>x/hD^{3\varepsilon}\log x$.

We begin by proving that with regard to the real exceptional zero $\beta' = 1-\delta'$ the additional conditions imply the estimate

$$(110)\qquad \delta' > D^{-\varepsilon_2}\qquad (D>D_0(\varepsilon_2))$$

where ε_2 (and ε_3 in the sequel) stands for an arbitrarily small positive constant.

Let first $\chi'\neq\chi_0$. Then by (9)

$$(111)\qquad \zeta(1,\chi') > D^{-c_3}$$

(12) If $q>1$ and c' is large enough, then taking $\varepsilon\leqslant c/2$ we get $\xi\geqslant\eta_0$, which is necessary in (114).

(cf. the proof of Lemma 22). By (109)

$$\zeta(1+2\varepsilon_1, \chi_0) \leqslant c_3(\varepsilon_1) D^{\varepsilon'} \int_1^\infty \frac{x^{1+\varepsilon_1}}{x^{2+2\varepsilon_1}} dx \leqslant c_4(\varepsilon_1) D^{\varepsilon'}, \tag{112}$$

which implies

$$|\zeta(1+2\varepsilon_1+it, \chi')| \leqslant c_4(\varepsilon_1) D^{\varepsilon'}. \tag{113}$$

Since by (12)

$$\zeta(1-\vartheta/2+it, \chi') \ll D^{c_5}(1+|t|),$$

using (113) and [5], Lemma 3, we deduce

$$|\zeta(s, \chi')| \leqslant c_6(\varepsilon_1) D^{c_7\varepsilon'} \quad \text{in} \quad (1-2\varepsilon_1 \leqslant \sigma \leqslant 1+2\varepsilon_1, |t| \leqslant 1),$$

whence in $|s-1| \leqslant \varepsilon_1$

$$|\zeta'(s, \chi')| \leqslant c_8(\varepsilon_1) D^{c_9\varepsilon'}$$

(cf. (68)). Now using (111) and arguing as in Lemma 20 we get (110) for $\chi' \neq \chi_0$.

If $\chi' = \chi_0$, then we use the function $G(s)$ of § 19, which is regular in $\sigma > 1-\vartheta$ and, by (12), (112), satisfies

$$|G(1+2\varepsilon_1+it)| < c_{10}(\varepsilon_1) D^{\varepsilon'}(1+|t|), \quad G(1-\vartheta/2+it) \ll D^{c_{11}}(1+|t|)^2.$$

By [5], Lemma 3,

$$|G(s)| \leqslant c_{12}(\varepsilon_1) D^{c_{13}\varepsilon'} \quad \text{in} \quad (1-2\varepsilon_1 \leqslant \sigma \leqslant 1+2\varepsilon_1, |t| \leqslant 1),$$

whence in $|s-1| \leqslant \varepsilon_1$

$$|G'(s)| \leqslant c_{14}(\varepsilon_1) D^{c_{15}\varepsilon'}$$

and observing that by (108) $G(1) > D^{-\varepsilon'}$, we may go on as in Lemma 20.

26. Now let us choose an integer $B \geqslant c_7+2$, c_7 being the constant of Lemma 19, and let

$$x = D^{\xi}, \quad \xi \geqslant 0;$$

$$y = \eta\nu^{-1}\log D, \quad \eta_0 \leqslant \eta \leqslant \eta_0+B \ll 1, \quad 1 \leqslant \nu = \nu(\xi) \leqslant \min(e^{\xi/C}, \log D)$$

where η_0, $C \ll 1$ are large enough. Then the remaining term in (94) is $< D^{-c_0}$ (cf. [4], p. 301). By the arguments used in § 22 (but now denoting by G_1 the part $|t| > \log D$ of G, and changing G_2, G_3 accordingly) we deduce that

$$I_B S' \ll e^{-(\frac{1}{3}\xi+\frac{1}{2}g\eta_0)\lambda_0} + D^{-c_0}$$

(cf. [4], pp. 302, 303). Supposing $q \ll 1$, in the next paragraph we shall prove that for a sufficiently large $\eta_0 \ll 1$ and for all $\xi \geqslant \partial\eta_0$

$$I_B S \geqslant \tfrac{1}{3}(1-e^{-\delta' y}), \tag{114}$$

whence (cf. [4], pp. 304, 305)

$$I_B S \geqslant \frac{\delta_0 \log D}{6A\nu}, \qquad U > \frac{\delta_0 \log D}{12A\nu}.$$

Using the number $z = xe^{4y}$ we divide the sum (90) into four partial sums as in § 24. By the arguments used there

$$S_0+S_1+hS_2 < c_3 x^g e^{\frac{1}{2}g^2 y},$$

whence

$$I_B \frac{1}{2\sqrt{\pi y}\, x^g e^{g^2 y}} h \sum_{\substack{b\in H \\ x<b<z}} \frac{\log b}{b^{\sigma_1}} \exp\left(-\frac{\log^2 b/x}{4y}\right) \geqslant U - I_B \frac{c_3}{2\sqrt{\pi y}} e^{-\frac{1}{2}g^2 y}$$
$$\geqslant \frac{\delta_0 \log D}{12A\nu} - c_4 e^{-\frac{1}{2}g^2\eta_0\nu^{-1}\log D}$$

and going on as in [4], pp. 308, 309, we prove the required results.

27. In this paragraph we shall be concerned with the proof of (114) when $q > 1$, $E'' = 0$, $E' = 1$ and $\chi'(H) = 1$ (the other cases may be treated as in § 23). Then by (91)

$$S = U_1 - U_2, \tag{115}$$

where

$$U_1 = 1 + \sum_{k=\pm 1,\ldots} e^{-\left(\frac{2k\pi}{\log q}\right)^2 y + \frac{4gk\pi i}{\log q} y},$$

$$U_2 = x^{-\delta'} e^{-\delta'(2g-\delta')y} \left\{1 + \sum_{k=\pm 1,\ldots} e^{-\left(\frac{2k\pi}{\log q}\right)^2 y + \frac{4(g-\delta')gk\pi i}{\log q} y}\right\}$$

are evidently real. Observing that the function

$$\varphi(s) = \frac{\log q}{1-q^{1-s}} = \log q \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} \qquad (\sigma > 1)$$

is regular in the whole plane, except for simple poles at $s = 1+2k\pi i/\log q$ $(k = 0, \pm 1, \ldots)$ with residue 1, we have, by (115)

$$(116)\qquad \log q \sum_{n=0}^{\infty} \frac{q^n - q^{n(1-\delta')}}{q^{n\sigma_1}} \exp\left(-\frac{1}{4y}\log^2 q^n/x\right)$$

$$= -i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \{\varphi(s)-\varphi(s+\delta')\} x^{s-\sigma_1} e^{(s-\sigma_1)^2 y} ds$$

$$= 2\sqrt{\pi y}\, x^g e^{g^2 y}(U_1 - U_2) - i\sqrt{\frac{y}{\pi}} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \{\varphi(s)-\varphi(s+\delta')\} x^{s-\sigma_1} e^{(s-\sigma_1)^2 y} ds.$$

Since by (17) on the line $\sigma = \sigma_1$ we have $\varphi(s)-\varphi(s+\delta') \ll \delta'$, the last term is $\ll \delta'$. For $x = q^{x_0}$ (x_0 being an integer $\geqslant 0$), the remainder after x_0 terms of the series in (116) satisfies

$$(117)\qquad \sum_{n \geqslant x_0} > x^g(1-x^{-\delta'})\{1+q^g e^{-(1^2/4y)\log^2 q} + q^{2g} e^{-(2^2/4y)\log^2 q} + \ldots\}.$$

The general term $q^{ng} e^{-(n^2/4y)\log^2 q} = f(n)$ (say) of the series in brackets attains its maxima at $n = n_1 = [2gy/\log q]$ or $[2gy/\log q]+1$ (or both). Since

$$f\left(\frac{2gy}{\log q}\right) = e^{g^2 y}, \qquad f\left(\frac{2gy}{\log q} \pm \frac{\sqrt{2y}}{\log q}\right) = e^{g^2 y - \frac{1}{2}}$$

for $q \ll 1$ and a sufficiently large $\eta_0 = \eta_0(q) \ll 1$ the sum of that series is

$$\geqslant 2\frac{\sqrt{2y}}{\log q} e^{g^2 y - \frac{1}{2}}.$$

Hence, by (116), (115)

$$S > \sqrt{\frac{2}{\pi e}}(1-x^{-\delta'}) > \tfrac{1}{3}(1-x^{-\delta'})$$

whence (since $\xi \geqslant \eta_0$ for $q > 1$) we can deduce (114).

NOTE. On (117) may be based another proof of (99) in the case of $1 < q < D^{\vartheta}$, $\chi'(H) = 1$. By the arguments of (116)

$$\{1+q^g e^{-(1^2/4y)\log^2 q} + q^{2g} e^{-(2^2/4y)\log^2 q} + \ldots\}\log q$$

$$= -i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \varphi(s) e^{(s-\sigma_1)^2 y} ds = 2\sqrt{\pi y} \sum_{k=0,\pm 1,\ldots} e^{\left(g+\frac{2k\pi i}{\log q}\right)^2 y} - i\sqrt{\frac{y}{\pi}} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty}$$

$$= 2\sqrt{\pi y}\, e^{g^2 y} \sum_{k=0,\pm 1,\ldots} e^{\left\{\frac{4gk\pi i}{\log q} - \left(\frac{2k\pi}{\log q}\right)^2\right\} y} + O(\log eq).$$

Hence, by (116) and (117)

$$S > (1-x^{-\delta'})\left\{\sum_{k=0,\pm 1,\ldots} e^{-\left(\frac{2k\pi}{\log q}\right)^2 y + \frac{4gk\pi i}{\log q} y} - O\left(\frac{\log eq}{\sqrt{y}\, e^{g^2 y}}\right)\right\},$$

whence

$$I_B S > (1-x^{-\delta'})\left\{1 - \left| I_B \sum_{k=\pm 1,\ldots} e^{-\left(\frac{2k\pi}{\log q}\right)^2 y + \frac{4gk\pi i}{\log q} y}\right| - c_3 \frac{\log eq}{\sqrt{\eta_0 \log D}\, e^{g^2 \eta_0 \log D}}\right\}$$

and we may proceed as in § 23: For sufficiently large η_0, $B \ll 1$ the expression in brackets is $> \frac{3}{4}$ and, if $\xi \geqslant 4g\eta_0$, (99) follows. Having proved the theorem for all $\xi \geqslant 4g\eta_0$ we get it for all $\xi \geqslant 0$ after increasing c by $4g\eta_0$.

Appendix. 28. In this paragraph we shall prove a statement of § 1 concerning the case where the coefficients a_i in (1) are unequal. It may be formulated as the following

LEMMA 26. *Let the classes H_i $(1 \leqslant i \leqslant h)$ satisfying* (1) *form a group K and let at least two of the coefficients $a = a_i > 0$ be different. Then there are two classes H, H' among the H_i such that $\pi(x, H)$ and $\pi(x, H')$ have different orders of magnitude as $x \to \infty$.*

EXAMPLE. Let H_1, H_2, H_3 be the classes of all the natural numbers n and, respectively, of the numbers $n\sqrt[3]{2}$, $n\sqrt[3]{4}$. Evidently they form a group and (1) holds with $a_1 = 1$, $a_2 = 2^{-1/3}$, $a_3 = 2^{-2/3}$, $\vartheta = 1$. The generators are $b_1 = \sqrt[3]{2}$ and all the odd primes p. There is no generator in H_2.

Proof. Let $\chi_1, \chi_2, \ldots, \chi_h$ be all the characters of the group K, the principal character being χ_1. We begin by proving that the matrix

$$M = \begin{pmatrix} \chi_2(H_1) & \ldots & \chi_2(H_h) \\ \ldots & \ldots & \ldots \\ \chi_h(H_1) & \ldots & \chi_h(H_h) \end{pmatrix}$$

has the rank $r(M) = h-1$.

Suppose, that $r(M) < h-1$. Then, on the one hand, all the determinants of the order $h-1$ belonging to M are zero, whence using the expansion corresponding to the first row in

$$D = \begin{vmatrix} \chi_1(H_1) & \ldots & \chi_1(H_h) \\ \chi_2(H_1) & \ldots & \chi_2(H_h) \\ \ldots & \ldots & \ldots \\ \chi_h(H_1) & \ldots & \chi_h(H_h) \end{vmatrix}$$

we deduce that $D = 0$. On the other hand, squaring D by the 'column by column' rule and observing that by (64)

$$\sum_{\chi} \chi(H_i)\chi(H_j) = \begin{cases} h & \text{if } H_i, H_j \text{ are inverse classes,} \\ 0 & \text{otherwise} \end{cases}$$

we get a determinant with the number h occurring once in each column and once in each row, all the other places being filled by zeros. This determinant being h^h, we get a contradiction. Hence $D \neq 0$ and thus $r(M) = h-1$.

If all the positive numbers $a_1, \ldots, a_h$ are equal, then in the system of equations

$$(118)\qquad \begin{aligned} &a_1\chi_1(H_1) + a_2\chi_1(H_2) + \ldots + a_h\chi_1(H_h) = c_1, \\ &a_1\chi_2(H_1) + a_2\chi_2(H_2) + \ldots + a_h\chi_2(H_h) = c_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_1\chi_h(H_1) + a_2\chi_h(H_2) + \ldots + a_h\chi_h(H_h) = c_h \end{aligned}$$

we have, by (14)

$$(119)\qquad c_1 > 0, \quad c_2 = \ldots = c_h = 0.$$

Suppose, that there is a set of positive and unequal numbers a_i such that in (118) $c_2 = \ldots = c_h = 0$. Dropping the first line we get a homogeneous system of linear equations with a set of unequal solutions, whence $r(M) < h-1$ — a contradiction. Hence, if in (118) $a_1, \ldots, a_h$ are positive unequal numbers, then next to $c_1 \neq 0$ at least one of the numbers $c_2, \ldots, c_h$ is $\neq 0$. Transposing (if necessary) the indices we may suppose that

$$c_1, c_2, \ldots, c_k \quad (k \geqslant 2)$$

are all the coefficients c in (118) which do not vanish. The corresponding characters $\chi_1, \ldots, \chi_k$ form a group Γ, by [11], Lemma A ([13]). The zeta functions

$$\zeta(s, \chi_j) \quad (1 \leqslant j \leqslant k)$$

with characters $\chi_j \in \Gamma$ (and these alone) have a simple pole at $s = 1$ (cf. § 3). Hence the functions $\zeta'/\zeta(s, \chi_j)$ have a simple pole at $s = 1$ with residue -1; the other $\zeta'/\zeta(s, \chi)$ are regular at $s = 1$. The main term in the asymptotical estimate of the sum $\sum \log b$ over the generators $b \leqslant x$ of any class H being

$$-h^{-1}\sum_{\chi} \bar{\chi}(H) \operatorname*{Res}_{s=1} \frac{x^s}{s}\cdot\frac{\zeta'}{\zeta}(s, \chi) = \frac{x}{h}\sum_{1\leqslant j\leqslant k} \bar{\chi}_j(H)$$

([13]) Although the starting point in [11] is different from that of the present paper (cf. ([2])), it does not affect the proof concerning Γ.

(cf. [17], p. 134), for the principal class H_1 is equal to

$$xk/h.$$

Hence the lemma would follow by proving that there is a class $H \in K$ such that

$$(120)\qquad \sum_{1\leqslant j\leqslant k} \bar{\chi}_j(H) = 0.$$

By the isomorphism of the group K' of the characters on the group K (cf. [12], p. 36) there is a sub-group $K_0 \subset K$ corresponding to $\Gamma \subset K'$. Hence the order of K_0 is $k > 1$ and thus there is a class $H \in K_0$ such that for $\varkappa_j$ running over all the characters of the group K_0 we have, by (64),

$$\sum_{1\leqslant j\leqslant k} \varkappa_j(H) = 0$$

whence

$$(121)\qquad \sum_{1\leqslant j\leqslant k} \bar{\varkappa}_j(H) = 0.$$

Let $\varkappa'$ run through all the characters of the factor-group K/K_0. Then

$$\varkappa_j\varkappa' \quad (j = 1, \ldots, k)$$

apparently represent all the characters χ of K. Since

$$\bar{\chi}_j(H) = \bar{\varkappa}'(H)\bar{\varkappa}_j(H)$$

(with an obvious interpretation of $\varkappa'(H)$), (120) follows from (121).

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The discrepancy of random sequences $\{kx\}$

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1. Introduction. It was R. Bellman [2] who first suggested the investigation of the limit distribution of

$$\sum_{k=1}^{N} f(y+kx;\ a,b) - N(b-a) \tag{1.1}$$

if the pair x, y is a random variable, uniformly distributed in the unit square and if, for $0 \leqslant a \leqslant b \leqslant 1$,

$$f(\xi;\ a,b) = \begin{cases} 1 & \text{if} \quad a \leqslant \xi \leqslant b, \\ 0 & \text{if} \quad 0 \leqslant \xi < a \text{ or } b < \xi \leqslant 1, \end{cases}$$

$$f(\xi+1;\ a,b) = f(\xi;\ a,b).$$

If $\{\xi\} = \xi - [x]$ denotes the fractional part of ξ, then $\sum_{k=1}^{N} f(y+kx;\ a,b)$ is simply the number of $k, 1 \leqslant k \leqslant N$, with $\{y+kx\} \epsilon [a,b]$ and (1.1) measures the deviation of this number from its average.

In [4] and [5] the author found the limiting distribution of (1.1). In this note those results are extended by studying the discrepancy

$$D_N(x) = \frac{1}{N} \sup_{0 \leqslant a \leqslant b \leqslant 1+a} \left| \sum_{k=1}^{N} f(kx;\ a,b) - N(b-a) \right|. \tag{1.2}$$

(If $1 < b \leqslant 1+a$ we define $f(\xi;\ a,b)$ in an obvious way, namely as $f(\xi;\ a,1)+f(\xi;\ 0,b-1)$.) Our main result is Theorem 2 below for which we consider x as a point from the measure space $[0,1]$ with Lebesgue measure.

THEOREM 2.

$$\frac{N \cdot D_N(x)}{\log N \cdot \log\log N} \to \frac{2}{\pi^2} \textit{ in measure on } [0,1] \textit{ as } N \to \infty.$$

The first part of the proof (section 2) gives an asymptotic expression for $D_N(x)$ in terms of the continued fraction denominators of x, which