(25) ist auf Grund von (3), (4), (6), (7) und (24) leicht zu beweisen.

Aus (5) und (7) folgt

$$\liminf_{n=+\infty}\frac{B(n)}{\sqrt{n}}>0 \tag{26}$$

und

$$B(n)=O(\sqrt{n}).$$

Aus der letzten Behauptung folgt wegen $A(n)B(n)\geqslant C(n)$ und $C(n)\sim n$

$$\liminf_{n=+\infty}\frac{A(n)}{\sqrt{n}}>0. \tag{27}$$

Aus (26) und (27) ergibt sich, daß $\mathfrak{A}$ und $\mathfrak{B}$ unendlich viele Elemente enthalten, und damit ist der Satz bewiesen.

Wir bemerken noch, daß die im Satz vorkommende Konstante c auch explizit berechenbar ist.

Es verlohnt sich, eine triviale Konsequenz des Satzes besonders zu erwähnen:

Satz II. *Ist für eine Folge* $\mathfrak{C}$

$$\limsup_{n=+\infty}\big(n-C(n)\big)\sqrt[3]{\frac{(\log n)^4}{n(\log\log n)^2}}<c$$

(wobei c die im Satz I. vorkommende Konstante ist), so ist die Folge ***nicht*** *totalprimitiv.*

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Some partition problems related to the Stirling numbers of the second kind

by

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1. Put ([4], Ch. 4)

$$\exp\{t(e^x-1)\}=\sum_{n=0}^{\infty}A_n(t)\frac{x^n}{n!}, \tag{1.1}$$

so that

$$A_n(t)=\sum_{r=0}^{n}a(n,r)t^r, \tag{1.2}$$

where

$$a(n,r)=\frac{1}{r!}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^n. \tag{1.3}$$

The $a(n,r)$ are the Stirling numbers of the second kind.

For fixed n, let $\theta_0(n)$ denote the number of $a(n,2r)$, $0\leqslant 2r\leqslant n$, that are odd and put

$$\theta_0(n+2)=\omega_0(n). \tag{1.4}$$

The writer has proved ([1], [2]) that $\omega_0(n)$ satisfies

$$\sum_{n=0}^{\infty}\omega_0(n)x^n=\prod_{n=0}^{\infty}(1+x^{2^n}+x^{2^{n+1}}) \tag{1.5}$$

and derived a number of additional properties of $\omega_0(n)$.

In the present paper we consider the corresponding problems for other prime moduli. Let $\theta_j(n)$ denote the number of $a(n,k)$, $0\leqslant k\leqslant n$, that are prime to p and such that

$$k\equiv j\pmod{p}\qquad(0\leqslant j\leqslant p-1), \tag{1.6}$$

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where p is an arbitrary prime. Put

$$\omega_0(n) = \theta_0(n+p) \tag{1.7}$$

and

$$W_0(x) = \sum_{n=0}^{\infty} \omega_0(n) x^n. \tag{1.8}$$

Then we show that

$$W_0(x) = \prod_{n=0}^{\infty} f(x^{p^n}), \tag{1.9}$$

where $f(x)$ is a polynomial of degree $p(p-1)$:

$$f(x) = \sum_{r=0}^{p(p-1)} c_r x^r, \tag{1.10}$$

where the c_r are equal to 0 or 1. In particular

$$f(x) = 1+x^2+x^3+x^4+x^5+x^6 \quad (p=3),$$
$$f(x) = 1+x^4+x^5+x^8+x^9+x^{10}+x^{12}+x^{13}+x^{14}+ +x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20} \quad (p=5).$$

For arbitrary p see § 7 below.

The $\theta_j(n)$ can be expressed in terms of $\theta_0(n)$; explicit results are obtained when $p = 3$ or 5.

The proof of (1.9) depends upon the recurrences

$$\theta_0(pn) = \sum_{s=0}^{p-1} \theta_0(n+s), \tag{1.11}$$

$$\theta_0(pn+k) = \sum_{s=1}^{k} \theta_0(n+s) \quad (1 \leqslant k \leqslant p-1). \tag{1.12}$$

In the proof we make frequent use of the congruence ([3], p. 52)

$$\binom{rp+s}{jp+k} \equiv \binom{r}{j}\binom{s}{k} \pmod{p}, \tag{1.13}$$

where $0 \leqslant s < p$, $0 \leqslant k < p$.

2. Touchard ([5], see also [4], p. 81) has proved the congruence

$$A_{n+p}(t) \equiv A_{n+1}(t) + t^p A_n(t) \pmod{p}, \tag{2.1}$$

where p is an arbitrary prime and $A_n(t)$ is the polynomial defined by (1.1).

Let $F = \mathrm{GF}(p, t)$ be the function field obtained by adjoining the indeterminate t to the finite field $\mathrm{GF}(p)$ and let

$$\alpha^p = \alpha + t^p, \tag{2.2}$$

so that α lies in a certain finite extension of F. Clearly the roots of (2.2) are given by

$$\alpha, \alpha+1, \ldots, \alpha+p-1. \tag{2.3}$$

Now put

$$\varphi_n = \varphi_n(t) = \sum_{c=0}^{p-1} (\alpha+c)^n. \tag{2.4}$$

Then by (2.2)

$$\varphi_{n+p}(t) = \varphi_{n+1}(t) + t^p \varphi_n(t). \tag{2.5}$$

Moreover it follows from (2.4) that

$$\sum_{n=0}^{\infty} \varphi_n x^n = \sum_{c=0}^{p-1} \frac{1}{1-(\alpha+c)x}.$$

A simple calculation leads to

$$\sum_{n=0}^{\infty} \varphi_n x^n = \frac{-x^{p-1}}{1-x^{p-1}-x^p t^p}. \tag{2.6}$$

Since

$$\begin{aligned}\frac{x^{p-1}}{1-x^{p-1}-x^p t^p} &= \sum_{r=0}^{\infty} x^{(p-1)(r+1)} (1+xt^p)^r \\ &= \sum_{r=0}^{\infty} x^{(p-1)(r+1)} \sum_{j=0}^{r} \binom{r}{j} x^j t^{pj} \\ &= \sum_{n=0}^{\infty} x^{n+p-1} \sum_{r} \binom{r}{n-(p-1)r} t^{p(n-(p-1)r)},\end{aligned}$$

it follows from (2.6) that

$$\varphi_{n+p-1}(t) = -\sum_{r} \binom{r}{n-(p-1)r} t^{p(n-(p-1)r)}; \tag{2.7}$$

the summation on the right is over all r such that

$$(p-1)r \leqslant n \leqslant pr. \tag{2.8}$$

It is clear from (2.4) that

$$\varphi_n(t) = 0 \quad (0 \leqslant n < p-1), \qquad \varphi_{n-1}(t) = -1. \tag{2.9}$$

Making use of (2.5) or (2.7) we get

$$\varphi_n(t) = 0 \quad (p \leqslant n < 2p-2), \qquad \varphi_{2p-2}(t) = -1. \tag{2.10}$$

Consider the polynomial

$$\bar{A}_n(t) = \varphi_n(t) - \sum_{r=0}^{p-1} A_{p-1-r}(t)\varphi_{n+r}(t). \tag{2.11}$$

It is clear from (2.5) that

$$\bar{A}_{n+p}(t) = \bar{A}_{n+1}(t) + t^p \bar{A}_n(t). \tag{2.12}$$

Moreover it follows from (2.9), (2.10) and (2.11) that

$$\bar{A}_n(t) = A_n(t) \quad (0 \leqslant n < p-1)$$

and

$$\bar{A}_{p-1}(t) = \varphi_{p-1}(t) - A_{p-1}(t)\varphi_{p-1}(t) - \varphi_{2p-2}(t) = A_{p-1}(t).$$

Therefore, by (2.1) and (2.12), we get

$$\bar{A}_n(t) = A_n(t) \quad (n = 0, 1, 2, \ldots).$$

Thus (2.11) becomes

$$A_n(t) = \varphi_n(t) - \sum_{r=0}^{p-1} A_{p-1-r}(t)\varphi_{n+r}(t). \tag{2.13}$$

3. We now take $p = 3$. Since

$$A_1(t) = t, \qquad A_2(t) = t+t^2,$$

(2.13) reduces to

$$A_n(t) = (1-t-t^2)\varphi_n(t) - t\varphi_{n+1}(t) - \varphi_{n+2}(t). \tag{3.1}$$

On the other hand, (2.7) becomes

$$\varphi_n(t) = -\sum_r \binom{r}{n-2r-2} t^{3(n-2r-2)}. \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\begin{aligned} A_n(t) &= -(1-t-t^2)\sum_r \binom{r}{n-2r-2} t^{3(n-2r-2)} + t\sum_r \binom{r}{n-2r-1} t^{3(n-2r-1)} + \\ &\quad + \sum_r \binom{r}{n-2r} t^{3(n-2r)} \\ &= \sum_r \binom{r}{n-2r-3} t^{3(n-2r-2)} + \\ &\quad + t\left\{\sum_r \binom{r}{n-2r-2} t^{3(n-2r-2)} + \sum_r \binom{n}{n-2r-1} t^{3(n-2r-1)}\right\} + \\ &\quad + t^2 \sum_r \binom{n}{3-2r-2} t^{3(n-2r-2)}. \end{aligned}$$

Comparison with (1.2) gives

$$\begin{aligned} a(n, 3j) &\equiv \binom{r}{j-1} && (j = n-2r-2), \\ a(n, 3j+1) &\equiv \binom{r}{j} && (j = n-2r-1 \text{ or } n-2r-2), \\ a(n, 3j+2) &\equiv \binom{r}{j} && (j = n-2r-2); \end{aligned} \tag{3.3}$$

the modulus 3 is understood in each congruence.

For fixed n, let $\theta_j(n)$ denote the number of $a(n, k)$, $0 \leqslant k \leqslant n$, that are prime to 3 and such that

$$k \equiv j \pmod 3 \quad (j = 0, 1, 2). \tag{3.4}$$

By the first of (3.3) we have

$$a(n+1, 3j+3) \equiv \binom{r}{j} \quad (j = n-2r-2),$$

so that

$$\theta_2(n) = \theta_0(n+1). \tag{3.5}$$

Also

$$a(n+2, 3j+3) \equiv \binom{r}{j} \quad (j = n-2r-1);$$

it follows that

$$\theta_1(n) = \theta_0(n+2) + \theta_2(n). \tag{3.6}$$

Combining this with (3.5) we have

$$\theta_1(n) = \theta_0(n+1) + \theta_0(n+2). \tag{3.7}$$

Thus it suffices to consider $\theta_0(n)$.

Replacing n by $3n$ in the first of (3.3) we get

$$a(3n, 3j) \equiv \binom{r}{j-1} \qquad (j = 3n-2r-2).$$

Thus $a(3n, 3j) \equiv 0$ unless $j \equiv r+1$. Hence if we put

$$r = 3r'+s, \qquad j-1 = 3(j'-1)+s \qquad (s = 0, 1, 2),$$

we get, by (1.13),

$$a(3n, 3j) \equiv \binom{r'}{j'-1} \qquad (j' = n-2r'-s)$$

with s at our disposal. We have therefore

$$\theta_0(3n) = \theta_0(n)+\theta_0(n+1)+\theta_0(n+2). \tag{3.8}$$

In the next place, since

$$a(3n+1, 3j) \equiv \binom{r}{j-1} \qquad (j = 3n-2r-1)$$

it is necessary that $j-1 \equiv r+1$; this requires

$$r = 3r'+2, \qquad j-1 = 3j',$$

so that, by (1.13),

$$a(3n+1, 3j) \equiv \binom{r'}{j'} \qquad (j' = n-2r'-2).$$

This implies

$$\theta_0(3n+1) = \theta_2(n);$$

in view of (3.5) this reduces to

$$\theta_0(3n+1) = \theta_0(n+1). \tag{3.9}$$

Finally, since

$$a(3n+2, 3j) \equiv \binom{r}{j-1} \qquad (j = 3n-2r),$$

we get $j \equiv r$. This requires

$$r = 3r'+s, \qquad j = 3j'+s \qquad (s = 1, 2),$$

so that

$$a(3n+2, 3j) \equiv s\binom{r'}{j'} \qquad (j' = n-2r'-s).$$

Then

$$\theta_0(3n+2) = \theta_2(n)+\theta_2(n+1),$$

which gives

$$\theta_0(3n+1) = \theta_0(n+1)+\theta_0(n+2). \tag{3.10}$$

4. It is convenient to define

$$\omega_0(n) = \theta_0(n+3). \tag{4.1}$$

Then (3.8), (3.9), (3.10) become

$$\omega_0(3n) = \omega_0(n)+\omega_0(n-1)+\omega_0(n-2), \tag{4.2}$$

$$\omega_0(3n+1) = \omega_0(n-1), \tag{4.3}$$

$$\omega_0(3n+2) = \omega_0(n)+\omega_0(n-1), \tag{4.4}$$

respectively. Since $\theta_0(1) = \theta_0(2) = 0$, these formulas hold for all $n = 0, 1, 2, \ldots$

If we now put

$$W_0(x) = \sum_{n=0}^{\infty} \omega_0(n)x^n, \tag{4.5}$$

it follows that

$$W_0(n) = \sum_0^{\infty} \omega_0(3n)x^{3n} + \sum_0^{\infty} \omega_0(3n+1)x^{3n+1} + \sum_0^{\infty} \omega_0(3n+2)x^{3n+2}$$

$$= \sum_0^{\infty} \big(\omega_0(n)+\omega_0(n-1)+\omega_0(n-2)\big)x^{3n} +$$

$$+ \sum_0^{\infty} \omega_0(n-1)x^{3n+1} + \sum_0^{\infty} \big(\omega_0(n)+\omega_0(n-1)\big)x^{3n+2}$$

$$= \sum_0^{\infty} \omega_0(n)(x^{3n}+x^{3n+3}+x^{3n+6}+x^{3n+4}+x^{3n+2}+x^{3n+5}).$$

We have therefore

$$W_0(x) = (1+x^2+x^3+x^4+x^5+x^6)W_0(x^3), \tag{4.6}$$

which implies

$$W_0(x) = \prod_{n=0}^{\infty} f(x^{3^n}), \tag{4.7}$$

where

$$f(x) = 1+x^2+x^3+x^4+x^5+x^6 = (1+x^2)(1+x^3+x^4). \tag{4.8}$$

5. We turn now to the case $p = 5$. Since

$$A_1 = t, \quad A_2 = t+t^2, \quad A_3 = t+3t^2+t^3, \quad A_4 = t+7t^2+6t^3+t^4,$$

we find that (2.13) reduces to

$$(5.1) \qquad A_n(t) = (1-t-2t^2-t^3-t^4)\varphi_n(t)-(t-2t^2+t^3)\varphi_{n+1}(t)- \\ -(t+t^2)\varphi_{n+2}(t)-t\varphi_{n+3}(t)-\varphi_{n+4}(t).$$

Substituting from (2.7) in (5.1) we get

$$(5.2) \qquad a(n, 5j) \equiv \binom{r}{j-1} \qquad (j = n-4r-4),$$

$$(5.3) \qquad a(n, 5j+1) \equiv \lambda\binom{r}{j} \qquad (j = n-4r-1, 2, 3, 4),$$

$$(5.4) \qquad a(n, 5j+2) \equiv \lambda\binom{r}{j} \qquad (j = n-4r-2, 3, 4),$$

$$(5.5) \qquad a(n, 5j+3) \equiv \binom{r}{j} \qquad (j = n-4r-3, 4),$$

$$(5.6) \qquad a(n, 5j+4) \equiv \binom{r}{j} \qquad (j = n-4r-4).$$

Congruences are now (mod 5); λ in (5.4) and (5.5) denotes a number prime to 5.

To illustrate we prove (5.4). It follows from (5.1) and (2.7) that we need consider only $\varphi_n(t)$, $\varphi_{n+1}(t)$, $\varphi_{n+2}(t)$. The coefficient λ is equal to -2, 2, -1, respectively.

For fixed n, let $\theta_j(n)$ denote the number of $a(n, k)$, $0 \leqslant k \leqslant n$, that are prime to 5 and such that

$$(5.7) \qquad k \equiv j \pmod 5 \qquad (j = 0, 1, 2, 3, 4).$$

Making use of (5.2)-(5.6) we find that

$$(5.8) \qquad \theta_3(n) = \theta_4(n)+\theta_4(n+1),$$

$$(5.9) \qquad \theta_2(n) = \theta_4(n)+\theta_4(n+1)+\theta_4(n+2),$$

$$(5.10) \qquad \theta_1(n) = \theta_4(n)+\theta_4(n+1)+\theta_4(n+2)+\theta_4(n+3).$$

On the other hand, since

$$a(n+1, 5j+5) \equiv -\binom{r}{j} \qquad (j = n-4r-4),$$

it is clear that

$$(5.11) \qquad \theta_4(n) = \theta_0(n+1).$$

Thus all $\theta_j(n)$ are expressible in terms of $\theta_0(n)$.

Replacing n by $5n$ in (5.2) we have

$$a(5n, 5j) \equiv \binom{r}{j-1} \qquad (j = 5n-4r-4).$$

Thus $j \equiv r+1$ and very much as in the case $p = 3$ we get

$$(5.12) \qquad \theta_0(5n) = \theta_0(n)+\theta_0(n+1)+\theta_0(n+2)+\theta_0(n+3)+\theta_0(n+4).$$

Similarly it follows from

$$a(5n+1, 5j) \equiv \binom{r}{j-1} \qquad (j = 5n-4r-3)$$

that $j \equiv r+2$. This requires $r \equiv 4$ and we get

$$(5.13) \qquad \theta_0(5n+1) = \theta_0(n+1).$$

Continuing in this way we get

$$(5.14) \qquad \theta_0(5n+2) = \theta_0(n+1)+\theta_0(n+2),$$

$$(5.15) \qquad \theta_0(5n+3) = \theta_0(n+1)+\theta_0(n+2)+\theta_0(n+3),$$

$$(5.16) \qquad \theta_0(5n+4) = \theta_0(n+1)+\theta_0(n+2)+\theta_0(n+3)+\theta_0(n+4).$$

Now put

$$(5.17) \qquad \theta_0(n+5) = \omega_0(n).$$

Then (5.12)-(5.16) become

$$(5.18) \qquad \omega_0(5n) = \omega_0(n)+\omega_0(n-1)+\omega_0(n-2)+\omega_0(n-3)+\omega_0(n-4),$$

$$(5.19) \qquad \omega_0(5n+1) = \omega_0(n-3),$$

$$(5.20) \qquad \omega_0(5n+2) = \omega_0(n-2)+\omega_0(n-3),$$

$$(5.21) \qquad \omega_0(5n+3) = \omega_0(n-1)+\omega_0(n-2)+\omega_0(n-3),$$

$$(5.22) \qquad \omega_0(5n+4) = \omega_0(n)+\omega_0(n-1)+\omega_0(n-2)+\omega_0(n-3),$$

respectively. The formulas (5.18)-(5.22) hold for all $n = 0, 1, 2, \ldots$

If we put

$$(5.23) \qquad W_0(n) = \sum_{n=0}^{\infty} \omega_0(n)x^n,$$

it follows that

$$W_0(n) = \sum_0^{\infty} \omega_0(5n)x^{5n}+\sum_0^{\infty} \omega_0(5n+1)x^{5n+1}+\sum_0^{\infty} \omega_0(5n+2)x^{5n+2}+ \\ +\sum_0^{\infty} \omega_0(5n+3)x^{5n+3}+\sum_0^{\infty} \omega_0(5n+4)x^{5n+4}.$$

Making use of (5.18)-(5.22) and simplifying we get

$$W_0(x) = \prod_{n=0}^{\infty} f(x^{5^n}), \tag{5.24}$$

where

$$f(x) = 1+x^4+x^5+x^8+x^9+x^{10}+x^{12}+x^{13}+ \\ +x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}. \tag{5.25}$$

6. For an arbitrary prime p, (2.13) is less explicit. It is convenient to rewrite (2.13) as

$$A_n(t) = \big(1-A_{p-1}(t)\big)\varphi_n(t) = \sum_{r=1}^{p-2} A_{p-1-r}(t)\varphi_{n+r}(t) - \varphi_{n+p-1}(t). \tag{6.1}$$

Since $a(n, 0) = 0$ for $n > 0$, it follows from (6.1) and (2.7) that

$$a(n, pj) \equiv \binom{r}{j-1} \quad (j = n-(p-1)(r+1)). \tag{6.2}$$

It follows also from (6.1) and (2.7) that

$$a(n, pj+p-1) \equiv \binom{r}{j} \quad (j = n-(p-1)(r+1)). \tag{6.3}$$

Since $a(n, 1) = 1$ for $n \geqslant 1$ we get also

$$a(n, pj+1) \equiv \binom{r}{j} \quad (j = n-(p-1)r-s, 1 \leqslant s \leqslant p-1). \tag{6.4}$$

Let $\theta_j(n)$ denote the number of $a(n, k)$, $0 \leqslant k \leqslant n$, that are prime to p and such that

$$k \equiv j (\mathrm{mod}\, p). \tag{6.5}$$

It follows immediately from (6.3) and (6.4) that

$$\theta_1(n) = \sum_{c=0}^{p-2} \theta_{p-1}(n+c). \tag{6.6}$$

Also, since by (6.2)

$$a(n+1, pj+p) \equiv \binom{r}{j} \quad (j = n-(p-1)(r+1)),$$

we have

$$\theta_{p-1}(n) = \theta_0(n+1). \tag{6.7}$$

Thus (6.6) becomes

$$\theta_1(n) = \sum_{c=1}^{p-1} \theta_0(n+c). \tag{6.8}$$

Returning to (6.2) we replace n by pn, so that

$$a(pn, pj) \equiv \binom{r}{j-1} \quad (j = pn-(p-1)(r+1)).$$

It follows that $j \equiv r+1$. We may therefore put

$$r = pr'+s, \quad j = pj'+s+1 \quad (s = 0, 1, \ldots, p-1).$$

Then, by (1.13),

$$\binom{r}{j-1} = \binom{pr'+s}{pj'+s} \equiv \binom{r'}{j'} \quad (j' = n-(p-1)r'-s-1).$$

Comparing this with (6.3) we get

$$\theta_0(pn) = \sum_{s=0}^{p-1} \theta_{p-1}(n+p-s-2);$$

in view of (6.7) this becomes

$$\theta_0(pn) = \sum_{s=0}^{p-1} \theta_0(n+s). \tag{6.9}$$

If in (6.2) we replace n by $pn+k$, $1 \leqslant k \leqslant p-1$, we get

$$a(pn+k, pj) \equiv \binom{r}{j-1} \quad (j = pn+k-(p-1)(r+1)).$$

Thus $j-1 \equiv k+r$. If $r \equiv s$, $0 \leqslant s < p-k$, it is evident that

$$\binom{r}{j-1} \equiv 0.$$

We may accordingly put

$$r = pr'+s, \quad j-1 = pj+s-k-p \quad (p-k \leqslant s \leqslant p-1).$$

Then, by (1.13),

$$\binom{r}{j-1} = \binom{pr'+s}{pj'+s+k-p} \equiv \binom{r'}{j'}\binom{s}{s+k-p}.$$

Since $p-k \leqslant s \leqslant p-1$ it follows that

$$\lambda = \binom{s}{s+k-p} \not\equiv 0.$$

We have therefore

$$a(pn+k, pj) \equiv \lambda \binom{r'}{j'} \qquad (j' = n-(p-1)r'-s). \tag{6.10}$$

Comparison with (6.3) yields

$$\theta_0(pn+k) = \sum_{s=p-1}^{p-k} \theta_{p-1}(n+p-s-1)$$

and therefore

$$\theta_0(pn+k) = \sum_{s=1}^{k} \theta_0(n+s) \qquad (1 \leqslant k \leqslant p-1). \tag{6.11}$$

7. Let

$$\theta_0(n+p) = \omega_0(n). \tag{7.1}$$

Then (6.9) becomes

$$\omega_0(pn) = \sum_{s=0}^{p-1} \omega_0(n-s) \tag{7.2}$$

while (6.11) becomes

$$\omega_0(pn+k) = \sum_{s=1}^{k} \omega_0(n-p+s+1) \qquad (1 \leqslant k \leqslant p-1). \tag{7.3}$$

If we put

$$W_0(x) = \sum_{n=0}^{\infty} \omega_0(n) x^n, \tag{7.4}$$

it follows that

$$\begin{aligned} W_0(x) &= \sum_{n=0}^{\infty} \omega_0(pn) x^{pn} + \sum_{k=1}^{p-1} \sum_{n=0}^{\infty} \omega_0(pn+k) x^{pn+k} \\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{p-1} \omega_0(n-s) x^{pn} + \sum_{k=1}^{p-1} \sum_{n=0}^{\infty} \sum_{s=1}^{k} \omega_0(n-p+s+1) x^{pn+k} \\ &= \sum_{s=0}^{p-1} \sum_{n=0}^{\infty} \omega_0(n) x^{pn+ps} + \sum_{s=1}^{p-1} \sum_{k=s}^{p-1} \sum_{n=0}^{\infty} \omega_0(n) x^{p(n+p-s-1)+k} \\ &= W_0(x^p) \sum_{s=0}^{p-1} x^{ps} + W_0(x^p) \sum_{s=1}^{p-1} \sum_{k=s}^{p-1} x^{p(p-s-1)+k}. \end{aligned}$$

We may put

$$f(x) = \sum_{s=0}^{p-1} x^{ps} + \sum_{s=1}^{p-1} \sum_{k=s}^{p-1} x^{p(p-s-1)+k} = \sum_{s=0}^{p-1} x^{ps} + \sum_{s=1}^{p-1} \sum_{k=p-s}^{p-1} x^{p(s-1)+k} \tag{7.5}$$

$$= \frac{1-x^{p^2}}{1-x^p} + \frac{1}{1-x}\left(\frac{1-x^{p(p-1)}}{1-x^{p-1}} - \frac{1-x^{p^2}}{1-x^p}\right) = \frac{1}{1-x}\left(\frac{1-x^{p(p-1)}}{1-x^{p-1}} - x\frac{1-x^{p^2}}{1-x^p}\right).$$

It is easily verified that when $p = 3$ or 5, $f(x)$ reduces to (4.8) or (5.25), respectively. We remark also that

$$f(x) = \sum_{r=0}^{p(p-1)} c_r x^r \qquad (c_0 = c_{p(p-1)} = 1), \tag{7.6}$$

where the c_r are either 0 or 1. More precisely $c_r = 1$ provided

$$r = ps \qquad (0 \leqslant s \leqslant p-1)$$

or

$$r = ps+k \qquad (0 \leqslant s < p-1;\ p-s-1 \leqslant k \leqslant p-1);$$

$c_s = 0$ otherwise. It can be verified that $f(x)$ is divisible by $(x^{p+1}-1)/(x^2-1)$.

Finally, we have

$$W_0(x) = \prod_{n=0}^{\infty} f(x^{p^n}). \tag{7.7}$$

8. Residues $(\bmod p)$ of $a(n, r)$.

$p = 2$; $1 \leqslant r \leqslant n \leqslant 10$.

1									
1	1								
1	1	1							
1	1	0	1						
1	1	1	0	1					
1	1	0	1	1	1				
1	1	1	0	0	1	1			
1	1	0	1	0	0	0	1		
1	1	1	0	1	0	0	0	1	
1	1	0	1	1	1	0	0	1	1

$p = 3$; $1 \leqslant r \leqslant n \leqslant 10$.

1									
1	1								
1	0	1							
1	1	0	1						
1	0	1	1	1					
1	1	0	2	0	1				
1	0	1	2	2	0	1			
1	1	0	0	0	2	1	1		
1	0	1	0	0	0	0	0	1	
1	1	0	1	0	0	0	0	0	1

$p = 5;\ 1 \leqslant r \leqslant n \leqslant 25$

1																								
1	1																							
1	3	1																						
1	2	1	1																					
1	0	0	0	1																				
1	1	0	0	0	1																			
1	3	1	0	0	1	1																		
1	2	1	1	0	1	3	1																	
1	0	0	0	1	1	2	1	1																
1	1	0	0	0	2	0	0	0	1															
1	3	1	0	0	2	2	0	0	0	1														
1	2	1	1	0	2	1	2	0	0	1	1													
1	0	0	0	1	2	4	2	2	0	1	3	1												
1	1	0	0	0	3	0	0	0	2	1	2	1	1											
1	3	1	0	0	3	3	0	0	0	3	0	0	0	1										
1	2	1	1	0	3	4	3	0	0	3	3	0	0	0	1									
1	0	0	0	1	3	1	3	3	0	3	4	3	0	0	1	1								
1	1	0	0	0	4	0	0	0	3	3	1	3	3	0	1	3	1							
1	3	1	0	0	4	4	0	0	0	1	0	0	0	3	1	2	1	1						
1	2	1	1	0	4	2	4	0	0	1	1	0	0	0	4	0	0	0	1					
1	0	0	0	1	4	3	4	4	0	1	3	1	0	0	4	4	0	0	0	1				
1	1	0	0	0	0	0	0	0	4	1	2	1	1	0	4	2	4	0	0	1	1			
1	3	1	0	0	0	0	0	0	0	0	0	0	0	1	4	3	4	4	0	1	3	1		
1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	1	2	1	1	
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

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Systems of three quadratic forms

by

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1. Introduction. Artin conjectured that a set of forms $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ respectively, in n variables over a $\mathfrak{p}$-adic field k has a common non-trivial zero in k provided that $n > \sum d_i^2$. This conjecture has been verified in the following cases: (i) one quadratic form ([9]), (ii) one cubic form ([7], [11], [6], [14]), (iii) two quadratic forms ([8], [2]), (iv) one quintic form ([1]) and (v) one form of degree 7 or 11 ([10]), provided in cases (iv) and (v) that the residue class field is large enough.

As Artin has shown, it is sufficient for the proof of the conjecture to show that it holds for the case of a single form of arbitrary degree. On the other hand, for example, if $N(x)$ is the reduced norm form of a division algebra of degree two over k, then N is a quadratic form over k in four variables that has only the trivial zero in k; and if $f_1, \dots, f_4$ are quadratic forms over k then $f = N(f_1, \dots, f_4)$ is a quartic form whose zeros in k are precisely the common zeros in k of $f_1, \dots, f_4$. Thus, in examining the truth of the conjecture for a single quartic we need to know whether the conjecture is valid for a system of quadratics. It is this last problem which we shall investigate in this note. Many of our results hold for any system of quadratics, but eventually the work becomes so involved that we restrict ourselves to three quadratics. Our arguments are in essence very similar to those used in [2]; however, our proof is far more involved, because in contrast to the case of two quadratics there does not seem to be any elegant utilizable property of a system of three quadratics which has only singular zeros (compare Lemma 2 of [2]). For reasons which will become clear later, our proof will work only if the residue class field is not too small and has odd characteristic.

Throughout this note, k will denote a $\mathfrak{p}$-adic field with ring of integers $\mathfrak{O}$ with maximal prime ideal $\mathfrak{p}$. The residue class field $\mathfrak{O}/\mathfrak{p}$ will be denoted by k^*. We denote the characteristic of k^* by p and the number

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