Appendix added June 5, 1964.

Our formula (6) is actually equivalent to

$$A_3(x) = 2\pi \sum_{1\leqslant q\leqslant x^{1/2}} \sideset{}{'}\sum_{h (\mathrm{mod}\, q)} \left(\frac{S(h,q)}{q}\right)^3 \sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q} + O(x^{3/4}\log x), \quad x\to +\infty.$$

This of course is (4) for $k=3$. In order to show this we need only prove that

$$(14)\qquad \sum_{2\leqslant q\leqslant x^{1/2}} \sideset{}{'}\sum_{h (\mathrm{mod}\, q)} \left(\frac{S(h,q)}{q}\right)^3 \sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q} = O(x^{3/4}\log x), \quad x\to +\infty.$$

By partial summation,

$$\Big|\sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q}\Big| \leqslant q\left(\frac{1}{h}+\frac{1}{q-h}\right)(1+[x])^{1/2}.$$

This together with (5) shows that the left hand side of (14) is

$$O\left(x^{1/2}\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sideset{}{'}\sum_{h (\mathrm{mod}\, q)}\left(\frac{1}{h}+\frac{1}{q-h}\right)\right) = O\left(x^{1/2}\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2}\log q\right) = O(x^{3/4}\log x), \quad \text{as} \quad x\to+\infty.$$

This proves (14) and hence (4) for the case $k=3$.

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On oscillations of certain means formed from the Möbius series II

by

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1. As announced in paper [1], the present work contains some new results concerning the distribution of values of $\mu(n)$ in relatively short intervals $a\leqslant n\leqslant b$. Briefly and roughly speaking, it will be proved that on Riemann hypothesis there exist infinitely many intervals $[U_1, U_2]$, $U_2^{1-o(1)}\leqslant U_1\leqslant U_2$, $U_2\to\infty$, such that

$$\sum_{U_1\leqslant n\leqslant U_2}\mu(n) > U_2^{1/2-o(1)},$$

and also that there exists an infinity of similar intervals $[U_3, U_4]$ with

$$\sum_{U_3\leqslant n\leqslant U_4}\mu(n) < -U_4^{1/2-o(1)}.$$

This result is a particular case of the following Theorem 1. As a by-product of the proof of this theorem, we will obtain the inequality (again on Riemann hypothesis)

$$\int_{T^{1-o(1)}}^{T}\frac{|M(x)|}{x}\,dx > T^{1/2-o(1)},$$

($M(x)$ being, as usual, $\sum_{n\leqslant x}\mu(n)$), which improves on my previous result ([2]).

2. In the following we will use two lemmas. Their proofs can be found respectively in [4] (proof of Lemma II) and in [3] (proof of Theorem 4.1). We call them Lemma 1 and Lemma 2.

Lemma 1. *Let $\beta_1,\beta_2,\dots$ be a real sequence and $a_1, a_2,\dots$ a similar one with the property that*

$$(2.1)\qquad |a_\nu|\geqslant U\ (>0)$$

and

$$\sum_{\nu}\frac{1}{1+|\alpha_\nu|^\gamma}\leqslant V\ (<\infty),\quad \text{where}\quad \gamma>1. \tag{2.2}$$

Then every real interval of length $\Delta>1/U$ contains a ξ-value such that for all ν's

$$\min_{\Omega\ \text{integer}}|\alpha_\nu\xi+\beta_\nu-\Omega|\geqslant\frac{1}{24V}\cdot\frac{1}{1+|\alpha_\nu|^\gamma}.$$

The second lemma pertains complex numbers $z_1, z_2, \ldots, z_n$ such that firstly

$$1=|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_n| \tag{2.3}$$

and secondly, with a $0<\varkappa\leqslant\pi/2$,

$$\varkappa\leqslant|\arg z_j|\leqslant\pi,\quad j=1,2,\ldots,n. \tag{2.4}$$

Further, we suppose that there are indices h and h_1, $h<h_1$, such that

$$|z_h|>\frac{4n}{m+n(3+\pi/\varkappa)} \tag{2.5}$$

and

$$|z_{h_1}|<|z_h|-\frac{2n}{m+n(3+\pi/\varkappa)}, \tag{2.6}$$

where m is an arbitrarily fixed non-negative integer. Finally, given a set of complex numbers $b_1, b_2, \ldots, b_n$, we put

$$A\overset{\text{def}}{=}\min_{h\leqslant j<h_1}\Big|\Re\sum_{\nu\leqslant j}b_\nu\Big| \tag{2.7}$$

and formulate

LEMMA 2. *If $A>0$, then there exist integers ν_1 and ν_2 with*

$$m+1\leqslant\nu_1,\nu_2\leqslant m+n(3+\pi/\varkappa) \tag{2.8}$$

such that

$$\Re\sum_{j=1}^{n}b_jz_j^{\nu_1}\geqslant\frac{A}{2n+1}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)}\left(\frac{n}{24(m+n(3+\pi/\varkappa))}\right)^{2n} \tag{2.9}$$

and

$$\Re\sum_{j=1}^{n}b_jz_j^{\nu_2}\leqslant-\frac{A}{2n+1}\left(\frac{|z_h|}{2}\right)^{m+n(3+\pi/\varkappa)}\left(\frac{n}{24(m+n(3+\pi/\varkappa))}\right)^{2n}. \tag{2.10}$$

Now we come to the theorems.

THEOREM 1. *Suppose all the ζ-zeros in $0<\sigma<1$, $|t|\leqslant\omega$, to lie on the line $\sigma=\frac{1}{2}$. Then for* (1)

$$c_1\leqslant T\leqslant e^{\omega^6} \tag{2.11}$$

there exist values U_1, U_2, U_3, U_4,

$$Te^{-6(\log T)^{5/6}}\leqslant\begin{matrix}U_1\leqslant U_2\\U_3\leqslant U_4\end{matrix}\leqslant Te^{6(\log T)^{5/6}} \tag{2.12}$$

such that

$$\sum_{U_1\leqslant n\leqslant U_2}\mu(n)>T^{1/2}e^{-(\log T)^{3/4}} \tag{2.13}$$

and

$$\sum_{U_3\leqslant n\leqslant U_4}\mu(n)<-T^{1/2}e^{-(\log T)^{3/4}}. \tag{2.14}$$

COROLLARY. *On Riemann hypothesis* (2.12), (2.13), (2.14) *hold for all T sufficiently large.*

THEOREM 2. *Under the same conditions as in Theorem* 1,

$$\int_{X_1}^{X_2}\frac{|M(x)|}{x}dx>T^{1/2}e^{-(\log T)^{3/4}}, \tag{2.15}$$

where $X_1=Te^{-6(\log T)^{5/6}}$, $X_2=Te^{6(\log T)^{5/6}}$.

3. In what follows, we shall denote by ϱ_0 the "earliest" zero of $\zeta(s)$ in the upper half-plane (which is simple)

$$\varrho_0=\tfrac{1}{2}+i\gamma_0=\tfrac{1}{2}+i\cdot 14.13\ldots, \tag{3.1}$$

and by ϱ_1 — the next one (again simple)

$$\varrho_1=\tfrac{1}{2}+i\gamma_1=\tfrac{1}{2}+i\cdot 21.02\ldots, \tag{3.2}$$

Further, we introduce

$$D\overset{\text{def}}{=}\frac{\operatorname{Arg}\zeta'(\varrho_0)}{\gamma_0} \tag{3.3}$$

and note that

$$\Re\frac{e^{D\varrho_0}}{\zeta'(\varrho_0)}=\frac{e^{D/2}}{|\zeta'(\varrho_0)|}=c_2>0. \tag{3.4}$$

(1) Throughout this paper $c_1, c_2, \ldots$ denote positive numerical constants.

It is well known (see e. g. [5], p. 185, Theorem 9.7) that for every $T \geqslant 2$ there exists a $t = t(T)$, $T \leqslant t \leqslant T+1$ such that

(3.5) $$\frac{1}{|\zeta(\sigma+it)|} \leqslant t^{c_3}, \quad -1 \leqslant \sigma \leqslant 2.$$

Putting

(3.6) $$Q \overset{\text{def}}{=} t(\log^{1/6}T - 1),$$

we start from the integral

(3.7) $$I_k \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{2-iQ}^{2+iQ} e^{Ds+k(s+b)^2}\frac{ds}{\zeta(s)},$$

where b and integer k are at the moment restricted only by

(3.8) $$c_4 \leqslant b \leqslant \tfrac{1}{5}\log^{1/3}T$$

and

(3.9) $$1 \leqslant k \leqslant \tfrac{1}{12}\log T.$$

Then, as is easy to see,

(3.10) $$I_k = \frac{1}{2\pi i}\int_{(2)} e^{Ds+k(s+b)^2}\frac{ds}{\zeta(s)} + O(e^{kb^2}\cdot T^{1/3}).$$

Substituting the Dirichlet series $\sum_n \mu(n)n^{-s}$ for $1/\zeta(s)$ in (3.10) and integrating term by term, we obtain

$$\frac{1}{2\pi i}\int_{(2)} e^{Ds+k(s+b)^2}\frac{ds}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{2\pi i}\int_{(2)} e^{k(s+b)^2 - s\log(ne^{-D})}ds$$
$$= \frac{e^{kb^2}}{2\pi i}\int_{(0)} e^{ks^2}ds \sum_{n=1}^{\infty}\mu(n)e^{-\frac{1}{4k}\{\log(ne^{-D})-2kb\}^2}$$
$$= \frac{e^{kb^2}}{2\sqrt{\pi k}}\sum_{n=1}^{\infty}\mu(n)e^{-\frac{1}{4k}\{\log(ne^{-D})-2kb\}^2}.$$

Hence, by (3.10)

(3.11) $$I_k = \frac{e^{kb^2}}{2\sqrt{\pi k}}\sum_{n=1}^{\infty}\mu(n)e^{-\frac{1}{4k}\{\log(ne^{-D})-2kb\}^2} + O(e^{kb^2}\cdot T^{1/3})$$
$$= \frac{e^{kb^2}}{2\sqrt{\pi k}}\int_{1-0}^{\infty} e^{-\frac{1}{4k}\{\log(xe^{-D})-2kb\}^2}dM(x) + O(e^{kb^2}\cdot T^{1/3}).$$

Partial integration gives

$$\int_{1-0}^{\infty} e^{-\frac{1}{4k}\{\log(xe^{-D})-2kb\}^2}dM(x)$$
$$= M(x)e^{-\frac{1}{4k}\{\log(xe^{-D})-2kb\}^2}\Big|_{1-0}^{\infty} - \int_1^{\infty} M(x)d_x(e^{-\frac{1}{4k}\log^2\frac{x}{\xi_k}}),$$

where

(3.12) $$\xi_k = e^{2kb+D}.$$

Thus, putting $\alpha_k = e^{3\sqrt{k\log\xi_k}}$ and using (3.8), (3.9), (3.12), we have

(3.13) $$\Re I_k = I_k = -\frac{e^{kb^2}}{2\sqrt{\pi k}}\int_1^{\infty} M(x)\frac{d}{dx}(e^{-\frac{1}{4k}\log^2\frac{x}{\xi_k}})dx + O(e^{kb^2}\cdot T^{1/3})$$
$$= \frac{e^{kb^2}}{2\sqrt{\pi k}}\int_{\xi_k\alpha_k^{-1}}^{\xi_k\alpha_k} M(x)\frac{d}{dx}(-e^{-\frac{1}{4k}\log^2\frac{x}{\xi_k}})dx + O(e^{kb^2}\cdot T^{1/3}).$$

Hence

$$\Re I_k \leqslant \frac{e^{kb^2}}{2\sqrt{\pi k}}\Big\{\max_{\xi_k\leqslant x\leqslant\xi_k\alpha_k} M(x)\int_{\xi_k}^{\xi_k\alpha_k}(-e^{-\frac{1}{4k}\log^2\frac{x}{\xi_k}})'dx -$$
$$- \min_{\xi_k\alpha_k^{-1}\leqslant x\leqslant\xi_k} M(x)\int_{\xi_k\alpha_k^{-1}}^{\xi_k}(e^{-\frac{1}{4k}\log^2\frac{x}{\xi_k}})'dx\Big\} + c_5e^{kb^2}\cdot T^{1/3}$$
$$= \frac{e^{kb^2}}{2\sqrt{\pi k}}(1-\xi_k^{-9/4})\{\max_{\xi_k\leqslant x\leqslant\xi_k\alpha_k} M(x) - \min_{\xi_k\alpha_k^{-1}\leqslant x\leqslant\xi_k} M(x)\} + c_5e^{kb^2}\cdot T^{1/3},$$

so that

(3.14) $$e^{-kb^2}\Re I_k \leqslant \frac{1}{2\sqrt{\pi k}}(1-\xi_k^{-9/4})\sum_{U_1\leqslant n\leqslant U_2}\mu(n) + c_5T^{1/3},$$

with certain U_1, U_2 satisfying

(3.15) $$\xi_k e^{-3\sqrt{k\log\xi_k}} \leqslant U_1 \leqslant U_2 \leqslant \xi_k e^{3\sqrt{k\log\xi_k}}.$$

Similarly, we come to

(3.16) $$e^{-kb^2}\Re I_k \geqslant \frac{1}{2\sqrt{\pi k}}(1-\xi_k^{-9/4})\sum_{U_3\leqslant n\leqslant U_4}\mu(n) - c_5T^{1/3},$$

with some U_3, U_4 satisfying

(3.17) $$\xi_k e^{-3\sqrt{k\log\xi_k}} \leqslant U_3 \leqslant U_4 \leqslant \xi_k e^{3\sqrt{k\log\xi_k}}.$$

4. By Cauchy's theorem of residues and by (3.5),

$$I_k = \sum_{|\Im\varrho|<Q} \operatorname*{Res}_{s=\varrho} e^{Ds+k(s+b)^2}\frac{1}{\zeta(s)} + \frac{1}{2\pi i}\int_{-1-iQ}^{-1+iQ} e^{Ds+k(s+b)^2}\frac{ds}{\zeta(s)} + O(e^{kb^2}),$$

whence, noting that

$$\int_{-1-iQ}^{-1+iQ} e^{Ds+k(s+b)^2}\frac{ds}{\zeta(s)} = O(e^{kb^2}),$$

and putting

(4.1) $$R_k \overset{\text{def}}{=} \sum_{|\Im\varrho|<Q} \operatorname*{Res}_{s=\varrho} e^{Ds+k(s+b)^2}\frac{1}{\zeta(s)},$$

we obtain

(4.2) $$I_k = R_k + O(e^{kb^2}).$$

Similarly to [1], we shall introduce "shifted" ϱ-zeros and a "shifted" ζ-function. We denote by $\varrho_j = \frac{1}{2}+i\gamma_j$, $0<\gamma_0<\gamma_1<\ldots<\gamma_r$ all the ζ-zeros in $0<t<Q$, the possible multiple zeros being counted only once. Next, we take an $\varepsilon>0$ subjected to

(4.3) $$\varepsilon < \min_{0\leqslant j\leqslant r-1}(\gamma_{j+1}-\gamma_j), \quad \varepsilon < Q-\gamma_r,$$

and for every ϱ_j (whose order of multiplicity is, say, ν) define ν "shifted zeros":

(4.4)

$$\varrho_j^{(1)} = \varrho_j = \frac{1}{2}+i\gamma_j, \quad \varrho_j^{(2)} = \frac{1}{2}+i\left(\gamma_j+\frac{\varepsilon}{\nu}\right), \quad \ldots, \quad \varrho_j^{(\nu)} = \frac{1}{2}+i\left(\gamma_j+\frac{\nu-1}{\nu}\varepsilon\right).$$

In the rectangle $0<\sigma<1$, $-Q<t<0$, we proceed similarly. Thus, we obtain a set of shifted zeros $s_\varepsilon(\varrho)$ such that there is a one-to-one correspondence between the ϱ's in $|t|<Q$ and the $s_\varepsilon(\varrho)$-numbers. We note also that

(4.5) $$|\varrho - s_\varepsilon(\varrho)| < \varepsilon,$$

and

(4.6) $$s_\varepsilon(\varrho_0) = \varrho_0, \quad s_\varepsilon(\bar\varrho_0) = \bar\varrho_0.$$

Finally, we define the "shifted" ζ-function by

(4.7) $$\zeta_\varepsilon(s) \overset{\text{def}}{=} \zeta(s)\prod_{|\Im\varrho|<Q}\frac{s-s_\varepsilon(\varrho)}{s-\varrho}.$$

Since $\zeta_\varepsilon(s)$ has only simple zeros in $|t|<Q$, we get

(4.8) $$R_k(\varepsilon) \overset{\text{def}}{=} \sum_{|\Im\varrho|<Q} \operatorname*{Res}_{s=s_\varepsilon(\varrho)} e^{Ds+k(s+b)^2}\frac{1}{\zeta(s)} = \sum_{|\Im\varrho|<Q}\frac{\exp\big(Ds_\varepsilon(\varrho)+k(s_\varepsilon(\varrho)+b)^2\big)}{\zeta_\varepsilon'(s_\varepsilon(\varrho))}.$$

Noting that

(4.9)
$$R_k(\varepsilon) = \frac{1}{2\pi i}\int_C \frac{e^{Ds+k(s+b)^2}}{\zeta_\varepsilon(s)}ds,$$
$$R_k = \frac{1}{2\pi i}\int_C \frac{e^{Ds+k(s+b)^2}}{\zeta(s)}ds,$$

where C is the boundary of the rectangle with vertices at $\pm iQ$, $2\pm iQ$, and also that (compare [1], section 3) for $\varepsilon\to 0$

$$\frac{1}{\zeta_\varepsilon(s)} \rightrightarrows \frac{1}{\zeta(s)}, \quad s\in C,$$

we conclude

(4.10) $$\lim_{\varepsilon\to 0} R_k(\varepsilon) = R_k.$$

5. We shall choose our b-value by means of Lemma 1. The rôle of the a_ν's is played by $\frac{1}{\pi}\Im(\varrho)$-numbers, that of β_ν's by $\frac{1}{2\pi}\Im(\varrho^2)$-numbers. Setting then $U=2$, $\gamma=\frac{11}{10}$, $\Delta=1$, we see by Lemma 1 that there exists a b with

(5.1) $$\tfrac{1}{6}(\log T)^{1/3} \leqslant b \leqslant \tfrac{1}{6}(\log T)^{1/3}+1,$$

such that for all ϱ in $|t|<Q$

(5.2) $$\min_{\Omega\text{ integer}}\left|\frac{1}{2\pi}\Im(\varrho^2+2b\varrho)-\Omega\right| > \frac{c_6}{Q^{11/10}} > \frac{1}{\sqrt[5]{\log T}}.$$

We can also put (5.2) in the form

(5.3) $$|\operatorname{Arg} e^{\varrho^2+2b\varrho}| > \frac{2\pi}{\sqrt[5]{\log T}}.$$

Making $\varepsilon>0$ small enough, we deduce from (5.3)

(5.4) $$\left|\operatorname{Arg}\exp\big(s_\varepsilon^2(\varrho)+2bs_\varepsilon(\varrho)\big)\right| > (\log T)^{-1/5}.$$

Then we introduce

$$(5.5)\quad \begin{aligned} z_j &= z_j(\varepsilon) \stackrel{\text{def}}{=} \exp\big(s_\varepsilon^2(\varrho)+2bs_\varepsilon(\varrho)+\gamma_0^2-b-\tfrac{1}{4}\big),\\ b_j &= b_j(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\zeta_\varepsilon'(s_\varepsilon(\varrho))} e^{Ds_\varepsilon(\varrho)}, \quad j=1,2,\ldots,n, \end{aligned}$$

where $z_1, z_2, \ldots, z_n$ are arranged so as to have

$$(5.6)\quad (1=)\ |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|.$$

Using this notation, we rewrite (4.8) as

$$(5.7)\quad R_k(\varepsilon) = \exp\{k(b+\tfrac{1}{4}-\gamma_0^2)+kb^2\}\sum_{j=1}^{n} b_j(\varepsilon)z_j^k(\varepsilon).$$

We shall use Lemma 2 with

$$(5.8)\quad m = \left[\frac{1}{2b}\log(Te^{-D})\right]$$

and (see (5.4))

$$(5.9)\quad \varkappa = (\log T)^{-1/5}.$$

We note also that for the number n of terms in (5.7) we have the bound

$$(5.10)\quad n \leqslant c_7(\log T)^{1/6}(\log\log T).$$

Putting $h=2$, $h_1=3$, we find

$$(5.11)\quad z_1(\varepsilon) = e^{i\gamma_0(1+2b)} = \bar{z}_2(\varepsilon)$$

and

$$(5.12)\quad z_3(\varepsilon) = e^{\gamma_0^2-\gamma_1^2}e^{i\gamma_1(1+2b)}.$$

Hence, and by (3.1), (3.2), (5.1), (5.8), (5.9), (5.10), the conditions (2.5) and (2.6) are readily verified. As to the number A of (2.7), we have

$$A = A(\varepsilon) = \Re\big(b_1(\varepsilon)+b_2(\varepsilon)\big) = 2\Re\frac{e^{D\varrho_0}}{\zeta_\varepsilon'(\varrho_0)},$$

so that, owing to

$$\lim_{\varepsilon\to 0}\zeta_\varepsilon'(\varrho_0) = \zeta'(\varrho_0)$$

and (3.4), conclude

$$(5.13)\quad A(\varepsilon) > c_2\ (>0).$$

By Lemma 2, there exists an integer $k=k_\varepsilon$ satisfying

$$(5.14)\quad m+1 \leqslant k_\varepsilon \leqslant m+n(3+\pi/\varkappa)$$

and such that

$$\Re R_{k_\varepsilon}(\varepsilon) > \frac{A(\varepsilon)}{2n+1}2^{-m-n(3+\pi/\varkappa)}\left(\frac{n}{24(m+n(3+\pi/\varkappa))}\right)^{2n}\exp\{k_\varepsilon(b+\tfrac{1}{4}-\gamma_0^2)+k_\varepsilon b^2\}.$$

Using (5.13), (5.10), (5.8), (5.9), (5.1), (5.14) and (3.1), we obtain

$$(5.15)\quad e^{-k_\varepsilon b^2}\Re R_{k_\varepsilon}(\varepsilon) > T^{1/2}e^{-610\log^{2/3}T}.$$

Clearly, there exists an integer k,

$$(5.16)\quad m+1 \leqslant k \leqslant m+n(3+\pi/\varkappa),$$

and a sequence $\varepsilon\to 0$ for which $k_\varepsilon = k$. Letting *these* ε's tend to zero in (5.15) and making use of (4.10), we get

$$(5.17)\quad e^{-kb^2}\Re R_k \geqslant T^{1/2}e^{-610\log^{2/3}T}.$$

(5.17) obviously implies (3.9), whence by (4.2) and (3.12), (3.14), (3.15)

$$(5.18)\quad \sum_{U_1\leqslant n\leqslant U_2}\mu(n) > T^{1/2}e^{-(\log T)^{3/4}}$$

with certain U_1, U_2 satisfying

$$(5.19)\quad e^{2kb+D-3\sqrt{2k^2b+Dk}} \leqslant U_1 \leqslant U_2 \leqslant e^{2kb+D+3\sqrt{2k^2b+Dk}}.$$

The above inequalities combined with (5.1), (5.8) and (5.16) lead straight to (2.12), so that the part (2.12) and (2.13) of Theorem 1 is settled. The part (2.12)-(2.14) follows similarly on applying (2.8)-(2.10) of Lemma 2 together with the relation (3.16) and (3.17).

6. Theorem 2 is much simpler and its proof does not require the complicated machinery of the previous sections. It would be enough to use a two-sided lemma in place of the one-sided Lemma 2 and dispense with Lemma 1 altogether. However, we prefer to give a *quicker* rather than a *simpler* proof and this is in fact possible by appealing to some points of the former proof. By (3.13)

$$|I_k| \leqslant e^{kb^2}\frac{3\sqrt{D+2kb}}{4k\sqrt{\pi}}\int_{\xi_k a_k^{-1}}^{\xi_k a_k}\frac{|M(x)|}{x}dx + O(e^{kb^2}\cdot T^{1/3}),$$

further, by (5.17) and (4.2),

$$e^{-kb^2}|I_k| \geqslant T^{1/2}e^{-610\log^{2/3}T} - c_8,$$

whence (2.15) follows at once.

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Одномерное решето

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1. В вопросе об оценке числа почти простых чисел в довольно широком классе последовательностей важную роль играет метод эратосфенова решета.

Класс последовательностей к которым успешно применяется метод решета можно охарактеризовать следующим образом. Он состоит из последовательностей a_n „в среднем" равномерно распределенных в прогрессиях. Точнее, для этой последовательности должны существовать мультипликативная функция $\psi(D)$ и число γ такие, что для всех $\alpha \leqslant \gamma - \varepsilon$ и любых A и $\varepsilon > 0$

$$\sum_{D \leqslant N^\alpha} \mu^2(D) \max_{\substack{l \bmod D\\ l \in U(D)}} \left| \sum_{\substack{n=1\\ a_n \equiv l (\bmod D)}}^{N} 1 - \frac{N\psi(D)}{D} \right| = O\left(\frac{N}{\log^A N}\right), \tag{1}$$

где $\mu(D)$ — функция Мёбиуса, $U(D)$ — множество тех l для которых сравнению $a_n \equiv l(\bmod D)$ удовлетворяет бесконечно много n. Кроме того должно выполняться равенство

$$\sum_{p \leqslant x} \psi(p)\log p = rx + O(xe^{-a\sqrt{\log x}}), \tag{2}$$

где r — натуральное число, $a = \text{Const} > 0$. $\psi(D)$ может зависить от N, но эта зависимость должна быть такой чтобы равенство (2) было равномерным по N при $x \leqslant N^B$, где $B = \text{Const}$. Схема применения решета и характер оценок при этом не зависят от тонкой арифметической природы последовательности a_n и вполне определяются числом r. В связи с этим условимся называть решето *r-мерным*, если оно применяется к последовательности a_n для которой выполнены условия (1) и (2).

Другим важным вопросом в применениях решета является вопрос об определении почти простоты. Условимся называть число *k-почти простым* и обозначать его P_k, если оно содержит не более k простых множителей в том числе и одинаковых.