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Lattice points in a sphere

by

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1. Introduction. In this paper we consider the classical lattice point problem for the three-dimensional sphere. The problem can be described as follows. Let x be a positive real number and let k be a positive integer. Consider a k-dimensional sphere of radius $\sqrt{x}$ and center $(0, \ldots, 0)$. Following the notation of Walfisz ([4]), we let $A_k(x)$ be the number of integer lattice points in this sphere. A simple geometric argument shows that as $x \to +\infty$, $A_k(x) \sim V_k(x)$, where $V_k(x)$ is the volume of the sphere in question. The problem then is to get an asymptotic estimate of the difference $R_k(x) = A_k(x) - V_k(x)$.

Here we are considering only $R_3(x) = A_3(x) - \frac{4}{3}\pi x^{3/2}$. We obtain the following results:

$$R_3(x) = O(x^{3/4}\log x), \qquad x \to +\infty, \tag{1}$$

$$R_3(x) = \Omega(x^{1/2}\log\log x), \qquad x \to +\infty. \tag{2}$$

Of course (1) is not new. Vinogradov ([3]) has in fact shown that $R_3(x) = O(x^{\frac{19}{28}+\varepsilon})$, $\varepsilon > 0$, an upper estimate better than (1) [1]. However this result depends upon his difficult theory of exponential sums. Our estimate (1), on the other hand, is better than the elementary result $A_3(x) = O(x)$ and depends only upon a fairly standard application of the circle method.

As far as we can ascertain (2) is new. It is based upon the Ω-estimate for $R_4(x)$ ([4], p. 95)

$$R_4(x) = \Omega(x\log\log x), \qquad x \to +\infty. \tag{3}$$

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[1] Added in proof. Chen Ting-run (Chinese Mathematics 4(1963), pp. 322-339) claims the result $R_3(x) = O(x^{2/3})$.

Walfisz ([4], p. 94) gives only $R_3(x) = \Omega(x^{1/2})$, $x \to +\infty$. In [1] it is shown that $\overline{\lim\limits_{x\to\infty}} R_3(x)x^{-1/2} = \overline{\lim\limits_{x\to\infty}}\left(-R_3(x)x^{-1/2}\right) = +\infty$, but this of course yields a weaker Ω-result than (3).

2. Preliminaries. Landau's formula for $A_k(x)$ $(k \geqslant 4)$ is ([4], p. 29)

$$A_k(x) = \frac{\pi^{k/2}}{\Gamma(k/2)} \sum_{1\leqslant q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{S(h,q)}{q}\right)^k \sum_{1\leqslant n\leqslant x} n^{k/2-1}e^{-2\pi i n h/q} + O(x^{k/4}\log x), \quad x\to+\infty. \tag{4}$$

Here $S(h,q) = \sum\limits_{a(\mathrm{mod}\, q)} e^{2\pi i h a^2/q}$ is the famous Gaussian sum about which we need only the fact that

$$|S(h,q)| \leqslant Kq^{1/2}, \tag{5}$$

where K is independent of h and q ([4], p. 10). The notation Σ' indicates that we are to sum over only those h such that $(h,q) = 1$.

If (4) held for $k = 3$ we could apply it to derive (1) without much difficulty. However, since the proof of (4) given in [4] fails for $k < 4$, we replace it for $k = 3$ with the following formula obtainable by the same general method

$$A_3(x) = 2\pi \sum_{n\leqslant x} n^{1/2} + O(x^{3/4}\log x), \quad x\to+\infty. \tag{6}$$

Once we have (6), (1) is easily obtainable.

We will also need the following standard result ([4], p. 25).

LEMMA 1. (Euler Summation Formula). *Let* $\Psi(t) = t - [t] - \frac{1}{2}$. *If* $f(t)$ *has a continuous derivative in the interval* $a \leqslant t \leqslant b$ $(a < b)$, *then*

$$\sum_{a<m\leqslant b} f(m) = \int_a^b f(t)\,dt + \Psi(a)f(a) - \Psi(b)f(b) + \int_a^b \Psi(t)f'(t)\,dt. \tag{7}$$

This is proved by integrating $\int_a^b \Psi(t)f'(t)\,dt$ by parts.

3. Proof of (6) and (1). Many of the calculations done in the proof of (4) ([4], pp. 29-35) are valid for $k = 3$. In particular we have ([4], p. 33, formula (21))

(8)

$$A_3(x) = \sum_{q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{S(h,q)}{q}\right)^3 \int_{\theta(h,q)} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y + \frac{h}{q}\right)\right\} dy + O(x^{3/4}\log x), \quad x\to+\infty.$$

In (8), $w = x^{-1} - 2yi$, and $\theta(h,q)$ is an interval described as follows. Let h'/q' and h''/q'' be the two Farey fractions of order $x^{1/2}$ closest to h/q with say $h'/q' < h/q < h''/q''$, and consider the interval $\left[\frac{h'+h}{q'+q}, \frac{h+h''}{q+q''}\right]$. Then $\theta(h,q)$ is obtained from this interval by translating h/q to the origin, that is,

$$\theta(h,q) = \left[\frac{h'+h}{q'+q} - \frac{h}{q}, \frac{h+h''}{q+q''} - \frac{h}{q}\right].$$

For our purpose here the essential fact about $\theta(h,q)$ is ([4], p. 30)

$$\begin{aligned} |y| &\leqslant q^{-1}x^{-1/2}, && \text{for } y \in \theta(h,q), \\ |y| &\geqslant 2^{-1}q^{-1}x^{-1/2}, && \text{for } y \notin \theta(h,q), \end{aligned} \tag{9}$$

for any Farey fraction h/q of order $x^{1/2}$.

By (8) we have

$$A_3(x) = \int_{\theta(0,1)} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n y\right\} dy + \sum_{2\leqslant q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{S(h,q)}{q}\right)^3 \int_{\theta(h,q)} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y + \frac{h}{q}\right)\right\} dy + O(x^{3/4}\log x), \quad x\to+\infty. \tag{10}$$

Again we observe that the calculations of [4] (pp. 33-34) are valid for $k = 3$. These yield

$$\int_{\theta(0,1)} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n y\right\} dy = \int_{-\infty}^{\infty} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n y\right\} dy + O(x^{3/4}), \quad x\to+\infty.$$

Now

$$\int_{-\infty}^{\infty} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n y\right\} dy = \sum_{n\leqslant x} e^{\pi n/x} \int_{-\infty}^{\infty} w^{-3/2} e^{-2\pi i n y}\,dy,$$

and by [4], p. 35 (again valid for $k = 3$),

$$\int_{-\infty}^{\infty} w^{-3/2} e^{-2\pi i n y}\,dy = \frac{\pi^{3/2}}{\Gamma(3/2)} e^{-\pi n/x} n^{1/2} = 2\pi e^{-\pi n/x} n^{1/2}.$$

Thus, we have

$$\int_{-\infty}^{\infty} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n y\right\} dy = 2\pi \sum_{n\leqslant x} n^{1/2};$$

and (10) becomes

$$(11)\qquad A_3(x) = 2\pi \sum_{n\leqslant x} n^{1/2} + $$
$$+ \sum_{2\leqslant q\leqslant x^{1/2}} \sum_{h (\mathrm{mod}\, q)}' \left(\frac{S(h,q)}{q}\right)^3 \int_{0(h,q)} w^{-3/2} \sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y + \frac{h}{q}\right)\right\} dy +$$
$$+ O(x^{3/4}\log x), \qquad x\to+\infty.$$

Let Σ denote the multiple sum on the right hand side of (11); to prove (6) it is sufficient to show that $\Sigma = O(x^{3/4}\log x)$, as $x\to+\infty$.

By (5) and (9),

$$(12)\qquad \left|\sum\right|$$
$$\leqslant K \sum_{2\leqslant q\leqslant x^{1/2}} q^{-3/2} \sum_{h (\mathrm{mod}\, q)}' \int_{|y|\leqslant q^{-1}x^{-1/2}} |w|^{-3/2} \left|\sum_{n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y + \frac{h}{q}\right)\right\}\right| dy.$$

We apply the familiar method of partial summation to estimate the inner sum. Let

$$T(n) = \sum_{1\leqslant k\leqslant n} e^{-2\pi i k (y + h/q)}.$$

Then since $T(n)$ is a geometric series

$$|T(n)| \leqslant 2\left|e^{\pi i (y+h/q)} - e^{-\pi i (y+h/q)}\right|^{-1} = \left|\sin\pi\left(y+\frac{h}{q}\right)\right|^{-1}.$$

Since $|y|\leqslant q^{-1}x^{-1/2}$, $q^{-1}(h - x^{-1/2}) \leqslant y + h/q \leqslant q^{-1}(h + x^{-1/2})$, while $q\geqslant 2$ implies that $1\leqslant h\leqslant q-1$; thus if $x\geqslant 1$ (say), $0\leqslant y+h/q\leqslant 1$. Therefore

$$\left|\sin\pi\left(y+\frac{h}{q}\right)\right|^{-1} \leqslant \max\left\{\frac{1}{2(y+h/q)}, \frac{1}{2(1-y-h/q)}\right\}.$$

Also, $qy + h \geqslant h - x^{-1/2} \geqslant h - \frac{1}{2}$, and $q - qy - h \geqslant q - h - x^{-1/2} \geqslant q - h - \frac{1}{2}$, if $x\geqslant 4$. We conclude that

$$|T(n)| \leqslant q\left\{\frac{1}{2h-1} + \frac{1}{2q-2h-1}\right\} \leqslant q\left\{\frac{1}{h} + \frac{1}{q-h}\right\}.$$

Now,

$$\sum_{1\leqslant n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y+\frac{h}{q}\right)\right\} = \sum_{1\leqslant n\leqslant x} e^{\pi n/x}\{T(n) - T(n-1)\}$$
$$= \sum_{1\leqslant n\leqslant x} T(n)\{e^{\pi n/x} - e^{\pi(n+1)/x}\} + e^{\pi([x]+1)/x} T([x]),$$

and we have

$$\left|\sum_{1\leqslant n\leqslant x} \exp\left\{\frac{\pi n}{x} - 2\pi i n\left(y+\frac{h}{q}\right)\right\}\right|$$
$$\leqslant q\left\{\frac{1}{h} + \frac{1}{q-h}\right\} \sum_{1\leqslant n\leqslant x} \{e^{\pi(n+1)/x} - e^{\pi n/x}\} + q\left\{\frac{1}{h} + \frac{1}{q-h}\right\} e^{\pi([x]+1)/x}$$
$$\leqslant 2q\left\{\frac{1}{h} + \frac{1}{q-h}\right\} e^{\pi(x+1)/x} \leqslant K' q\left\{\frac{1}{h} + \frac{1}{q-h}\right\},$$

where K' is independent of h, q, and x. This, with (12), leads to

$$\sum = O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h (\mathrm{mod}\, q)}' \left\{\frac{1}{h} + \frac{1}{q-h}\right\} \int_0^{q^{-1}x^{-1/2}} |w|^{-3/2} dy\right), \qquad x\to+\infty.$$

But

$$|w|^{-3/2} = x^{3/2}(1+4x^2y^2)^{-3/4} \leqslant \min\{x^{3/2}, (2y)^{-3/2}\},$$

so that

$$\sum = O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h (\mathrm{mod}\, q)}' \left\{\frac{1}{h} + \frac{1}{q-h}\right\} \left\{\int_0^{x^{-1}} x^{3/2} dy + \int_{x^{-1}}^{q^{-1}x^{-1/2}} y^{-3/2} dy\right\}\right)$$
$$= O\left(\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h (\mathrm{mod}\, q)}' \left\{\frac{1}{h} + \frac{1}{q-h}\right\} x^{1/2}\right)$$
$$= O\left(x^{1/2} \sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2}\log q\right) = O(x^{3/4}\log x), \qquad \text{as} \qquad x\to+\infty,$$

and (6) is proved.

To obtain (1) we simply apply (7) to $\sum_{1\leqslant n\leqslant x} n^{1/2}$. This gives

$$\sum_{1\leqslant n\leqslant x} n^{1/2} = \int_0^x t^{1/2} dt - \Psi(x) x^{1/2} + \tfrac{1}{2}\int_0^x \Psi(t) t^{-1/2} dt$$
$$= \tfrac{2}{3} x^{3/2} + O(x^{1/2}), \qquad x\to+\infty.$$

Together with (6), this implies

$$A_3(x) = \tfrac{4}{3}\pi x^{3/2} + O(x^{3/4}\log x), \qquad x \to +\infty,$$

and the proof of (1) is complete.

4. Proof of (2). We begin with two lemmas (cf. [4], pp. 49-50).

LEMMA 2.

$$A_k(x) = \sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} A_{k-1}(x-m^2), \qquad \textit{for} \qquad k \geqslant 2.$$

Proof. Clear.

LEMMA 3.

$$\sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} (x-m^2)^{k/2} = \int_{-\sqrt{x}}^{\sqrt{x}} (x-t^2)^{k/2}dt + O(x^{(k-1)/2}), \qquad x \to +\infty.$$

Proof. By Lemma 1,

$$\sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} (x-m^2)^{k/2} = \sum_{-\sqrt{x}< m\leqslant\sqrt{x}} (x-m^2)^{k/2}$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} (x-t^2)^{k/2}dt - k\int_{-\sqrt{x}}^{\sqrt{x}} \Psi(t)(x-t^2)^{\frac{k}{2}-1}t\,dt.$$

But by the second mean value theorem of the integral calculus,

$$\int_{-\sqrt{x}}^{\sqrt{x}} \Psi(t)(x-t^2)^{\frac{k}{2}-1}t\,dt = O(x^{\frac{k}{2}-1+\frac{1}{2}}) = O(x^{\frac{k-1}{2}}), \qquad \text{as} \qquad x \to +\infty,$$

since $\int_{-\sqrt{x}}^{\sqrt{x}} \Psi(t)\,dt$ is bounded, independently of x.

To prove (2) we assume

(13) $$R_3(x) = o(x^{1/2}\log\log x), \qquad x \to +\infty,$$

and show that this leads to a contradiction. By Lemma 2, and the definition of $R_3(x)$,

$$A_4(x) = \sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} A_3(x-m^2) = \tfrac{4}{3}\pi \sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} (x-m^2)^{3/2} + \sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} R_3(x-m^2).$$

By (13), given any $\varepsilon > 0$ there exists $N > 3$ such that if $x > N$, then $|R_3(x)| < \varepsilon x^{1/2}\log\log x$. Also (13) implies that for any $x > 3$, $|R_3(x)| < Kx^{1/2}\log\log x$, where K is independent of x.

Therefore, assuming that $x > N$, we have

$$\Big|\sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} R_3(x-m^2)\Big| \leqslant \sum_{-\sqrt{x-N}<m<\sqrt{x-N}} |R_3(x-m^2)| + \sum_{\sqrt{x-N}\leqslant|m|\leqslant\sqrt{x}} |R_3(x-m^2)|$$

$$< 2\varepsilon(x-N)^{1/2}x^{1/2}\log\log x + \frac{KN}{(x-N)^{1/2}}x^{1/2}\log\log x + R_3(0)+R_3(1)+R_3(2),$$

where we have used the fact that $x^{1/2}\log\log x$ is monotone and observed that there are at most $N/(x-N)^{1/2}$ integers in the range $\sqrt{x-N} \leqslant |m| \leqslant \sqrt{x}$. Now holding N fixed and letting $x \to +\infty$, we have

$$\overline{\lim_{x\to+\infty}} \frac{\Big|\sum_{-\sqrt{x}<m<\sqrt{x}} R_3(x-m^2)\Big|}{x\log\log x} \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} R_3(x-m^2) = o(x\log\log x), \qquad \text{as} \qquad x \to +\infty,$$

so that

$$A_4(x) = \tfrac{4}{3}\pi \sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} (x-m^2)^{3/2} + o(x\log\log x), \qquad x \to +\infty.$$

Lemma 3, with $k = 3$, implies that

$$\sum_{-\sqrt{x}\leqslant m\leqslant\sqrt{x}} (x-m^2)^{3/2} = \tfrac{3}{8}\pi x^2 + O(x), \qquad x \to +\infty,$$

and we get

$$A_4(x) = \tfrac{1}{2}\pi x^2 + o(x\log\log x), \qquad x \to +\infty,$$

in contradiction to (3). Thus (13) is impossible, and the proof of (2) is complete.

Remarks. 1. The method used here is the derivation of a o-estimate for $R_4(x)$ from an assumed o-estimate for $R_3(x)$. Thus an improved Ω-estimate for $R_4(x)$ would immediately give an improvement on (2), by the same method.

2. This process can be applied to give an O-estimate for $R_3(x)$, given an O-estimate for $R_2(x)$. If we start with Vinogradov's result ([2])

$$R_2(x) = O(x^{\frac{17}{53}+\varepsilon}), \qquad \varepsilon > 0,\ x \to +\infty,$$

we get

$$R_3(x) = O(x^{\frac{87}{106}+\varepsilon}), \qquad \varepsilon > 0,\ x \to +\infty,$$

an estimate which is, however, weaker than (1).

Appendix added June 5, 1964.

Our formula (6) is actually equivalent to

$$A_3(x) = 2\pi \sum_{1\leqslant q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{S(h,q)}{q}\right)^3 \sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q} + O(x^{3/4}\log x), \quad x\to+\infty.$$

This of course is (4) for $k = 3$. In order to show this we need only prove that

$$(14)\qquad \sum_{2\leqslant q\leqslant x^{1/2}} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{S(h,q)}{q}\right)^3 \sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q} = O(x^{3/4}\log x), \quad x\to+\infty.$$

By partial summation,

$$\Big|\sum_{1\leqslant n\leqslant x} n^{1/2} e^{-2\pi i n h/q}\Big| \leqslant q\left(\frac{1}{h}+\frac{1}{q-h}\right)(1+[x])^{1/2}.$$

This together with (5) shows that the left hand side of (14) is

$$O\left(x^{1/2}\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2} \sum_{h(\mathrm{mod}\, q)}{}' \left(\frac{1}{h}+\frac{1}{q-h}\right)\right) = O\left(x^{1/2}\sum_{2\leqslant q\leqslant x^{1/2}} q^{-1/2}\log q\right) = O(x^{3/4}\log x), \quad \text{as} \quad x\to+\infty.$$

This proves (14) and hence (4) for the case $k = 3$.

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On oscillations of certain means formed from the Möbius series II

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1. As announced in paper [1], the present work contains some new results concerning the distribution of values of $\mu(n)$ in relatively short intervals $a\leqslant n\leqslant b$. Briefly and roughly speaking, it will be proved that on Riemann hypothesis there exist infinitely many intervals $[U_1, U_2]$, $U_2^{1-o(1)}\leqslant U_1\leqslant U_2$, $U_2\to\infty$, such that

$$\sum_{U_1\leqslant n\leqslant U_2}\mu(n) > U_2^{1/2-o(1)},$$

and also that there exists an infinity of similar intervals $[U_3, U_4]$ with

$$\sum_{U_3\leqslant n\leqslant U_4}\mu(n) < -U_4^{1/2-o(1)}.$$

This result is a particular case of the following Theorem 1. As a by-product of the proof of this theorem, we will obtain the inequality (again on Riemann hypothesis)

$$\int_{T^{1-o(1)}}^{T}\frac{|M(x)|}{x}\,dx > T^{1/2-o(1)},$$

($M(x)$ being, as usual, $\sum_{n\leqslant x}\mu(n)$), which improves on my previous result ([2]).

2. In the following we will use two lemmas. Their proofs can be found respectively in [4] (proof of Lemma II) and in [3] (proof of Theorem 4.1). We call them Lemma 1 and Lemma 2.

Lemma 1. *Let $\beta_1, \beta_2, \ldots$ be a real sequence and $a_1, a_2, \ldots$ a similar one with the property that*

$$(2.1)\qquad |a_\nu| \geqslant U \ (>0)$$