Lattice points in a sphere

by

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1. Introduction. In this paper we consider the classical lattice point problem for the three-dimensional sphere. The problem can be described as follows. Let \( x \) be a positive real number and let \( k \) be a positive integer. Consider a \( k \)-dimensional sphere of radius \( \sqrt{x} \) and center \( (0, \ldots, 0) \). Following the notation of Walfisz (\cite{4}), we let \( A_k(x) \) be the number of integer lattice points in this sphere. A simple geometric argument shows that as \( x \to +\infty \), \( A_k(x) \sim V_k(x) \), where \( V_k(x) \) is the volume of the sphere in question. The problem then is to get an asymptotic estimate of the difference \( R_k(x) = A_k(x) - V_k(x) \).

Here we are considering only \( R_k(x) = A_k(x) - \frac{4}{3} \pi x^{\frac{3}{2}} \). We obtain the following results:

\[
1. \quad R_k(x) = O(x^{\frac{3}{2}} \log x), \quad x \to +\infty,
\]

\[
2. \quad R_k(x) = \Omega(x^{\frac{3}{2}} \log x), \quad x \to +\infty.
\]

Of course (1) is not new. Vinogradov (\cite{3}) has in fact shown that \( R_k(x) = O(x^{\frac{3}{2}}), \quad x \to +\infty, \) an upper estimate better than (1)'s. However this result depends upon his difficult theory of exponential sums. Our estimate (1), on the other hand, is better than the elementary result \( A_k(x) = O(x) \) and depends only upon a fairly standard application of the circle method.

As far as we can ascertain (2) is new. It is based upon the \( \Omega \)-estimate for \( R_k(x) \) (\cite{4}, p. 95)

\[
3. \quad R_k(x) = \Omega(x \log x), \quad x \to +\infty.
\]

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\(^{(0)}\)Added in proof. Chen Ting-run (Chinese Mathematics 4 (1963), pp. 322-329) claims the result \( R_k(x) = O(x^{1/2}) \).
Walfisz ([4], p. 94) gives only $R_q(x) = \Omega(x^{1/2})$, $x \to +\infty$. In [1] it is shown that $\lim_{x \to \infty} R_q(x) = -1 - R_q(x) = +\infty$, but this of course yields a weaker $O$-result than (3).

2. Preliminaries. Landau’s formula for $A_k(x)$ (k ≥ 4) is ([4], p. 29)

$$A_k(x) = \frac{\pi^{k/2}}{\Gamma(k/2)} \sum_{\lambda \subseteq \mathbb{Z}/x} \sum_{\lambda \subseteq \mathbb{Z}/x} \left( \frac{S(\lambda, q)}{q} \right) \sum_{\alpha \subseteq \mathbb{Z}/q} \alpha^{\lambda \cdot \alpha q} \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$

Here $S(\lambda, q) = \sum_{\alpha \subseteq \mathbb{Z}/q} e^{2\pi i \lambda \cdot \alpha q}$ is the famous Gaussian sum about which we need only the fact that

$$|S(\lambda, q)| \leq K q^{1/2},$$

where $K$ is independent of $\lambda$ and $q$ ([4], p. 10). The notation $\mathcal{S}$ indicates that we are to sum over only those $\lambda$ such that $(\lambda, q) = 1$.

If (4) held for $k = 3$ we could apply it to derive (1) without much difficulty. However, since the proof of (4) in [4] fails for $k < 4$, we replace it for $k = 3$ with the following formula obtainable by the same general method

$$A_3(x) = 2\pi \sum_{\alpha \subseteq \mathbb{Z}/q} \alpha^{\lambda \cdot \alpha q} + O(x^{3/10} \log x), \quad x \to +\infty. $$

Once we have (6), (1) is easily obtainable.

We will also need the following standard result ([4], p. 25).

**Lemma 1.** (Euler Summation Formula). Let $\Psi(t) = t - \lfloor t \rfloor - 1/2$. If $f(t)$ has a continuous derivative in the interval $a < t < b$ ($a < b$), then

$$\sum_{a < n < b} f(n) = \int_a^b f(t) dt + \Psi(t) f(a) f(b) + \frac{1}{2} \Psi'(t) f'(t) dt. $$

This is proved by integrating $\frac{1}{2} \Psi'(t) f'(t) dt$ by parts.

3. Proof of (6) and (1). Many of the calculations done in the proof of (4) ([4], pp. 29–35) are valid for $k = 3$. In particular we have ([4], p. 33, formula (21))

$$A_3(x) = \sum_{\alpha \subseteq \mathbb{Z}/x} \sum_{\alpha \subseteq \mathbb{Z}/x} \left( \frac{S(\lambda, q)}{q} \right) \int_{\alpha \subseteq \mathbb{Z}/x} \exp \left( \frac{\alpha}{x} - 2\pi i a \frac{\alpha + \lambda}{q} \right) \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$

In (8), $\omega = x^{1/2} - 2y_1$, and $\theta(h, q)$ is an interval described as follows. Let $h'/q'$ and $h''/q''$ be the two Farey fractions of order $x^{1/2}$ closest to $h/q$ with say $h'/q' < h/q < h''/q''$, and consider the interval $[h' + h'' - (h' + h'') - h'/q', h' + h'' - (h' + h'') - h'/q'']$.

Then $\theta(h, q)$ is obtained from this interval by translating $h/q$ to the origin, that is,

$$\theta(h, q) = \left[ \frac{h' + h'' - h'}{q' + q'' - q} \right].$$

For our purpose here the essential fact about $\theta(h, q)$ is ([4], p. 30)

$$|y| \leq q^{-1/2}, \quad \text{for} \quad y \notin \theta(h, q),$$

$$|y| \geq 2q^{-1/2}, \quad \text{for} \quad y \notin \theta(h, q),$$

for any Farey fraction $h/q$ of order $x^{1/2}$.

By (8) we have

$$A_3(x) = \int_{\alpha \subseteq \mathbb{Z}/x} - \int_{\alpha \subseteq \mathbb{Z}/x} \exp \left( \frac{\alpha}{x} - 2\pi i a \frac{\alpha + \lambda}{q} \right) \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$

Again we observe that the calculations of [4] (pp. 33–34) are valid for $k = 3$. These yield

$$\int_{\alpha \subseteq \mathbb{Z}/x} - \int_{\alpha \subseteq \mathbb{Z}/x} \exp \left( \frac{\alpha}{x} - 2\pi i a \frac{\alpha + \lambda}{q} \right) \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$

Now

$$\int_{\alpha \subseteq \mathbb{Z}/x} - \int_{\alpha \subseteq \mathbb{Z}/x} \exp \left( \frac{\alpha}{x} - 2\pi i a \frac{\alpha + \lambda}{q} \right) \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$

and by [4], p. 35 (again valid for $k = 3$),

$$\int_{\alpha \subseteq \mathbb{Z}/x} - \int_{\alpha \subseteq \mathbb{Z}/x} \exp \left( \frac{\alpha}{x} - 2\pi i a \frac{\alpha + \lambda}{q} \right) \alpha + O(x^{3/10} \log x), \quad x \to +\infty. $$
Thus, we have
\[
\int_{-\infty}^{\infty} e^{-it\lambda} \sum_{n \in \mathbb{C}} \exp \left( \frac{t}{q} \left( y + \frac{1}{q} \right) \right) dy = 2\pi \sum_{n \in \mathbb{C}} n^{1/2},
\]
and (10) becomes
\[
(11) \quad A_1(x) = 2\pi \sum_{n \in \mathbb{C}} n^{1/2} + \sum_{2 \leq n \leq x} \sum_{k \in \mathbb{C}} \frac{2^{1/2}}{q} \int_{\mathbb{R}} e^{-it\lambda} \sum_{n \in \mathbb{C}} \exp \left( \frac{t}{q} \left( y + \frac{1}{q} \right) \right) dy + O(\lambda^{1/2} \log \lambda), \quad \lambda \to +\infty.
\]
Let \( \Xi \) denote the multiple sum on the right hand side of (11); to prove (6) it is sufficient to show that \( \Xi = O(\lambda^{1/2} \log \lambda) \), as \( x \to +\infty \).

By (5) and (9),
\[
(12) \quad \left| \sum_{n \in \mathbb{C}} n^{1/2} \right| \leq K \sum_{q \leq x^{1/2}} \sum_{y \leq x} \left| \sum_{n \mod q} \frac{1}{n - y} \right| = K \sum_{q \leq x^{1/2}} \sum_{y \leq x^{1/2}} \left| \sum_{n \mod q} \frac{1}{n - y} \right|.
\]
We apply the familiar method of partial summation to estimate the inner sum. Let
\[
T(n) = \sum_{n \in \mathbb{C}} n^{1/2}(q + 1/2) q^{1/2}.
\]
Then since \( T(n) \) is a geometric series
\[
|T(n)| \leq 2|e^{i(q + 1/2) \lambda} - e^{-i(q + 1/2) \lambda}| = |\sin \pi (y + \frac{h}{q})|^{1/2}.
\]
Since \( y \leq q^{1/2} \), \( q^{-1}(h - q + 1/2) \leq y + h/q \leq q^{-1}(h + q - 1/2) \), while \( q \geq 2 \) implies that \( 1 \leq h \leq q - 1 \); thus if \( x \geq 1 \) (say), \( 0 \leq y + h/q \leq 1 \). Therefore
\[
|\sin \pi (y + \frac{h}{q})|^{1/2} \leq \max \left\{ 1, \frac{1}{2(y + h/q)} \right\} \left( \frac{2}{1 - y + h/q} \right).
\]
Also, \( y + h/q \geq h - q^{1/2} \geq h - 1 \), and \( y - h/q \geq q - h - q^{1/2} \geq q - h - 1 \), if \( x \geq 2 \). We conclude that
\[
|T(n)| \leq \frac{1}{4} \left( \frac{1}{2h - 1} + \frac{1}{2y - 2h + 1} \right) \left( \frac{1}{h + \frac{1}{q - h}} \right).
\]
Now,
\[
\sum_{1 \leq n \leq x} \exp \left( \frac{tn}{x} - 2\pi \sin \left( y + \frac{1}{q} \right) \right) = \sum_{1 \leq n \leq x} e^{i\pi/n} (T(n) - T(n - 1))
\]
and we have
\[
\left| \sum_{1 \leq n \leq x} \exp \left( \frac{tn}{x} - 2\pi \sin \left( y + \frac{1}{q} \right) \right) \right| \leq \left( \frac{1}{h + \frac{1}{q - h}} \right) \sum_{1 \leq n \leq x} \left( e^{\pi x - h} - e^{-\pi x} \right) + \left( \frac{1}{h + \frac{1}{q - h}} \right) \sum_{1 \leq n \leq x} e^{\pi x - h} \left( e^{\pi x - h} - e^{-\pi x} \right) \left( h^{1/2} \log h \right), \quad h \to +\infty.
\]
where \( K' \) is independent of \( h, q, \) and \( x \). This, with (12), leads to
\[
\sum_{1 \leq n \leq x} \left( \sum_{1 \leq y \leq x} \left( \sum_{1 \leq n \leq x} \frac{1}{n - y} \right) \right) \left( h^{1/2} \log h \right), \quad x \to +\infty.
\]
But
\[
\left| n \right|^{1/2} \leq \left( 1 + 4x^2 \right)^{-1/2} \leq \min \left( (a^{1/2}), (d^{1/2}) \right),
\]
so that
\[
\sum_{1 \leq n \leq x} \left( \sum_{1 \leq y \leq x} \left( \sum_{1 \leq n \leq x} \frac{1}{n - y} \right) \right) \left( h^{1/2} \log h \right) = \sum_{1 \leq n \leq x} \left( \sum_{1 \leq y \leq x} \left( \sum_{1 \leq n \leq x} \frac{1}{n - y} \right) \right) \left( h^{1/2} \log h \right) = \sum_{1 \leq n \leq x} \left( \sum_{1 \leq y \leq x} \left( \sum_{1 \leq n \leq x} \frac{1}{n - y} \right) \right) \left( h^{1/2} \log h \right) = \sum_{1 \leq n \leq x} \left( \sum_{1 \leq n \leq x} \frac{1}{n - y} \right) \left( \log h \right) = O(h^{1/2} \log h), \quad x \to +\infty.
\]
and (9) is proved.

To obtain (1) we simply apply (7) to \( \sum_{1 \leq n \leq x} n^{1/2} \). This gives
\[
\sum_{1 \leq n \leq x} n^{1/2} = \int_{0}^{x} t^{1/2} dt - W(x) x^{1/2} + \frac{3}{2} x^{1/2} \log x = \frac{3}{4} x^{1/2} + O(x^{1/2}), \quad x \to +\infty.
\]
Together with (6), this implies

\[ A_4(x) = \frac{4}{3} \pi x^3 + O(x^{3+\varepsilon}), \quad x \to +\infty, \]

and the proof of (1) is complete.

4. Proof of (2). We begin with two lemmas (cf. [4], pp. 49-50).

Lemma 2.

\[ A_k(x) = \sum_{-\sqrt{x}} \sum_{m \in \mathbb{Z}^k, m \neq 0} \frac{x - m^k}{x - m^k}, \quad k \geq 2. \]

Proof. Clear.

Lemma 3.

\[ \sum_{-\sqrt{x}} (x - m^k)^{k/2} = \int_{-\sqrt{x}}^x (x - t^{k/2}) dt + O(x^{\varepsilon}), \quad x \to +\infty. \]

Proof. By Lemma 1,

\[ \sum_{-\sqrt{x}} (x - m^k)^{k/2} = \sum_{-\sqrt{x}} (x - m^k)^{k/2} = \int_{-\sqrt{x}}^x (x - t^{k/2}) dt - k \int_{-\sqrt{x}}^x \Psi(t)(x - t^{k/2})^{-1} dt. \]

But by the second mean value theorem of the integral calculus,

\[ \int_{-\sqrt{x}}^x \Psi(t)(x - t^{k/2})^{-1} dt = O(x^{\varepsilon}), \quad x \to +\infty, \]

since \( \int_{-\sqrt{x}}^x \Psi(t) dt \) is bounded, independently of \( x \).

To prove (2) we assume

\[ R_k(x) = o(x^{3/2} \log \log x), \quad x \to +\infty, \]

and show that this leads to a contradiction. By Lemma 2, and the definition of \( R_k(x) \),

\[ A_k(x) = \sum_{-\sqrt{x}} \sum_{m \in \mathbb{Z}^k, m \neq 0} \frac{x - m^k}{x - m^k} + \sum_{-\sqrt{x}} \sum_{m \in \mathbb{Z}^k, m \neq 0} R_k(x - m^k). \]

By (13), given any \( \varepsilon > 0 \) there exists \( N > 3 \) such that if \( x > N \), then \( |R_k(x)| < c x^{3/2} \log \log x \). Also (13) implies that for any \( x > 3 \), \( |R_k(x)| < K x^{3/2} \log \log x \), where \( K \) is independent of \( x \).

Therefore, assuming that \( x > N \), we have

\[ \left| \sum_{-\sqrt{x}} R_k(x - m^k) \right| \leq \sum_{-\sqrt{x}} |R_k(x - m^k)| + \sum_{-\sqrt{x}} |R_k(x - m^k)| \]

\[ < 2x \sum_{-\sqrt{x}} |m^k|^{3/2} \log \log x + \frac{K x^{3/2} \log \log x + 1}{\left( x - N \right)^{3/2} \log \log x} + R_k(0) + R_k(1) + R_k(2), \]

where we have used the fact that \( |m^k|^{3/2} \log \log x \) is monotone and observed that there are at most \( N \left( x - N \right)^{3/2} \) integers in the range \( \sqrt{x} - N < |m| < \sqrt{x} \). Now holding \( N \) fixed and letting \( x \to +\infty \), we have

\[ \lim_{x \to +\infty} \frac{\sum_{-\sqrt{x}} R_k(x - m^k)}{\log x} \leq 2x. \]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that

\[ \sum_{-\sqrt{x}} R_k(x - m^k) = o(x^{3/2} \log \log x), \quad x \to +\infty, \]

so that

\[ A_k(x) = \frac{4}{3} \pi x^3 + o(x^{3/2} \log \log x), \quad x \to +\infty. \]

Lemma 3, with \( k = 3 \), implies that

\[ \sum_{-\sqrt{x}} (x - m^3)^{3/2} = \pi \frac{1}{4} x^2 + o(x), \quad x \to +\infty, \]

and we get

\[ A_3(x) = \pi \frac{1}{4} x^2 + o(x), \quad x \to +\infty, \]

in contradiction to (3). Thus (13) is impossible, and the proof of (2) is complete.

Remarks. 1. The method used here is the derivation of a \( o \)-estimate for \( E_k(x) \) from an assumed \( o \)-estimate for \( R_k(x) \). Thus an improved \( O \)-estimate for \( E_k(x) \) would immediately give an improvement on (2), by the same method.

2. This process can be applied to give an \( O \)-estimate for \( E_k(x) \), given an \( O \)-estimate for \( R_k(x) \). If we start with Vinogradov’s result (22)

\[ E_k(x) = O(x^{3/2} \log \log x), \quad x \to +\infty, \]

we get

\[ B_k(x) = O(x^{3/2 + \varepsilon}), \quad x \to +\infty, \]

an estimate which is, however, weaker than (1).
Our formula (6) is actually equivalent to
\[
A_{x}(\varrho) = 2\pi \sum_{1 \leq g \leq x} \sum_{1 \leq q \leq \varrho \left( \varrho, g \right)} \left( \frac{S(b, q)}{q} \right)^{3} \sum_{1 \leq r \leq \varrho} n^{1/2} e^{-2\pi i n b r} + O(x^{3/2} \log x), \quad x \to +\infty.
\]
This of course is (4) for \( k = 3 \). In order to show this we need only prove that
\[
\sum_{1 \leq g \leq x} \sum_{1 \leq q \leq \varrho \left( \varrho, g \right)} \left( \frac{S(b, q)}{q} \right)^{3} \sum_{1 \leq r \leq \varrho} n^{1/2} e^{-2\pi i n b r} = O(x^{3/2} \log x), \quad x \to +\infty.
\]
By partial summation,
\[
\left| \sum_{1 \leq r \leq \varrho} n^{1/2} e^{-2\pi i n b r} \right| \leq \varrho \left( 1 + \frac{1}{q} \right)^{1/2}.
\]
This together with (5) shows that the left hand side of (14) is
\[
O \left( x^{1/2} \sum_{1 \leq g \leq x} \varrho^{1/3} \sum_{1 \leq q \leq \varrho \left( \varrho, g \right)} \left( 1 + \frac{1}{q} \right)^{1/2} \right) = O \left( x^{1/2} \sum_{1 \leq g \leq x} \varrho^{1/3} \log \varrho \right) = O \left( x^{3/2} \log x \right), \quad x \to +\infty.
\]
This proves (14) and hence (4) for the case \( k = 3 \).

References


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On oscillations of certain means formed from the Möbius series II

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1. As announced in paper [1], the present work contains some new results concerning the distribution of values of \( \mu(n) \) in relatively short intervals \( a < n < b \). Briefly and roughly speaking, it will be proved that on Riemann hypothesis there exist infinitely many intervals \([U_{1}, U_{2}]\), \( U_{1}^{1/3} \to 0 \to U_{1} \to U_{2} \to \infty \), such that
\[
\sum_{U_{2}^{1/3} < n \leq U_{1}} \mu(n) > U_{1}^{1/3 - \epsilon(n)},
\]
and also that there exists an infinity of similar intervals \([U_{3}, U_{4}]\) with
\[
\sum_{U_{2}^{1/3} < n \leq U_{1}} \mu(n) < -U_{1}^{1/3 - \epsilon(n)}.
\]
This result is a particular case of the following Theorem 1. As a by-product of the proof of this theorem, we will obtain the inequality (again on Riemann hypothesis)
\[
\int_{2}^{T} \frac{|M(x)|^2}{x} \, dx > T^{1/3 - \epsilon(1)},
\]
(\( M(x) \) being, as usual, \( \sum_{n \leq x} \mu(n) \)), which improves on my previous result ([2]).

2. In the following we will use two lemmas. Their proofs can be found respectively in [4] (proof of Lemma II) and in [3] (proof of Theorem 4.1). We call them Lemma 1 and Lemma 2.

Lemma 1. Let \( \beta_{1}, \beta_{2}, \ldots \) be a real sequence and \( a_{1}, a_{2}, \ldots \) an analogous one with the property that
\[
|a_{n}| > U \quad (> 0)
\]