[7] E. Landau, *Vorlesungen über Zahlentheorie II*, Leipzig 1927.
[8] K. Prachar, *Primzahlverteilung*, Berlin 1957.
[9] A. Selberg, *On an elementary method in the theory of primes*, Norske Videnskabers Selskab Forhandlinger XIX, N 18 (1946), pp. 64-67.
[10] P. Turán, *On a density theorem of Yu. V. Linnik*, Publications of the Mathem. Institute of the Hungarian Academy of Sci, VIA (1961), pp. 165-179.
[11] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, London and New York 1954.

Reçu par la Rédaction le 13. 1. 1964

Errata to the part I of this paper (Acta Arithm. 10(1964), pp. 137-182),

p. 165[3]: read J_χ instead of y_χ,

p. 172[3]: read $\vartheta\sum$ instead of $e\sum$,

p. 177$_3$: read $4(g-\delta')k\pi i$ instead of $4(g-\delta')gk\pi i$.

ACTA ARITHMETICA
X (1965)

Some remarks on a series of Ramanujan

by

W. Staś (Poznań)

1. In my previous papers [7], [8] I was concerned with the Ramanujan series

$$S(\beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-(\beta/n)^2} \tag{1.1}$$

where $\mu(n)$ was the function of Möbius and β a real parameter.

G. H. Hardy and J. E. Littlewood have proved (see [1]) that

$$S(\beta) = O(\beta^{-\frac{1}{2}}), \quad \beta \to \infty, \tag{1.2}$$

is equivalent to the conjecture of Riemann.

At present we shall prove by Turán's methods the following theorem, which is stronger than my previous result (see [8]), based on Riemann's hypothesis and on the conjecture that the ζ-function has only simple zeros.

THEOREM. *Suppose Riemann's conjecture. Then for $T > C$*

$$\max_{T^{1-o(1)} \leqslant \beta \leqslant T} |S(\beta)| \geqslant T^{-\frac{1}{2}-o(1)}. \tag{1.3}$$

In the proof we shall apply the method of Turán, namely we shall use the following modification ([2]) of Turán's Satz X ([11]):

LEMMA 1. *Suppose that $m \geqslant 0$, $z_1, z_2, \ldots, z_N$ are complex numbers with*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_{h_1}| \geqslant \ldots \geqslant |z_N| \tag{1.4}$$

and

$$|z_h| > 2\frac{N}{N+m}, \quad |z_{h_1}| < |z_h| - \frac{N}{m+N}. \tag{1.5}$$

Then there exists an integer μ with

$$m \leqslant \mu \leqslant m+N \tag{1.6}$$

such that

$$(1.7)\qquad \frac{|b_1z_1^{\mu}+b_2z_2^{\mu}+\ldots+b_Nz_N^{\mu}|}{(\frac{1}{2}|z_h|)^{\mu}} \geqslant \min_{h\leqslant j<h_1}|b_1+b_2+\ldots+b_j|\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^{N}.$$

2. Before we turn to the proof we shall list some known properties of the functions $\Gamma(s)$ and $\zeta(s)$ which will be used in the following:

There is a constant $c_1\geqslant 2$ such that each interval $(\Omega,\Omega+1)$ contains a value of $t=t(\Omega)$ for which

$$(2.1)\qquad |\zeta(\sigma+it)|>t^{-c_1+1},\qquad -1\leqslant\sigma\leqslant 2,\ t=t(\Omega)$$

([9], *Theorem* 9.7).

With a $1\leqslant k\leqslant\sqrt{1+t^2}$, $0<\sigma<1$,

$$(2.2)\qquad |\Gamma(\sigma+it)|=\frac{k\Gamma(1+\sigma)}{\sqrt{\sigma^2+t^2}}\sqrt{\frac{2\pi t}{e^{\pi t}-e^{-\pi t}}}.$$

Now let $N(\tau)$ stand for the number of zeros of $\zeta(s)$ in $0<\sigma<1$, $0<t\leqslant\tau$, $\tau\geqslant 2$.

We have

$$(2.3)\qquad N(\tau)<c_2\tau\log\tau.$$

The function of Riemann satisfies the functional equation

$$(2.4)\qquad \Gamma(s)\zeta(2s)=\pi^{-\frac{1}{2}+2s}\Gamma(\tfrac{1}{2}-s)\zeta(1-2s).$$

Suppose that η is a constant, $0<\eta<2$. Then by Riemann's hypothesis

$$(2.5)\qquad |\zeta(\tfrac{1}{2}+\eta+it)|^{-1}\leqslant t^{c_3/(\eta\log^{\eta/3}t)},\qquad t\geqslant 11$$

([5], p. 164).

3. We turn to the proof of the theorem. Let us write

$$(3.1)\qquad \eta=(\log\log T)^{-1},$$

$$(3.2)\qquad \omega=2\log T.$$

Integer ν will be supposed to satisfy the inequality

$$(3.3)\qquad \varphi_1(T)\overset{\text{def}}{=}(1-\eta)\frac{\log T}{\log\log T}\leqslant\nu\leqslant\frac{\log T}{\log\log T}\overset{\text{def}}{=}\varphi_2(T).$$

Further, put

$$(3.4)\qquad l=\frac{\log T}{(\log\log T)^4},$$

$$(3.5)\qquad \omega=B\nu+c_0,$$

where c_0 satisfies at the moment only

$$(3.6)\qquad 1\leqslant c_0\leqslant 2.$$

Owing to (3.2) we then have from (3.5)

$$(3.7)\qquad B=\frac{2\log T-c_0}{\nu}.$$

We start from the formula

$$(3.8)\qquad \beta S(\beta)=\frac{1}{2\pi i}\int_{(\frac{1}{2}+\eta)}\beta^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds,$$

which follows by (1.1) and by the well-known formula of Cahen and Mellin

$$(3.9)\qquad \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\Gamma(s)y^{-s}ds=e^{-y}\qquad (x>0,\ \mathrm{re}\,y>0).$$

Substituting

$$(3.10)\qquad \beta=e^{\omega_1/2}$$

we integrate (3.8) ν times from 0 to $\omega_2,\omega_3,\ldots,\omega_\nu,\omega$, respectively. On the one hand, we then have

$$(3.11)\qquad I_\omega=\int_0^{\omega}\int_0^{\omega_\nu}\ldots\int_0^{\omega_3}\int_0^{\omega_2}\big(e^{\omega_1/2}S(e^{\omega_1/2})\big)\,d\omega_1d\omega_2\ldots d\omega_\nu,$$

and this gives clearly

$$(3.12)\qquad |I_\omega|\leqslant\frac{\omega^{\nu}}{\nu!}\max_{0\leqslant\omega_1\leqslant\omega}|e^{\omega_1/2}S(e^{\omega_1/2})|.$$

On the other hand,

$$(3.13)\qquad I_\omega=\frac{1}{2\pi i}\int_{(\frac{1}{2}+\eta)}\frac{e^{\omega s}\Gamma(\frac{1}{2}-s)}{s^{\nu}\zeta(2s)}\,ds-\frac{1}{2\pi i}\sum_{j=1}^{\nu}\frac{\omega^{\nu-j}}{(\nu-j)!}\int_{(\frac{1}{2}+\eta)}\frac{\Gamma(\frac{1}{2}-s)}{s^{j}\zeta(2s)}\,ds.$$

Further we have

$$(3.14)\qquad \frac{1}{2\pi i}\int_{(\frac{1}{2}+\eta)}\frac{\Gamma(\frac{1}{2}-s)}{s^{j}\zeta(2s)}\,ds=O(\log\log T)$$

and

$$(3.15)\qquad \frac{1}{2\pi i}\int_{(\frac{1}{2}+\eta)}\frac{e^{\omega s}\Gamma(\frac{1}{2}-s)}{s^{\nu}\zeta(2s)}\,ds=\int_{\frac{1}{2}+\eta-il}^{\frac{1}{2}+\eta+il}(\ldots)\,ds+O\left(\frac{e^{\omega(\frac{1}{2}+\eta)}}{l^{\nu}}\right),$$

where l is to be chosen as in (3.4).

We apply Cauchy's theorem of residues to the right integral:

$$(3.16)\qquad \frac{1}{2\pi i}\int\limits_{\frac12+\eta-il}^{\frac12+\eta+il}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}\,ds$$

$$=\frac{1}{2\pi i}\int\limits_{\frac14-\eta-il}^{\frac14-\eta+il}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}\,ds+\sum_{|\Im\varrho|<l}\operatorname*{Res}_{s=\varrho/2}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}+O\left(\frac{e^{(\frac12+\varepsilon)\omega}}{l^\nu}\right).$$

But again

$$(3.17)\qquad \frac{1}{2\pi i}\int\limits_{\frac14-\eta-il}^{\frac14-\eta+il}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}\,ds=O\big(e^{(\frac14-\eta)\omega}\big).$$

Owing to (3.12)-(3.17) we have the inequality

$$(3.18)\qquad \frac{\omega^\nu}{\nu!}\max_{0\leqslant\omega_1\leqslant\omega}|e^{\omega_1/2}S(e^{\omega_1/2})|+c_4\frac{\nu}{\eta}\left(\frac{\omega e}{\nu}\right)^\nu+c_5\frac{e^{\omega(\frac12+\eta)}}{l^\nu}+c_6e^{(\frac14-\eta)\omega}$$

$$\geqslant\left|\sum_{|\Im\varrho|<l}\operatorname*{Res}_{s=\varrho/2}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}\right|.$$

4. Let us write further

$$(4.1)\qquad F_\nu(s)\overset{\text{def}}{=}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta(2s)}$$

and

$$(4.2)\qquad L_\nu\overset{\text{def}}{=}\sum_{|\Im\varrho|<l}\operatorname*{Res}_{s=\varrho/2}F_\nu(s).$$

We shall prove the following

Lemma 2. *By Riemann's conjecture and for* $T>c_4$

$$(4.3)\qquad \max_{\varphi_1(T)\leqslant\nu\leqslant\varphi_2(T)}|L_\nu|>\sqrt{T}e^{-3\frac{\log T\log\log\log T}{\log\log T}},$$

by $\varphi_1(T),\varphi_2(T)$, *from* (3.3).

I am using in the proof of the lemma an interesting idea of S. Knapowski, the so-called "shifted zeros" (see [4]). This idea seems very useful from the point of view of some applications connected with the function $1/\zeta(s)$.

Let $\varrho_j=\frac12+i\gamma_j$, $j=1,2,3,\ldots,r$ run through the set of ζ-zeros in $0<\sigma<1$, $0<t<l$, so that

$$0<\gamma_1<\gamma_2<\ldots<\gamma_r<l,$$

the possible multiple zeros being, however, counted only once.

Suppose that $\varepsilon>0$ satisfies the inequalities

$$(4.4)\qquad \varepsilon<\min_{1\leqslant j\leqslant r-1}(\gamma_{j+1}-\gamma_j),\qquad \varepsilon<l-\gamma_r.$$

If the order of multiplicity of ϱ_j is, for instance, k, we define k "shifted zeros" corresponding to ϱ_j:

$$(4.5)\qquad \varrho_j^{(1)}=\varrho_j,\quad \varrho_j^{(2)}=\varrho_j+i\frac{\varepsilon}{k},\quad\ldots,\quad \varrho_j^{(k)}=\varrho_j+i\frac{k-1}{k}\varepsilon.$$

In this way we get a set E of "shifted zeros", so that to each ϱ with $|\Im\varrho|<l$ there corresponds an ϱ_ε. This definition implies in particular that

$$(4.6)\qquad |\varrho-\varrho_\varepsilon|<\varepsilon.$$

Now we introduce

$$(4.7)\qquad \zeta_\varepsilon(2s)\overset{\text{def}}{=}\zeta(2s)\prod_{|\Im\varrho|<l}\frac{2s-\varrho_\varepsilon}{2s-\varrho}.$$

It is easy to see by (4.4) and (4.5) that $\zeta_\varepsilon(2s)$ has only simple zeros (namely those at $\frac12\varrho_\varepsilon$'s) in the rectangle $0<\sigma<1$, $|t|<l$.

Writing

$$(4.8)\qquad F_\nu^{(\varepsilon)}(s)\overset{\text{def}}{=}\frac{e^{\omega s}\Gamma(\frac12-s)}{s^\nu\zeta_\varepsilon(2s)}$$

and

$$(4.9)\qquad L_\nu^{(\varepsilon)}\overset{\text{def}}{=}\sum_{|\Im\varrho|<l}\operatorname*{Res}_{s=\frac12\varrho_\varepsilon}F_\nu^{(\varepsilon)}(s)$$

we see at once that

$$(4.10)\qquad L_\nu^{(\varepsilon)}=2^{\nu-1}\sum_{|\Im\varrho|<l}\frac{e^{\frac12\omega\varrho_\varepsilon}\Gamma(\frac12-\frac12\varrho_\varepsilon)}{\varrho_\varepsilon^\nu\zeta_\varepsilon'(\varrho_\varepsilon)}.$$

But let us note that by (4.2) and (4.10)

$$(4.11)\qquad L_\nu=\frac{1}{2\pi i}\int\limits_C F_\nu(s)\,ds,\qquad L_\nu^{(\varepsilon)}=\frac{1}{2\pi i}\int\limits_C F_\nu^{(\varepsilon)}(s)\,ds,$$

where the contour of integration C consists of

$$0\leqslant\sigma\leqslant\tfrac32,\quad t=\pm l;\qquad \sigma=\tfrac32,\quad |t|<l,$$

$$\sigma=0,\quad \tfrac14\leqslant|t|\leqslant l;\qquad \sigma^2+t^2=\tfrac1{16},\quad |t|\leqslant\tfrac14.$$

One can show (see [4], section 3) that

$$\lim_{\varepsilon\to\infty} L_\nu^{(\varepsilon)} = L_\nu. \tag{4.12}$$

In order to prove (4.3) we shall first find by Turán's methods a lower estimate of $|L_\nu^{(\varepsilon)}|$ and then, after passing to the limit as $\varepsilon\to 0$, we shall get (4.3) at once.

5. In view of (3.5) and (3.6), putting

$$\beta_0 = e^{\frac{1}{2}c_0}, \qquad \sqrt{e}\leqslant\beta_0\leqslant e, \tag{5.1}$$

we have from (4.10)

$$L_\nu^{(\varepsilon)} = 2^{\nu-1}\sum_{|\Im\varrho|<l}\beta_0^{\varrho_\varepsilon}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\varrho_\varepsilon)}{\zeta_\varepsilon'(\varrho_\varepsilon)}\left(\frac{e^{\frac{1}{2}B\varrho_\varepsilon}}{\varrho_\varepsilon}\right)^\nu. \tag{5.2}$$

Let us denote by $\varrho_\varepsilon^{(0)}$ that zero at which

$$\left|\frac{e^{\frac{1}{2}B\varrho_\varepsilon}}{\varrho_\varepsilon}\right|, \qquad |\Im\varrho|<l \tag{5.3}$$

attains its maximum.

We put (5.2) in the form

$$L_\nu^{(\varepsilon)} = 2^{\nu-1}\left|\frac{e^{\frac{1}{2}B\varrho_\varepsilon^{(0)}}}{\varrho_\varepsilon^{(0)}}\right|^\nu\left|\sum_{|\Im\varrho|<l}\beta_0^{\varrho_\varepsilon}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\varrho_\varepsilon)}{\zeta_\varepsilon'(\varrho_\varepsilon)}\left(\frac{e^{\frac{1}{2}B(\varrho_\varepsilon-\varrho_\varepsilon^{(0)})}}{\varrho_\varepsilon/\varrho_\varepsilon^{(0)}}\right)^\nu\right| \tag{5.4}$$

and define

$$z_j = \frac{e^{\frac{1}{2}B(\varrho_\varepsilon-\varrho_\varepsilon^{(0)})}}{\varrho_\varepsilon/\varrho_\varepsilon^{(0)}}, \qquad b_j = \beta_0^{\varrho_\varepsilon}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\varrho_\varepsilon)}{\zeta_\varepsilon'(\varrho_\varepsilon)}, \tag{5.5}$$

arranging them as required in lemma (1.4)-(1.7).

Using (2.3) and (3.4) we can choose

$$N = c_2\frac{\log T}{(\log\log T)^3}. \tag{5.6}$$

Putting $\Omega = \Omega_T = (\log\log T)^\lambda$ with $\lambda = 1/(2c_1+3)$ in (2.1), we define

$$\chi(T) \overset{\text{def}}{=} t(\Omega_T).$$

Then obviously

$$\chi_1(T) \overset{\text{def}}{=} (\log\log T)^\lambda < \chi(T) < (\log\log T)^\lambda + 1 = \chi_1(T)+1 \tag{5.7}$$

and by (2.1)

$$|\zeta(\sigma+i\chi(T))| > \frac{1}{(\chi(T))^{c_1-1}}, \qquad -1\leqslant\sigma\leqslant 2. \tag{5.8}$$

Let

$$z_h = e^{\frac{1}{2}B(\varrho_\varepsilon^{(h)}-\varrho_\varepsilon^{(0)})}\frac{\varrho_\varepsilon^{(0)}}{\varrho_\varepsilon^{(h)}} \tag{5.9}$$

denote that one of our z_j's corresponding to ϱ's with $|\Im\varrho|<\chi(T)$ which has the maximal index $j=h$.

Further, let

$$z_{h_1} = e^{\frac{1}{2}B(\varrho_\varepsilon^{(h_1)}-\varrho_\varepsilon^{(0)})}\frac{\varrho_\varepsilon^{(0)}}{\varrho_\varepsilon^{(h_1)}} \tag{5.10}$$

be any of z_j's with $|\Im\varrho|>\chi(T)$.

Writing

$$\varrho_\varepsilon^{(h)} = \tfrac{1}{2}+i\gamma_\varepsilon^{(h)}, \qquad \varrho_\varepsilon^{(h_1)} = \tfrac{1}{2}+i\gamma_\varepsilon^{(h_1)} \tag{5.10}$$

it is easy to show, using (5.8), (5.7) (see [8], p. 267) that

$$|\gamma_\varepsilon^{(h_1)}|-|\gamma_\varepsilon^{(h)}| > \frac{1}{(\chi_1(T))^{2c_1}}. \tag{5.11}$$

Choosing

$$m = (1-\eta)\frac{\log T}{\log\log T}, \qquad \eta = \frac{1}{\log\log T}, \tag{5.12}$$

N as in (5.6), and using (5.11), (5.6) we can prove that conditions (1.5) are satisfied (see [8], pp. 268, 269).

6. We can now apply the modified lemma of Turán (1.4)-(1.7), with $h_1 = h+1$.

We then get with a ν satisfying (3.3)

$$|L_\nu^{(\varepsilon)}| \geqslant 2^{\nu-1}\left|\frac{e^{\frac{1}{2}B\varrho_\varepsilon^{(0)}}}{\varrho_\varepsilon^{(0)}}\right|^\nu(\tfrac{1}{2}|z_h|)^\nu\left|\sum_{j=1}^{h}b_j\right|\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N. \tag{6.1}$$

But owing to (5.5) and the definition of h in (5.9) we have

$$\left|\sum_{j=1}^{h}b_j\right| = \left|\sum_{|\Im\varrho|<X(T)}\beta_0^{\varrho_\varepsilon}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\varrho_\varepsilon)}{\zeta_\varepsilon'(\varrho_\varepsilon)}\right| \overset{\text{def}}{=} |P(\varepsilon)|. \tag{6.2}$$

In view of (3.1), (3.3), (3.7), (5.6), (5.9), (5.12) and (6.2) we have from (6.1)

$$(6.3)\qquad |L_\nu^{(\varepsilon)}| \geqslant \frac{1}{2}\,\frac{e^{\frac{1}{4}B_\nu}}{|\varrho_\varepsilon^{(h)}|^\nu}\, e^{-\frac{\log T\log\log\log T}{\log\log T}}|P(\varepsilon)|$$

$$\geqslant \frac{1}{2}\,\frac{e^{\frac{1}{4}(2\log T-c_0)}}{\left(\sqrt{\frac{1}{4}+(\chi_1+1)^2}\right)^\nu}\, e^{-\frac{\log T\log\log\log T}{\log\log T}}|P(\varepsilon)|$$

$$\geqslant \sqrt{T}\, e^{-2\frac{\log T\log\log\log T}{\log\log T}}|P(\varepsilon)|,$$

$P(\varepsilon)$ can be considered as the sum of residues of the function

$$\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta_\varepsilon(2s)},$$

and we can use the integral representation of it,

$$(6.4)\qquad P(\varepsilon) = \frac{1}{2\pi i}\int_{C_0}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta_\varepsilon(2s)}\,ds,$$

where C_0 is a slight modification of the contour C defined in (4.11).

Writing

$$(6.5)\qquad P \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{C_0}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$

we get as in (4.12)

$$(6.6)\qquad \lim_{\varepsilon\to 0}P(\varepsilon) = P.$$

Suppose that for a β_0

$$(6.7)\qquad |P| \geqslant e^{-\frac{\log T\log\log\log T}{\log\log T}},\qquad T > c_7.$$

Then after passing to the limit $\varepsilon\to 0$ we get our lemma (4.3) at once from (6.3), (6.6), (6.7) and (4.12). It is necessary to note that ν in (6.1), (6.3) really depends on ε. But an easy reasoning leads to the conclusion that one can dispense with this dependence. Namely we can choose an ε-sequence tending to zero by which the ν_ε-sequence is a constant sequence.

In order to finish the proof of (4.3) it is necessary to show (6.7).

We start with the integral

$$(6.8)\qquad I_0 = \frac{1}{2\pi i}\int_{(\frac{1}{4}+\varepsilon_0)-i\chi(T)}^{(\frac{1}{4}+\varepsilon_0)+i\chi(T)}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds,$$

where $\varepsilon_0 = \frac{1}{10}$.

On the one hand, we have from (3.8)

$$(6.9)\qquad I_0 = \beta_0 S(\beta_0) - \frac{1}{2\pi i}\int_{\frac{1}{4}+\varepsilon_0+i\chi(T)}^{\frac{1}{4}+\varepsilon_0+i\infty}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds - \frac{1}{2\pi i}\int_{\frac{1}{4}+\varepsilon_0-i\infty}^{\frac{1}{4}+\varepsilon_0-i\chi(T)}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds.$$

On the other hand, applying Cauchy's theorem of residues to the integral (6.8) and replacing the residues of the function under consideration by the integral representation, we have from (6.5) in view of (2.1), (2.2), (2.4), (2.5) and (6.5)

$$(6.10)\qquad I_0 = P + \frac{1}{2\pi i}\int_{(\frac{1}{4}-\varepsilon_0)}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds + O\left(\frac{1}{e^{\chi(T)}}\right).$$

But

$$(6.11)\qquad \frac{1}{2\pi i}\int_{(\frac{1}{4}-\varepsilon_0)}\beta_0^{2s}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds = \sqrt{\pi}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(\pi/\beta_0 n)^2} = \sqrt{\pi}\,S\left(\frac{\pi}{\beta_0}\right)$$

(see [1], p. 157). (6.10) and (6.11) combined with (6.9) gives, after estimating the integrals in (6.9),

$$(6.12)\qquad |P| \geqslant \left|\beta_0 S(\beta_0) - \sqrt{\pi}\,S\left(\frac{\pi}{\beta_0}\right)\right| - \frac{c_8}{e^{\chi(T)}} \overset{\text{def}}{=} \left|\varphi\left(\beta_0, \frac{\pi}{\beta_0}\right)\right| - \frac{c_8}{e^{\chi(T)}}.$$

We can obviously suppose (see [8], pp. 262, 270)

$$\left|\varphi\left(\beta_0, \frac{\pi}{\beta_0}\right)\right| = \max_{\sqrt{e}\leqslant\beta\leqslant e}\left|\varphi\left(\beta, \frac{\pi}{\beta}\right)\right| = c_9 \neq 0.$$

Hence by (5.7) we have (6.7) at once, and then by (4.12) owing to (6.3) follows (4.3).

In order to finish the proof of theorem (1.3) we combine (4.3) with (3.18).

Owing to (3.10), (3.1)-(3.4) we get the estimate

$$\max_{1\leqslant\beta\leqslant T}|\beta S(\beta)| > \sqrt{T}\, e^{-5\frac{\log T\log\log\log T}{\log\log T}},\qquad T > c_{10}.$$

Now using (1.2) we obviously get (1.3).

References

[1] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. 41 (1918).

[2] S. Knapowski, *Contributions to the theory of the distribution of prime numbers in arithmetical progressions I*, Acta Arith. 6 (1961), pp. 415-434.

[3] S. Knapowski, *Mean-value estimations for the Möbius function II*, Acta Arithm. 7 (1962), pp. 337-343.

[4] — *On oscillations of certain means formed from the Möbius series I*, Acta Arithm. 8 (1963), pp. 311-320.

[5] E. Landau, *Vorlesungen über Zahlentheorie*, Bd. II, Leipzig 1927.

[6] N. Nielsen, *Handbuch der Theorie der Gammafunction*, Leipzig 1906.

[7] W. Staś, *Zur Theorie der Möbiusschen μ-Funktion*, Acta Arithm. 7 (1962), pp. 409-416.

[8] — *Über eine Reihe von Ramanujan*, Acta Arithm. 8 (1963), pp. 216-271.

[9] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

[10] — *The theory of functions*, Oxford 1932.

[11] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953.

INSTITUTE OF MATHEMATICS OF THE ADAM MICKIEWICZ UNIVERSITY, POZNAŃ

Reçu par la Rédaction le 8. 2. 1964

ACTA ARITHMETICA
X (1965)

Lattice points in a sphere

by

M. N. Bleicher and M. I. Knopp* (Madison, Wis.)

1. Introduction. In this paper we consider the classical lattice point problem for the three-dimensional sphere. The problem can be described as follows. Let x be a positive real number and let k be a positive integer. Consider a k-dimensional sphere of radius $\sqrt{x}$ and center $(0, \ldots, 0)$. Following the notation of Walfisz ([4]), we let $A_k(x)$ be the number of integer lattice points in this sphere. A simple geometric argument shows that as $x \to +\infty$, $A_k(x) \sim V_k(x)$, where $V_k(x)$ is the volume of the sphere in question. The problem then is to get an asymptotic estimate of the difference $R_k(x) = A_k(x) - V_k(x)$.

Here we are considering only $R_3(x) = A_3(x) - \frac{4}{3}\pi x^{3/2}$. We obtain the following results:

(1) $$R_3(x) = O(x^{3/4}\log x), \quad x \to +\infty,$$

(2) $$R_3(x) = \Omega(x^{1/2}\log\log x), \quad x \to +\infty.$$

Of course (1) is not new. Vinogradov ([3]) has in fact shown that $R_3(x) = O(x^{\frac{19}{28}+\varepsilon})$, $\varepsilon > 0$, an upper estimate better than (1) [1]. However this result depends upon his difficult theory of exponential sums. Our estimate (1), on the other hand, is better than the elementary result $A_3(x) = O(x)$ and depends only upon a fairly standard application of the circle method.

As far as we can ascertain (2) is new. It is based upon the Ω-estimate for $R_4(x)$ ([4], p. 95)

(3) $$R_4(x) = \Omega(x\log\log x), \quad x \to +\infty.$$

*The authors would like to thank the National Science Foundation for financial assistance.

[1] Added in proof. Chen Ting-run (Chinese Mathematics 4(1963), pp. 322-339) claims the result $R_3(x) = O(x^{2/3})$.