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On the abstract theory of primes II

by

E. FOGELS (Riga)

Introduction

1. In the present paper we shall be concerned with an infinite semigroup $\mathbb{G}$ on a countable number of generators $g$, the elements of $\mathbb{G}$ generally being denoted by $a$. We suppose that they are distributed into classes $H_j (1 \leq j \leq k)$ forming a group $H'$, and that the number of classes satisfies

$$1 \leq k \leq D,$$

where $D$ is parameter $D_\theta > 2$ which may increase indefinitely. Using a homomorphism $N$ of $\mathbb{G}$ into the multiplicative semigroup of real numbers $\mathbb{R}$ we denote the images $Na, N\beta, \ldots$ (called norms of $a, \beta, \ldots$) by $a, b, \ldots$

We take for granted that for any $\alpha > 1$

$$\sum_{x \leq D_\alpha} \frac{1}{x} = \log D_\alpha + O(D_\alpha^{1-\theta}), \quad x = D,'$$

where the constants $I, \epsilon, \theta$ do not depend on $\theta$ ($0 < \theta < 1; 0 < \epsilon < 1; 0 < \theta < 1$). In a previous paper (see (4)) it has been proved in particular that for $\theta > \frac{1}{2}$ (actually for any $\theta > 0$) in every class $H_i$ there is a generator $g$ which in norm does not exceed $D^{2\theta}$. In the case of an even $k$ and $\theta < \frac{1}{2}$ the same estimate was proved supposing that for a suitable constant $\epsilon > 0$ we have

$$\lim_{x \rightarrow \infty} \left( \sum_{x \leq D} \frac{1}{x} \right) - \sum_{x \leq D} \frac{1}{x} > D^{-\epsilon},$$

where $D'$ denotes any subgroup of the group $H'$ with the index 2.

In the present paper we shall use a homomorphism into the multiplicative semigroup of complex numbers

$$\left( a = Na \geq 1, \quad 0 < \alpha < 1 \right).$$
We suppose that there is a single $a \in \mathfrak{a}$ with $a = 1$ and that $a = 1$ implies $a = 0$. (For this particular $a$ we shall sometimes write $1$.) Now we take for granted that the images (4) of the elements $a \in \mathfrak{a}$ next to (1), (2) satisfy

$$
\sum_{a \in \mathfrak{a}} 1 = \mathfrak{a} = O(D^2\log -a), \quad 0 < \theta' < \theta(1)
$$

(with $\theta'$ independent of $f$) uniformly in $0 < \varphi < 1$.

Our present task is the proof of some estimate for the least norm of a generator $b$ lying in a fixed angular region

$$
\mathfrak{a} \{(a = a_0 + \theta \mathfrak{S}(\text{mod} 1)) , \quad 0 \leq \theta < 1, \quad D^{-\epsilon} < D < 1 \}
$$

with an arbitrarily large constant $c_0 < 1$. To this end we shall prove the following

**Theorem.** (i) If $\theta < \frac{1}{\nu}$, then there is a positive constant $c$ (depending merely on $c_0, c_1, 1, \theta, \theta'$) such that for any $x > 1$ and any $H_j$ in the region

$$
(x < a < xD_1, \ a \in \mathfrak{a}), \quad D_1 = D^2 \log xlD
$$

there is a generator $b \in H_j$. For an odd class number $h$ the conclusion holds as well in the case of $\theta > \frac{1}{\nu}$.

(ii) Let $b$ be even and $b < \frac{1}{\nu}$. If (3) is true, then the conclusion of (i) holds (with the constant $c$ depending also on $c_0$).

For $D = 1$ the theorem reduces to that of the previous paper [4] (with $g = 1$). The result has been announced in [5].

**Corollary.** Let $\pi(x, a_0, \mathfrak{a}, H_j)$ denote the number of generators $b \in H_j$ with $a \in \mathfrak{a}$ and $b < x$. For appropriate constants $c_0, c_1 > 0$ and any $x > D_0^2$ we have

$$
\pi(x, a_0, \mathfrak{a}, H_j) > xD^2 \log x.
$$

If $x \to \infty$ and some other conditions are satisfied, then the theorem holds for regions

$$
(x < a < xD_1, \ a \in \mathfrak{a}),
$$

with arbitrarily small positive $\alpha$. This will be proved in §§ 13, 14.

---

(1) In order to illustrate the inequality $\theta' < \theta$ let us take, for example, the semigroup $\mathfrak{S}$ (with $\lambda = 1$) of the ordinary complex integers $a$ with norms $a = |a|^2$, considering the integers $a$ and $a'$ as identical. Then $\pi = \pi'$ and we may take $D = 4$. The number of integers $a$ in the sector

$$
S_{a_0}(a < x, a_0 = a_0 + \theta \mathfrak{S}(\text{mod} 1)) \quad (0 < \theta < 1)
$$

is evidently $\mathfrak{N} + O(1)$ for $a_0 = 0$ and $\mathfrak{N} + O(1)$ for any other $a_0$. In the present instance (5) holds with $\theta' > \frac{1}{2}$ (cf. (7), (75)) but it is not true for $\theta' > \frac{1}{2}$ (since there is no $a$ in $S_{a_0}$ with a positive $a_0 < 1/3\mathfrak{S}^2$). On the other hand (2) holds with $\theta = \frac{1}{2}$ (see, for example, [7], (83)) and thus $\theta' < \theta$.

---

The theorem is of interest chiefly in the case of $D \to \infty$ and $1/D < 1$ (see (6)). If on the contrary $1/D \to \infty$ and $D < 1$, then better results can be obtained by a simpler method. I hope to return to the latter case in another paper.

An application of the present theorem for primes representable by binary quadratic forms will be given in a continuation of this paper.

The method used in this second paper is the same as that of the previous one (the density method of Ya. V. Linnik). For the proof of the main auxiliary theorems we will use the method of Turán ([10]). Further on $D, \epsilon, c_0, c_1, c_2, \ldots$ denote positive constants which may depend on $1, \epsilon, \theta, \theta', c_0, c_1, c_2$ (if $h$ is even and $\theta < \frac{1}{2}$). Generally they retain their meaning only throughout the same paragraph.

By $b|a$ we mean that $b|a \in \mathfrak{a}$ and that there is an $a' \in \mathfrak{a}$ such that $a = b a'$. If $b$ is in norm the largest element of $\mathfrak{a}$ for which $b|a$, and $b|a_0$, then we write $b = (a_0, a_2)$. By $(a_0, a_1, a_2)$ we shall denote the corresponding number in the semigroup of norms (or sometimes the interval $a_0 < a < a_2$). For the norms of $a, b, b, \ldots \in \mathfrak{a}$ we shall write $a, b, a_0, \ldots$, respectively.

The complex variable will be denoted by $s = x + it$ ($x = \Re s, t = \Im s$).

The functions $\zeta(s, X)$ and their zeros near the line $s = 1$

2. Lemma 1. Let the sequence of real numbers $a_n$ $(n = 1, 2, \ldots, N)$ be distributed uniformly modulo 1 with the remaining term $< R$, that is to say, for any $\varphi 

\pi = \frac{\varphi N}{2}\pi < R$. Then for any integer $m \neq 0$

\begin{equation}
\sum_{a \leq N}\psi(a) \approx \left\lfloor \frac{\varphi N}{2}\pi \right\rfloor R.
\end{equation}

**Proof.** Writing

\begin{equation}
N_\pi - \varphi N = R\varphi
\end{equation}

we have

\begin{equation}
R = \max_{\pi \leq N\epsilon} R(\varphi).
\end{equation}

By (9) and Abel's identity (see [8], p. 371)

\begin{equation}
\sum_{a \leq N^{\epsilon}} \psi(a) \approx - \frac{1}{\pi} \left( R\varphi + R(\varphi) \right) \psi(N) \psi(N) = - \frac{1}{\pi} \int R(\varphi) e^{i\varphi t} d\varphi.
\end{equation}

Let $R(\varphi)$ (for any fixed positive integer $m$) denote the last term in (9) when the sequence $a_n$ is replaced by $a_{m\pi}$. Then, by (11),

\begin{equation}
\sum_{a \leq N^{\epsilon}} \psi(a) = - 2\pi \int \left( \frac{1}{\pi} R_N(\varphi) e^{i\varphi t} d\varphi.
\end{equation}
Let \( A_\epsilon \) (for any fixed \( \epsilon \in [0, 1] \)) be the number of solutions \( a_n \) of the inequality
\[
\epsilon a_n = \delta \rho \pmod{1} \quad (0 \leq \delta < 1).
\]
By the definition of \( R_\epsilon(\delta) \) we have
\[
A_\epsilon = \delta p N + R_\epsilon(\delta).
\]
Now, (13) is equivalent to \( m a_n = k + \epsilon \rho \pmod{1} \), i.e.
\[
a_n = \frac{k}{m} + \frac{\epsilon \rho}{m}.
\]
The last condition is satisfied only by those \( a_n \) which lie in any of the following intervals mod 1:
\[
\left[ 0, \frac{\epsilon \rho}{m} \right], \left[ \frac{1}{m}, \frac{1 + \epsilon \rho}{m} \right], \ldots, \left[ \frac{m-1}{m}, \frac{m-1 + \epsilon \rho}{m} \right].
\]
Hence, according to (9), the number of the solutions \( a_n \) of (15) is
\[
\frac{\rho}{m} N + R \left( \frac{\epsilon \rho}{m} \right) + R \left( \frac{1}{m} \right) + \ldots + \left( \frac{m-1}{m}, \frac{m-1 + \epsilon \rho}{m} \right) - R \left( \frac{1}{m} \right) - \ldots - \left( \frac{m-1}{m}, \frac{m-1 + \epsilon \rho}{m} \right).
\]
From this and (14) we deduce
\[
R_\epsilon(\delta) = B \left( \frac{\epsilon \rho}{m} \right) + \sum_{1 \leq \epsilon < \rho} \left[ B \left( \frac{\epsilon \rho}{m} \right) - B \left( \frac{\epsilon}{m} \right) \right].
\]
Hence, by (10),
\[
\max_{0 < \epsilon < \rho} |R_\epsilon(\delta)| < 2m B.
\]
Using this estimate in (19) we get (8) for a positive \( m \). And the case of \( m < 0 \) can be reduced to that of \( m > 0 \).

3. Let \( x \) have a higher order of magnitude than \( (D^4 |n|^{1/2}) \). Then by (2) and (5) the numbers
\[
a = a_n \quad (n = 1, \ldots, N; \quad N = x + O(D^4 |n|^{1/2}))
\]
corresponding to the points (4) (with \( a \leq x \)) of any class \( H = H_\delta \) are distributed uniformly mod 1 with the remaining term \( R \ll D^4 |n|^{1/2} \).
Hence, writing
\[
\xi = \xi(a) = e^{2\pi i a},
\]
we have by Lemma 1 for any integer \( m \neq 0 \)
\[
\sum_{a \in C \cap m} \xi^m \ll |m| D^4 |n|^{1/2}. \tag{16}
\]
Now let us introduce the function
\[
\xi(s, H, \xi^m) = \sum_{n=1}^{\infty} \frac{x(n) a^{-m}}{n^s}, \quad (s > 1)
\]
and the number
\[
\delta_1 = \left| \frac{\phi - \eta (0 < \eta < \frac{1}{2})}{\frac{1}{2} \phi^2} \right| \quad \text{if} \quad m = 0,
\]
\[
\delta_2 = \left| \frac{\phi - \eta}{\frac{1}{2} \phi^2} \right| \quad \text{if} \quad m \neq 0.
\]

**Lemma 2.** The function (17) is regular in the half-plane \( s > 1 - \delta_1 \), except for a simple pole at \( s = 1 \) with residue \( x \) in the case of \( m = 0 \) and we have uniformly in the strip \( G \) \((1 - \delta_2 < s < 2)\)
\[
\xi(s, H, \xi^m) \ll \eta^{-1} D^4 |n|, \quad (m = 0).
\]

**Proof.** For \( m = 0 \) this has been proved in [4], § 3. Further let \( m \) be a fixed integer \( \neq 0 \) and \( f(s) \) denote the sum (16). Then in \( s > 1 \)
\[
\xi(s, H, \xi^m) = \sum_{n=1}^{\infty} \xi(n) a^{-m} = \int_{1}^{\infty} \frac{f(x)}{x^{s-1}} dx = s \sum_{n=1}^{m} g_n(s), \tag{20}
\]
where
\[
g_n(s) = \int_{\frac{1}{2}}^{s} \frac{f(x)}{x^{s-1}} dx.
\]
Now we can find a number \( a_0 = a_0(m) > 1 \) such that
\[
\xi(s, H, \xi^m) = \sum_{n=1}^{a_0(m)} \xi(n) a^{-m} + \sum_{n=a_0(m)}^{\infty} \xi(n) a^{-m} = \sum_{n=a_0(m)}^{\infty} \xi(n) a^{-m}.
\]
Then, by (16), for any \( n \geq a_0 \) we have in \( G \)
\[
g_n(s) \ll D^4 |n| \int_{1}^{s} \frac{x^{1/2}}{|s-1|} ds \ll D^4 |n| \int_{1}^{s} \frac{n^{-1/2} - (n+1)^{-1/2}}{n^{s-1}} ds.
\]
Hence by (20) \( \xi(s, H, \xi^m) \) is regular in \( s > 1 - \delta_1 \) and (19) holds, since by (20), (16), (3), (18) and (21) in \( G \)
\[
\sum_{n=a_0}^{\infty} g_n(s) = \sum_{a=a_0}^{s} f(x) dx \ll \sum_{a=a_0}^{s} \frac{x+D^4 |n|^{1/2}}{|s-1|} dx \ll \sum_{a=a_0}^{s} \frac{x}{|s-1|} dx + D^4 \]
\[
\ll \frac{a_0^2}{|s-1|} + D^4 + D^4 |n| \ll |m| D^4.
\]

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4. Let \( \chi \) denote the characters of the group \( G \), \( \chi_a \) being principal character, and let \( \chi(a) = \chi(H) \) for all \( a \in H \). Write
\[
X(a) = \chi(a) \xi(a)^m = \chi(a) a^{\text{even}}.
\]

Now we introduce the function
\[
\zeta(s, X) = \zeta(s, X; \xi) = \sum_{H} \chi(H) \zeta(s, H; \xi^m) \]
\[
= \sum_{a} \chi(a) \xi(a)^m a^s = \sum_{a} X(a) a^s \quad (\sigma > 1).
\]

Since
\[
\sum_{H} \chi(H) = \begin{cases} h & \text{if } \chi = \chi_{\text{triv}}, \\ 0 & \text{otherwise,} \end{cases}
\]
by Lemma 2 the function (23) is regular in \( \sigma > 1 - \theta_1 \), except for a simple pole at \( s = 1 \) with residue \( h_0 \) in the case of \( X = X_{\text{triv}} \) (when \( \chi = \chi_{\text{triv}} \) and \( m = 0 \)). By (1), (23), (19) we have in \( G(1 - \theta_1 \leq \sigma \leq 2) \)
\[
\zeta(s, X; \xi) - e_0 h_0 (s \rho - 1)^{-1} \ll (\theta_1 - \theta_1)^{-1} D(1 + |m|)(1 + |l|)
\]
where
\[
e_0 = \begin{cases} 1 & \text{if } \chi = \chi_{\text{triv}} \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

\( G \) being a semigroup with the generators \( b_i \) in the half-plane \( \sigma > 1 \) we have
\[
\zeta(s, X) = \sum_{b_i} (1 - X(b_i)^{-1})^{-1}
\]
whence there are no zeros of \( \zeta(s, X) \) in \( E \).

Let \( \mu(a) = (-1)^r \) if \( a \) is a product of \( r \) distinct generators, and \( = 0 \) if \( F[a] \). Further let
\[
A(a) = \begin{cases} \log b & \text{if } a = b^a \quad (a \geq 1), \\ 0 & \text{otherwise.} \end{cases}
\]

Using the product-form of \( \zeta(s, X) \) we can prove that
\[
\zeta(s, X) = \sum_{a} X(a) \mu(a) a^{-s},
\]
\[
1/\zeta(s, X) = \sum_{a} X(a) A(a) a^{-s} \quad (\sigma > 1).
\]

Hence by (24) for any \( \eta \leq 0 \)
\[
1/\zeta(1 + \eta + it, X) > \lambda_0 h_0 |\xi(0)| (1 + |m|)(1 + |l|).
\]

Now taking
\[
\eta = 1/|D(1 + |m|)(1 + |l|)|, \quad s_0 = 1 + \eta + it,
\]
and using (24) and (26) we can prove (cf. [4], (19)) that in \( |a - a_0| \leq \frac{1}{4} \theta_1 \)
\[
\zeta'/\zeta(s, X) + e_0 h_0 - \sum_{\eta < \sigma < \eta' \leq l + \eta} (s - \sigma)^{-1} \ll \log D(1 + |m|)(1 + |l|)
\]
where \( \eta \) runs through the zeros of \( \zeta(s, X) \). By (27) and the arguments used in [2], § 11
\[
|\zeta'/\zeta(a, X_0)| < \frac{1}{c_0}(c_0 - 1)^{-1}
\]
where \( e_0 = 1 + c_0/\log D \) and \( c_0 \) is small enough. For any positive \( r \leq 1 \) we have (cf. [4], (21))
\[
|\zeta'/\zeta(1 + r, X_0)| \ll 1/r + c_0 \log D.
\]

Let \( \nu = \nu(\nu, X, t_0) \) denote the number of zeros of \( \zeta(s, X) \) in \( |s - 1 - t_0| \leq r \). If
\[
e_0 \log D(1 + |m|)(1 + |l|) \ll r < \theta_1/4 - 1/|D(1 + |m|)(1 + |l|)|,
\]
then (cf. [2], § 16)
\[
r \ll \log D(1 + |m|)(1 + |l|).
\]

By the arguments of [4], § 6 (with \( \theta_1 \) instead of \( \theta_0 \)) we can prove that the number of zeros of \( \zeta(s, X) \) in the rectangle \( 1 - \theta_1/2 \leq \sigma \leq 1, \quad \|l - t_0| \leq \frac{1}{4} \) does not exceed \( \log D(1 + |m|)(1 + |l|) \).

5. Now we can repeat the arguments of [4], §§ 7-10 with \( D(1 + |m|), X \) instead of \( D, X \) (and with \( q = 1 \)). Considering that by (22) a real \( X \) implies \( m = 0 \) we get the following

**Fundamental Lemma 3.** For appropriate \( \nu \) in the region
\[
\sigma \geq 1 - c/\log D(1 + |m|)(1 + |l|) \quad (\nu \geq \sigma', 2/4)
\]

there are no zeros of \( \zeta(s, X) \) with a complex \( X = \chi(a) \xi(a)^m \). For at most one real \( X \) in (29) with \( m = 0, t = 0 \) there may be a simple real zero
\[
\nu' = 1 - \nu' \leq 1.
\]

\( \nu' \) (if it exists) will be called the exceptional zero of \( \zeta(s, X) \). If the conditions of the theorem of § 1 are satisfied, then we have in (30)
\[
\nu' > \delta - \delta \quad (\text{see [4], § 19, Lemma 22}).
\]
Further on in (22) let \(m\) satisfy
\[
|m| \leq D^{\varepsilon_{4}},
\]
\(q_{1}\) being defined by (6). Then by the arguments used in [4], §§ 20, 21 we can prove the following

**Fundamental Lemma 4.** Let \(B\) be defined by (30). For appropriate \(A \ll 1\) and
\[
\delta_{6} = \min(B, A, \log D), \quad \lambda_{A} = A \log \frac{eA}{\delta_{6} \log D} - A, \quad \phi \log D
\]
there are in \((1 - \lambda_{A} \log D \leq \sigma \leq 1, \vert t \vert \leq D)\) no other zeros of the function
\[
\sum_{\chi \neq \chi}(\varepsilon, X) \text{ (with } m \text{ satisfying (31)) than at most the exceptional zero (36)}.
\]

An upper bound for the number of generators

6. **Lemma 5.** Let
\[
a_{n} = (n = 1, 2, \ldots, N)
\]
be a set of elements \(q \in \mathfrak{G}\) such that for any fixed \(q \in \mathfrak{G}\) we have
\[
\sum_{q \in \mathfrak{G}} 1 = N f(q) + R_{n}
\]
where \(f(q)\) is a positive function satisfying \(f(q_{1}q_{2}) = f(q_{1})f(q_{2})\) whenever \((q_{1}, q_{2}) = 1\). Further let \(N_{q}\) (for any \(q > 1\)) denote the number of those elements \(a_{n}\) of (33) which are not divisible in \(\mathfrak{G}\) with any generator \(b\) in norm less than \(q\). Write
\[
F(a) = \sum_{q \in \mathfrak{G}} \mu(a) f(q), \quad S_{a}(a) = \sum_{q \in \mathfrak{G}} \frac{\mu^{2}(a)q}{F(q)}, \quad \lambda_{a} = \frac{\mu(a)}{\sum_{q \in \mathfrak{G}} (1 - 1/f(b))^{-1} S_{b}(a)} S_{a}(a), \quad \text{if } a \leq z,
\]
\[
\lambda_{a} = 0 \quad \text{otherwise}.
\]

Then
\[
N_{q} \leq N_{q} + \sum_{q \in \mathfrak{G}} \left| \lambda_{a} B_{n}(q, a) \right|.
\]

This may be proved by the sieve method of A. Selberg ([9]). Cf. [3], § 3.

7. **Lemma 6.** Let (33) in the previous lemma be all the elements of any class \(M_{q}\) of \(\mathfrak{G}\) for which in (4)
\[
a \leq z, \quad a \equiv a_{n} \mod 1, \quad \theta \leq 1; \quad \varphi \text{ fixed, } D^{1/\varphi} < \varphi < 1, \quad \text{and (4)}
\]

If \(x \geq x_{0}^{1/8}\) and \(x \geq D^{1/4}\) (where \(c_{1} < 1\) is large enough), then the main term in (35) does not exceed \(c_{0}x^{1/2} \log x\).

**Proof.** By (5) we have
\[
N = x^{1/2} + O(D^{1/4} x^{1/8}).
\]

By (5) and (36) the number of elements (33) with \(q \in \mathfrak{G}\) for any \(q \in \mathfrak{G}\) is
\[
N = \frac{x}{q} + O(D^{1/2} x^{1/6} - \sqrt{3}) = \frac{N}{q} + O(D^{1/2} x^{1/6} - \sqrt{3}) = \frac{N}{q} + O(D^{1/2} x^{1/6} - \sqrt{3})
\]

Hence (34) holds for
\[
f(q) = q, \quad R_{n} \leq D^{1/2} x^{1/6} - \sqrt{3}
\]

and thus
\[
S_{a} = \sum_{q \in \mathfrak{G}} \frac{\mu^{2}(a)}{q^{1/2}} = \sum_{q \in \mathfrak{G}} \frac{\mu^{2}(a)}{q^{1/2}} \left( \prod_{q < b < q} (1 + b^{1/2} + \cdots) \right) = \sum_{q \in \mathfrak{G}} \frac{1}{a^{2}}
\]

where \((e)\) denotes the set of elements \(e \in \mathfrak{G}\) such that the product of all different generators of any \(e\) is in norm \(< \frac{1}{z}\). Hence
\[
S_{a} > \sum_{a \leq z} 1 > \sum_{a \leq z} 1 \int_{a}^{z} \frac{\varphi(y)}{y} dy
\]

By (2) \(\varphi(y) \geq \frac{1}{4} \log y\) (for a sufficiently large \(a_{1}\) and thus
\[
S_{a} > \frac{1}{4} \int_{a}^{z} \frac{\varphi(y)}{y} dy > \frac{1}{20} \log x, \quad \frac{N}{S_{a}} < c_{1} x^{1/2} \log x = c_{1} x^{1/2} \log x.
\]

8. **Lemma 7.** Let \(W_{e}\) denote the remaining term in (35) and let in Lemma 6
\[
z^{\varphi} = \varphi \frac{x^{1/2}}{\log x}, \quad c_{1} = 1 + c_{2} + \max(c_{1}, 1), \quad \varphi \geq z^{\varphi}
\]

for any positive constant \(c_{2} < \varphi\) and for \(x > D^{1/4}\) with a sufficiently large \(c_{3} = c_{1}(\varphi) < 1\). Then
\[
W < c_{1} x^{1/2} \log x, \quad c_{1} = c_{1}(\varphi, c_{2}).
\]
Proof. Since $|\lambda| \leq 1$ (cf. (3), (38)), by (35), (37),
$$W \leq D^q \sum_{a \leq \sqrt x} \frac{x}{a} \sum_{x_2 \leq a_2 \leq a_\lambda} \frac{(a_1, a_2)}{a_1 a_2} \frac{1}{1 - \nu},$$
and thus we have to prove that
$$\sum_{x_1 \leq a_1 \leq a_\lambda} \frac{(a_1, a_2)}{a_1 a_2} \frac{1}{1 - \nu} < \frac{\nu}{hD^q \log a}.$$  
From (2) we can deduce
$$\sum_{a \leq \sqrt x} x^{1 - \nu} \ll \log x, \quad \sum_{a \leq \sqrt x} x^{1 - \nu} \ll D^q - a_\nu,$$
whence
$$\sum_{x_1 \leq a_1 \leq a_\lambda} \frac{(a_1, a_2)}{a_1 a_2} \frac{1}{1 - \nu} \ll \left( \sum_{a \leq \sqrt x} x^{1 - \nu} \right)^2 \ll \log x \log^2 x.$$  
Writing
$$S_b(z) = \sum_{a \leq \sqrt x} \frac{(a_1, a_2)}{a_1 a_2} \frac{1}{1 - \nu},$$
we have, by (41),
$$S_b(z) = z^{-\nu} S_1(z) = \frac{a}{d} - z^{-\nu} h^2 x^2 \left( \frac{z}{d} \right)^{2\nu} = h^2 x^{2\nu} d^{-1 - \nu}.$$  
Hence, by (40) and (38),
$$\sum_{x_1 \leq a_1 \leq a_\lambda} \frac{(a_1, a_2)}{a_1 a_2} \frac{1}{1 - \nu} \ll S_b(z) \ll (\log x)^{2\nu} \sum_{a \leq \sqrt x} a^{-1 - \nu} \ll D^{q - \nu} (\log x)^{2\nu} \frac{\nu^\nu}{hD^q \log a},$$
which proves (39).

9. **Lemma 8.** Let $\pi_H(x, \varphi, a_\nu)$ denote the number of generators $b \leq H$ with
$$b \leq \sqrt x, \quad a \equiv a_\nu \oplus \theta \varphi \pmod{1}, \quad 0 \leq \theta < 1,$$
where $x^{-\phi} \varphi \leq 1$ (for any positive constant $\theta_\nu < \theta$, $x > D^q$ and $a_\nu = a_\nu(\theta_\nu)$ large enough). Then for appropriate $a_\mu$ (which does not depend on $a_\nu$)
$$\pi_H(x, \varphi, a_\nu) < \phi x \log x.$$

Proof. Since, by (38), $x < \sqrt x$, it follows from Lemma 6 that all the generators larger in norm than $x$ and satisfying the conditions of the present lemma are in the set of the $N_\nu$ elements as defined in Lemma 5. Hence, by (2) and Lemma 7
$$\pi_H(x, \varphi, a_\nu) \leq N_\nu + \pi_H(x, \varphi, a_\nu) < N_\nu + \frac{\psi x}{D^q (x^{-\phi})^\nu} < \phi x \log x.$$  

**Corollary.** If $\varphi > \phi^{1/\delta}, \varphi > D^q$, then
$$\sum_{\text{all } a \leq \sqrt x \atop a \equiv a_\nu \oplus \theta \varphi \pmod{1}} \log b \ll \phi x \log x,$$
whence
$$\sum_{\text{odd } a \leq \sqrt x \atop a \equiv a_\nu \oplus \theta \varphi \pmod{1}} A(a) < \phi x \log x.$$  

Proof. The left-hand side of (43) being $< \pi_H(x, \varphi, a_\nu) \log x$, the estimate holds by (42). By (43) and the definition of $A(a)$, (44) follows from the estimate
$$\sum_{\text{all } a \leq \sqrt x \atop a \equiv a_\nu \oplus \theta \varphi \pmod{1}} \log b + \sum_{\text{all } a \leq \sqrt x \atop a \equiv a_\nu \oplus \theta \varphi \pmod{1}} \log b + \ldots < \phi x \log x,$$
which is evident (for a sufficiently large $a_\nu < 1$), since the number of terms is $< D^q \log x$, none of them exceeding $(\psi \varphi x + (D^q x^{1-\phi})) \log x$ (cf. [4], § 14).
Proof. Writing

$$R(a) = \frac{1}{2\pi i} \int_{2-\infty}^{1+\infty} \left( e^{2\pi ia} - e^{2\pi ika} \right) e^{-\pi a^2 \sigma} ds,$$

we have, by (45) and (25),

$$J_X(\tau, k, A) = \sum_{a, \nu} A(a) R(a) = \sum_{a, \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} X(a)$$

(say), since $R(a) = 0$ outside the interval $\phi^A < a < \phi^{AB}$ (see [4], (61)). Let $S$ denote the left-hand side of (46). Then

$$S = \sum_{\tau, \nu} \sum_{a, \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} X(a) X(a) + \sum_{\nu \geq \nu} e_{a} X(a)$$

$$= h \sum_{\nu} \sum_{\nu \geq \nu} \sum_{a, \nu} e_{a} \Theta_{e_{a} X(a)}(\nu, \nu + a).$$

Since

$$\sum_{\nu} \Theta_{\nu} = \begin{cases} h & \text{if } \nu \text{ is the principal class } H, \\ 0 & \text{otherwise}. \end{cases}$$

Now, using an integer $N > 1$ and considering that $e_{a} = A(a) e^{-\nu a}$ does not depend on $a$, we can write the sum $U$ in brackets of (47) as follows:

$$U = \sum_{\nu \geq \nu} \sum_{a, \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} \Theta_{e_{a} X(a)}(\nu, \nu + a) + \sum_{\nu \geq \nu} e_{a} X(a)$$

Since for any real $\tau(0, 1)$

$$\sum_{\nu \geq \nu} e_{\nu} \Theta_{\nu} \leq \min \left\{ M, M/\min(\nu, 1 - \nu) \right\}$$

(cf. [8], p. 189), we have

$$U < \sum_{\nu \geq \nu} \sum_{a, \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} \Theta_{e_{a} X(a)}(\nu, \nu + a) + \sum_{\nu \geq \nu} \sum_{a, \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} X(a)$$

Using (44) and the estimate

$$|R(a)| < \phi^{A} A \quad \text{for } \phi^{A} < a < \phi^{AB}$$

(see [4], (61)), we deduce

$$\sum_{\nu \geq \nu} \sum_{a, \nu} |e_{a}| \leq \phi^{A} A \sum_{\nu \geq \nu} \sum_{a, \nu} \frac{A(a)}{a} \leq \phi^{AB} A \sum_{\nu \geq \nu} \sum_{a, \nu} \frac{\phi^{A} A}{a}$$

$$< \phi^{AB} A \sum_{\nu \geq \nu} \frac{\phi^{A} A}{a}$$

Hence

$$U < \sum_{\nu \geq \nu} \sum_{a, \nu} \frac{\phi^{A} A}{a} \leq \phi^{AB} A \sum_{\nu \geq \nu} \sum_{a, \nu} \frac{\phi^{A} A}{a}$$

$$= \frac{\phi^{AB} A}{h} \sum_{\nu \geq \nu} \sum_{\nu \geq \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} \Theta_{e_{a} X(a)}(\nu, \nu + a)$$

Taking $N = M$ we get

$$U < \phi^{AB} A \sum_{\nu \geq \nu} \sum_{\nu \geq \nu} \sum_{a, \nu} \sum_{\nu \geq \nu} e_{a} \Theta_{e_{a} X(a)}(\nu, \nu + a)$$

From this and (47) follows (48).

Now let $\nu = \nu(\tau, k, A)$ for any selected $\tau \epsilon [-D, D]$ and let $\lambda(\epsilon, \nu, \log D)$ (with appropriate $\epsilon, \nu, \log D$) denote the number of functions $f(\tau, k, X)$ (with $|\epsilon| < M, M < D^{2\nu}$) having at least one zero $e = e(X)$ in the square $Q(\nu, \log D) = \nu < 1, |e| < \lambda(\log D)$. Then for at least $\nu(\epsilon, \nu, \log D)$ functions $f(\tau, k, X)$ we have, by [4], (49),

$$|J(\tau, k, \lambda^{-1} \log D)| < e^{-\nu(\epsilon, \nu, \log D)}$$

with the same $k = k_{e} < \epsilon, \nu$. Hence, by (46),

$$\frac{\nu}{\epsilon, \nu} \lambda^{-1} \log D < \sum_{\nu} \sum_{\nu} \sum_{\nu} \sum_{\nu} e_{\nu} \Theta_{e_{\nu} X(\nu)}(\nu, \nu + a) + \sum_{\nu} \sum_{\nu} \sum_{\nu} \sum_{\nu} e_{\nu} X(\nu)$$

whence

$$\nu < \phi^{AB} A \sum_{\nu} \sum_{\nu} \sum_{\nu} \sum_{\nu} e_{\nu} \Theta_{e_{\nu} X(\nu)}(\nu, \nu + a) + \sum_{\nu} \sum_{\nu} \sum_{\nu} \sum_{\nu} e_{\nu} X(\nu)$$

Combining this with (28) and arguing as in [4], §18, we can prove the following.

**Fundamental Lemma 9.** Let $N$, denote the number of zeros of the function $H(\tau, k, X)$ (with $m$ in (22) satisfying $|m| < M, M < D^{2\nu}$) in the
rectangle

\[ B_1 \left( 1 - \frac{1}{4} \log D \leq \sigma \leq 1, \frac{1}{4} \leq \log D \right) \quad \left( 0 < \lambda \leq \frac{1}{4} \theta' \log D. \right) \]

Then for appropriate \( C(\epsilon) \)

\[ N_1 < \theta^2 \log M. \]  

(49)

**Proof of the theorem**

11. **Lemma 10.** Let \( r \) denote any fixed integer \( \geq 1 \), and let \( 0 < \delta < \frac{1}{2} \), \( 0 \leq \sigma_1 - \sigma_2 < 1 < 2 \Delta \). There is a periodic function \( f(a) \) of the real variable \( a \) with the period 1 such that (i) \( f(a) = 1 \) in \( a_0 \leq a < a_1 \), \( f(a) = 0 \) in \( a_1 < a < a_2 + \delta \), \( f(a) \leq 1 \) for other \( a_1 \); (ii) it has the Fourier expansion

\[ f(a) = \sum_{m=0}^{\infty} d_m \exp(i m a), \]

where

\[ (50) \quad d_0 = a_2 - a_1 + \delta, \quad d_m = \min(d_m, |m|^{-1}, \Delta^{-m}|m|^{-\epsilon}) \quad \text{for} \quad |m| \geq 1. \]

This is an immediate consequence of I. M. Vinogradov's lemma ([11], Lemma 12). Cf. [9], p. 514.

Now let \( L_{\epsilon,m} \) denote a broken line in the strip

\[ 1 - \frac{1}{4} \theta' + 1 < \log D (1 + \| \|) < \sigma < 1 - \frac{1}{4} \theta' \]

(with appropriate \( \sigma \) such that (i) for any \( \sigma = \sigma + \theta + \epsilon L_{\epsilon,m} \) we have \( \zeta' \zeta(s, X) < \log^2 D (1 + \| \|) \) (when \( m \) in (22) satisfies (31)) and (ii) the length of the piece of \( L_{\epsilon,m} \) between any two of its points \( \sigma + \theta \), \( \sigma' + \theta' + 1 \)) is < 2. Write

\[ g = \frac{1}{4} \theta' \quad \left( \leq \frac{1}{4}, \right) \quad \sigma_1 = 1 - g. \]

From the identity

\[ \sum_{a} \frac{X(a)A(a)}{a^{\sigma_1}} \exp \left( - \frac{\log a}{4y} \right) = \frac{\sqrt{y}}{\pi} \int_{-\infty}^{\infty} \frac{\zeta'(s, X)}{\zeta(s, X)} e^{i \theta' \log z} ds \]

\((x > 1, y > 0)\)

(\( \epsilon \)) Any possible improvement of (49) would imply a corresponding improvement in the theorem of § 1. If, for example, the factor \( \log M \) could be dropped, then the theorem would be true for \( D_1 = D \). Cf. the proof of (60) and (63).

(cf. [1], p. 299) and (22), (48) we deduce

\[ (51) \quad h \sum_{a \leq x} \frac{A(a)}{a^{\sigma_1}} \exp \left( - \frac{\log^2 a}{4y} \right) \exp \left( i \int_{-\infty}^{\infty} \frac{\zeta'(s, X)}{\zeta(s, X)} e^{i \theta' \log z} ds \right) \]

Let \( J \) denote the interval

\[ a = a_1 - \frac{1}{2} \theta + 2 \Delta (\mod 1), \quad 0 < \theta < \frac{1}{2} \]

with a fixed \( a_1 \) and \( \Delta \) satisfying the conditions of Lemma 10. By (51), (22) and Lemma 10,

\[ (52) \quad h \sum_{a \leq x} \frac{A(a)}{a^{\sigma_1}} \exp \left( - \frac{\log^2 a}{4y} \right) \geq h \sum_{a \leq x} \frac{A(a)}{a^{\sigma_1}} f(a) \exp \left( - \frac{\log^2 a}{4y} \right) \]

\[ = h \sum_{a \leq x} \frac{A(a)}{a^{\sigma_1}} e^{i \theta' \log z} \sum \sum d_m \varepsilon(a)^m \]

\[ = h \sum_{a \leq x} \sum_{m<\frac{2}{\sigma_1}} + h \sum_{a \leq x} \sum_{m<\frac{2}{\sigma_1}} = U_1 + U_1, \]

say, where \( M \) stands for \( D^{\sigma_1} \), which is the right-hand side of (31). By (51)

\[ (53) \quad U_1 = \int \frac{\sqrt{y}}{\pi} \sum_{m} \sum \frac{1}{\zeta(s, X)} \sum_{m<\frac{2}{\sigma_1}} \sum_{m<\frac{2}{\sigma_1}} d_m \varepsilon(a)^m \exp \left( - \frac{\log^2 a}{4y} \right) \exp \left( i \theta' \log z \right) ds \]

\[ = 2 \sqrt{\frac{\log^2 a}{4y}} d_m - 2 \sqrt{\frac{\log^2 a}{4y}} \sum_{m<\frac{2}{\sigma_1}} d_m \exp \left( \frac{1}{2} \theta' \log z \right) \]

\[ + i \int \frac{\sqrt{y}}{\pi} \sum_{m<\frac{2}{\sigma_1}} \sum_{m<\frac{2}{\sigma_1}} d_m \varepsilon(a)^m \exp \left( - \frac{\log^2 a}{4y} \right) \exp \left( i \theta' \log z \right) ds \]

where \( d_m \) runs through the zeros of \( \zeta(s) \chi(X = \zeta^m) \) on the right of \( L_{\epsilon,m} \). Supposing \( 1 < y < \log^2 D \), the remaining term in (53) satisfies

\[ \leq h \sqrt{y} \sum_{m<\frac{2}{\sigma_1}} |d_m| \log^2 D \int_{0}^{\infty} e^{-\theta' \log z} (2 + i) dt \leq h \log^2 D \log M \leq D \log^2 D, \]
by (1). Further on we shall use the notation
\[ \varepsilon = 1 - \delta + \varepsilon', \quad \varepsilon' = 1 - \delta', \]
\( \varepsilon \) being a typical zero of \( \zeta(s, X) \) and \( \varepsilon' \) the exceptional zero of \( \zeta(s, X) \) with a real exceptional character \( X = \varepsilon' \) (cf. (30)). Writing
\[ S = 1 - E_1 X' \langle H \rangle = \varepsilon^{-\theta} e^{-\varepsilon\alpha_2 - \varepsilon'w}, \]
where \( E_1 = 1 \) if \( \varepsilon' \) exists, and \( E_1 = 0 \) otherwise) and
\[ S' = \sum_{x} \Xi(x) \sum_{d_m \leq \lambda_0} \sum_{\eta_m \leq \eta_0} x^{-\varepsilon\alpha_2 - \varepsilon'w} = \sum_{x} \Xi(x) \sum_{d_m \leq \lambda_0} \sum_{\eta_m \leq \eta_0} x^{-\varepsilon\alpha_2 - \varepsilon'w}, \]
we have
\[ U_1 = 2 \sqrt{y} \pi y^2 x^{-\theta} (d_n - S') + O(D \log^2 D). \]

Let \( \mathcal{B} \) denote the integration \( B \) times repeated with respect to \( \eta \), the range of integration being \( (\eta_0, \eta_0 + 1) \). Then, by (56) and (55)
\[ |I_{\mathcal{B}} - \mathcal{B} \Xi(x) S - |I_{\mathcal{B}} S| - \gamma \Xi(d_{\mathcal{B}} = \omega_{\mathcal{B}})| \log^2 \mathcal{D}| = T_1 + T_2 + T_3, \]

where \( d_\omega \) (with \( \omega = \omega_{\mathcal{B}} \)) stands for the \( d_\omega \), and \( T_1, T_2, T_3 \) denote the parts of the previous sum obtained by dissection of the region \( \mathcal{G} \) (say) on the right of every \( \mathcal{B} \) as follows.

Let \( \mathcal{G}_1 \) denote the remaining part of \( \mathcal{G} \) and let \( \lambda_0 \leq \log \log D \) (otherwise there is no \( \varepsilon + \varepsilon' \) in \( \mathcal{G}_1 \)). Then, by (58) and (60)
\[ T_1 = e^{\theta + \varepsilon \alpha_2 \lambda_0} \lambda_0 \log \log D. \]

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\[ T_1 = e^{\theta + \varepsilon \alpha_2 \lambda_0} \lambda_0 \log \log D. \]

Let \( \lambda_0 \leq \log \log D \) (otherwise there is no \( \varepsilon + \varepsilon' \) in \( \mathcal{G}_1 \)). Then, by (58) and (60)
\[ T_1 = e^{\theta + \varepsilon \alpha_2 \lambda_0} \lambda_0 \log \log D. \]
and $a_d > D^{-a}$, by § 5, whereas by (32)

$$T_1 + T_2 < a_d e^{-(1+10^{-6})a} \log 1/d \leq a_d e^{-10^{-6}a} \log 1/d$$

$$= a_d e^{-(1+10^{-6})a} \log 1/d$$

$$= a_d e^{-10^{-6}a} \log D,$$

provided that

$$a_d e^{-10^{-6}a} \log 1/d < a_d e^D A \log D,$$

whence

$$\eta_d > a_d \log 1/d.$$  

(62)

Let $U$ denote the left-hand side of (57). Then, by (61) and (58)

$$U > a_d e^D A \log D.$$  

(63)

12. Introducing the number

$$\eta = \eta^{\eta_d} = D^{1+\eta},$$

we divide the sum $U_\eta$ (say) on the left of (52) up into four sums

$$U_\eta = S + h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \log b \exp \left( - \frac{\log b |a|}{4y} \right) + S_1 + hS_2,$$

where

$$S = h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \log b \exp \left( - \frac{\log b |a|}{4y} \right),$$

$$S_1 = h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{A(a)}{a^s},$$

$$S_2 = h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{A(a)}{a^{s+1}} \exp \left( - \frac{\log a |a|}{4y} \right).$$

(64)

For a sufficiently large $\eta < 1$ and any $T > D^\eta$ we have, by (44),

$$\sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{A(a)}{a^s} < Ak^{-1} \left( \frac{T}{D^\eta} \right)^{\frac{1}{s-1+\eta}} dt + \frac{T}{D^\eta} < Ak^{-1} T^s,$$

whence

$$S_1 < D \int \frac{1}{t} \exp \left( - \frac{\log t |a|}{4y} \right) \log \frac{t}{x} \log \frac{t}{y} \int \frac{dt}{t^\eta}$$

$$= \int \frac{1}{t} \exp \left( - \frac{\log t |a|}{4y} \right) \frac{\log t |a|}{2yt} \frac{\log t}{t} \int dt$$

$$< 2D \int \frac{1}{t} \exp \left( - \frac{\log t |a|}{4y} \right) \frac{\log t |a|}{2yt} \frac{\log t}{t} \int dt$$

$$= 2D \exp \left( - \frac{\log t |a|}{4y} \right).$$

(67)

By (65), (66) and (5)

$$S_2 < h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{\log b}{b^s} < h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{1}{a^s} + h \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{1}{a^{s+1}}$$

$$< \Delta \Delta^\prime a^s + \Delta a^s < \Delta a^s \eta_d a^s.$$  

(68)

Since $a_\eta = 1 - g$, $0 < g < \frac{1}{3}$, we have for any positive constant $g' < 1 - 2g$

$$\sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{\log b}{b^s} \exp \left( - \frac{\log b |a|}{4y} \right) < \sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{1}{a^{s+1}} + D^\eta$$

$$< \int \frac{dt}{t^{s+1}} \int \frac{dt}{t^{s+1}} \int \frac{dt}{t^{s+1}} \int \frac{dt}{t^{s+1}} < D^\eta,$$

whence for a sufficiently large $\eta < 1$

$$\sum_{a \in \mathcal{C}, 1 < a < \lambda, \eta_d} \frac{A(a)}{a^s} < \Delta a^s \eta_d a^s.$$  

(69)

And thus

$$hS_2 < \Delta a^s \eta_d.$$  

(70)

Having chosen the number $\Delta (\frac{1}{2} D^{-\eta_d}, \frac{1}{2})$, we take $M = D^\eta_x$ and $r \geq 4$. Then for any $a$ in absolute value $\geq M$ we have $|a| \Delta a^s \eta_d$.
whence by (50)
\[ \sum_{n \leq M} |d_n| < M^{-\eta_0} \]
and thus in (52)
\[ U_1 < h \sum \frac{A(a)}{a^\eta} \exp \left( -\frac{\log^2 a/\pi}{4y} \right) \sum_{n \leq M} |d_n| \]
\[ < D^{-\eta_0} h \sum \frac{A(a)}{a^\eta} \exp \left( -\frac{\log^2 a/\pi}{4y} \right) \]
\[ = D^{-\eta_0} \zeta \sqrt{\frac{y}{\pi}} \sum_{n \leq M} \Lambda(n) \sum_{x \leq \eta_0} \frac{L(x)}{L(\eta_0)} \sum_{x \leq \eta_0} \frac{L(x)}{L(\eta_0)} x^{-\eta_0} \exp \left( -\frac{\log^2 x/\pi}{4y} \right) \]
(say). In order to get an upper bound for \( P \) we may use the same method of residues as the one we used in dealing with \( U_1 \) (except that in the present case \( m = 0, d_0 = 1 \) and there are no other coefficients \( d_n ) \). Arguing as in (53)-(61) we can prove that

\[ I_B = \frac{U_1}{2\sqrt{\log x}} \exp \left( -\frac{\log^2 x/\pi}{4y} \right) \]

Let us denote by \( J_B \) the operation

\[ I_B = \frac{1}{2\sqrt{\log x}} \exp \left( -\frac{\log^2 x/\pi}{4y} \right) \]

Then for sufficiently large \( r, \eta_0 < 1 \) we have by (63), (64), (70), (62) and (71)

\[ J_B U_1 < \sum_{n \leq M} |d_n| \exp \left( -\frac{\log^2 n/\pi}{4y} \right) \]

\[ = J_B U_1 - J_B(S_0 + S_1 + hS_2) \]
\[ > J_B(U_1 + U_3) - J_B(S_0 + S_1 + hS_2) = J_B(U_1 + J_B U_1) - J_B(S_0 + S_1 + hS_2) \]
\[ > |J_B U_1| - |J_B U_1| - |J_B(S_0 + S_1 + hS_2)| > d_0 (\eta_0/4A) \log D - \eta_0 D^{-\eta_0} - \kappa_0 D^{-\eta_0} \]
\[ > d_0 (\eta_0/4A) \log D. \]

Since \( z < zD^\rho \) with \( E = 4(\eta_0 + B) \) and \( \eta_0 = \eta_0 \log \log 1/\Delta \), by (62), the theorem follows. If \( z > D^\rho \) (with appropriate \( \eta_0 \)), then by (3) the left-hand side of (72) does not exceed

\[ \pi(z, \zeta, H) \frac{D}{z^{1-\rho}} \]

which implies (7).

**Improvement of the theorem for \( z \to \infty \)**

15. Further on we suppose that (3) holds for any \( \xi > 0 \), \( D > D_1(\xi) \) and that for arbitrarily small positive constants \( \eta_1, \eta_2 \) and any \( D > D_2(\eta_1, \eta_2) \), \( x > 1 \) we have

\[ h \leq D^{-\xi}, \quad \sum \frac{1}{n \leq x} \leq c(n) D^x x^{-1+\eta_1}. \]

These conditions imply the estimate for exceptional zero \( \xi' = 1 - \delta' \)

\[ \xi' > D^{-\eta_1} \quad (D > D_1(\xi)) \]

with an arbitrarily small constant \( \xi > 0 \) (see [4], § 25).

We shall prove that if (73) holds (which is certainly true for an odd class number), then there are positive constants \( \alpha, \alpha' \) depending on \( \xi_0, \xi, \xi' \) such that for any positive \( \xi \leq \xi \) and any \( D > D_1(\xi, x) > x \) (where \( D_1 = D(\min(\eta_1, \eta_2)) \) in every class \( H \) there is a generator \( b \) with the image \( 4a^2 \) lying in the region

\[ (x < a < xD^\rho, a \in \mathcal{A}) \]

\( \mathcal{A} \) being defined by (6). And if \( a > aD^\rho \), then \( \pi(z, \mathcal{A}, H) > x D^\rho \log x. \)

The proof begins as in § 11. But now we use

\[ y = \frac{\eta_0}{\eta} \log D, \quad \eta_0 < \eta < \log \log 8/\Delta, \]
\[ 1 < \eta = v(x) \leq \min(\eta_0, \log D), \quad k = \frac{1}{4A} \log \log 8/\Delta, \]

where \( \alpha_0 < 1 \) and \( \eta_0 \) are large enough. Considering that

\[ I_B y^{-1/2} e^{-2\pi x/|D|} \leq e^{-\rho_0(\eta_0) \log D} \leq \begin{cases} 1 & \text{if } x > x_1 = (3 + \alpha_0)/\delta \delta_0 \\ D^{-\rho_0(\eta_0)} D^{-1-\eta_0} & \text{if } x < x_1, x_1 = v(x_1), \end{cases} \]

we deduce that the remaining term in (57) satisfies

\[ I_B y^{-1/2} e^{-\rho_0(\eta_0) \log D} D^{-1} \leq d_0 D^{-1}. \]
Let $G_1$ be the part of $G$ with $|x| \geq \log D$. Then in (58)

$$
T_1 \ll \sum_{D \geq p \geq D} e^{-(\log p)^{1+\log D}} \ll h M \int_{\log D} \int_{\log D} e^{-(\log p)^{1+\log D}} \log D \log D dt
$$

$$
< D^{1+2\log D} \int_{\log D} \int_{\log D} e^{-(\log p)^{1+\log D}} \log D \log dt
$$

$$
< D^{2\log D} \int_{\log D} e^{-p^2 \log D} dt < D^{3\log D} \int_{\log D} e^{-p^2 \log D} \log D \log D dt = \frac{D^{3\log D}}{c_1 \log D} D^{-3\log D},
$$

whence

$$
T_1 \ll d_4 D^{-1}.
$$

In the previous definition of $G_3$ we now take $\tau_1(\lambda) = \min(\lambda^3, \log^2 D)$. The corresponding part of (58) is

$$
T_2 \ll (2\pi)^{B \sum D_{p \leq D}} |d_3| e^{-(\log p)^{1+\log D}} (\log D)^{-B} \ll (2\pi) \sum D_{p \leq D} |d_3| e^{-(\log p)^{1+\log D}}
$$

$$
< (2\pi) \left[ \sum_{\xi \leq \log D} (\xi + \log p) e^{-(\log p)^{1+\log D}} \psi(\xi + 2 \log D) \right] (2 \log D)
$$

$$
< (2\pi) e^{-(\log p)^{1+\log D}} d_4 \log D \log D if \xi \geq 3C
$$

$$
< (2\pi) e^{-(\log p)^{1+\log D}} d_4 \log D \log D if \xi < 3C, \eta_p > 2^{-r} C(3C).
$$

In any case

$$
T_2 \ll d_4 e^{-(\log p)^{1+\log D}} \log D.
$$

Next, we have

$$
T_3 \ll (2\pi)^{B \sum D_{p \leq D}} |\lambda| e^{-(\log p)^{1+\log D}} |	au_1^{-B} = \left( \frac{2\pi}{\log D} \sum_{D_{p \leq D}} |\lambda| \right) e^{-(\log p)^{1+\log D}}
$$

$$
< 2\pi \sum_{D_{p \leq D}} \psi(\xi) e^{-\xi D} D \log D dt + e^{-(\log p)^{1+\log D}} \psi(2 \log D)
$$

$$
< 2\pi e^{-(\log p)^{1+\log D}} d_4 \log D.
$$

Hence, for a large $\eta < 1$

$$
T_3 \ll d_4 e^{-(\log p)^{1+\log D}} \log D.
$$

This proves the following analogue of (57):

$$
(75) \quad U = \frac{U_1}{2} \geq \frac{d_1 (I_{12} S - c_0 e^{-24\eta_p^{2\log D}} \log 1/D - c_3 D^{-1})},
$$

where

$$
I_{12} S \geq 1 - \exp(-24\eta_p^{2\log D}) D.
$$

Hence, by Lemma 4,

$$
(76) \quad I_{12} S \geq 1 - \exp(-24\eta_p^{2\log D}) D \geq 1 - \exp(-24\eta_p^{2\log D}) D \geq \frac{d_1 \log D}{2 A \varepsilon}. \tag{75}
$$

We may suppose that $k < A/3$ (see (74)). Then there is a number $\xi_1 (0 < \xi_1 < \log \log 8/D)$ which is the least positive $\xi$ such that

$$
\left\{ \begin{array}{l}
\xi \geq 1, \\
\eta_p + \log(4c_0 \log 1/D).
\end{array} \right.
$$

If $\xi \geq \xi_1$, then

$$
e^{-\eta_p^{2\log D}} \leq \frac{1}{4c_0 \log 1/D} \left( \eta_p^{2\log D} \right).
$$

whence, by (74)

$$
c_0 e^{-(\log D) D} \log 1/D \leq \frac{1}{4c_0 \log 1/D} \left( \eta_p^{2\log D} \right).
$$

Taking $\eta_p$ large enough:

$$
(77) \quad c_0 e^{-(\log D) D} \log 1/D \leq \frac{1}{4c_0 \log 1/D} \left( \eta_p^{2\log D} \right).
$$

we have

$$
\eta_p e^{-\eta_p^{2\log D}} \geq \left\{ \begin{array}{l}
1, \\
\eta_p^{2} + \log(4c_0 \log 1/D),
\end{array} \right.
$$

whence for any $\xi \in [0, \xi_1]$

$$
e^{-\eta_p^{2\log D}} \leq \frac{1}{4c_0 \log 1/D} \left( \eta_p^{2\log D} \right). \tag{75}
$$
and thus (cf. (77))
\[ c \tilde{c} \sim (\delta_2 \log D) \sim \frac{\delta_2 \log D}{A} \]

Hence in any case (77) holds for appropriate \( \eta_4 \) satisfying (78).
Since by (73)
\[ c_4 D^{-1} < (\delta_3 / 8 \pi) \log D, \]
from (75) (76) and (77) it follows that for any \( c_4 < 1 / 8 \pi \)
\[ U > \delta_4 (c_4 / \pi) \delta_4 \log D. \]

14. Using the number \( x = a^c \theta \) we divide up the sum \( U_0 \) on the left of (52) into four partial sums as in (64). By the arguments of § 12 we prove that
\[ S_1 + S_2 + h S_3 < AD^{-c \omega_4}, \]
whence
\[ I_2 = \frac{1}{2V} \sum_{x \in C} \left( S_1 + S_2 + h S_3 \right) < AD^{-c \omega_4}. \]

Hence, by (71) and (79)
\[ I_2 = \frac{1}{2V} \sum_{x \in C} \left( S_1 + S_2 + h S_3 \right) < AD^{-c \omega_4}. \]

Write
\[ E = \delta_4 (c_4 / \pi) \delta_4 \log D, \]
For any fixed positive \( \varepsilon \) let
\[ \varepsilon = \delta_4 (c_4 / \pi) \delta_4 \log D. \]

Then \( 1 < \varepsilon < \log D \), if \( D > D_0 (\varepsilon) \). Further on let \( x \geq D_0 (\varepsilon) \); then \( \varepsilon \geq h^{-1} \log (c / \varepsilon) \), whence
\[ c / \varepsilon \leq e^{D_1 (\varepsilon)}, \]
Some remarks on a series of Ramanujan

by

W. Staś (Poznań)

1. In my previous papers [7], [8] I was concerned with the Ramanujan series

\[ S(\beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\beta}} \]

where \( \mu(n) \) was the function of Möbius and \( \beta \) a real parameter.

G. H. Hardy and J. E. Littlewood have proved (see [1]) that

\[ S(\beta) = O(\beta^{-1}), \quad \beta \to \infty, \]

is equivalent to the conjecture of Riemann.

At present we shall prove by Turán’s methods the following theorem, which is stronger than my previous result (see [3]), based on Riemann’s hypothesis and on the conjecture that the \( \zeta \)-function has only simple zeros.

Theorem. Suppose Riemann’s conjecture. Then for \( T > C \)

\[ \max_{\tau_1, \ldots, \tau_l \in \epsilon} |S(\beta)| \gg T^{-\frac{1}{2} - \epsilon}. \]

In the proof we shall apply the method of Turán, namely we shall use the following modification ([2]) of Turán’s Satz X ([11]):

Lemma 1. Suppose that \( m \geq 0, \tau_1, \ldots, \tau_m \) are complex numbers with

\[ |\tau_1| \geq |\tau_2| \geq \ldots \geq |\tau_m| \geq \ldots \geq |\tau_2| \]

and

\[ |\tau_k| \geq 2 \frac{N}{N+m}, \quad |\tau_k| < |\tau_1| - \frac{N}{m+N}. \]

Then there exists an integer \( \mu \) with

\[ m \leq \mu \leq m+N \]