ACTA ARITHMETICA
X (1965)

On the abstract theory of primes II

by

E. Fogels (Riga)

Introduction

1. In the present paper we shall be concerned with an infinite semigroup $\mathfrak{G}$ on a countable number of generators $\mathfrak{b}$, the elements of $\mathfrak{G}$ generally being denoted by $\mathfrak{a}$. We suppose that they are distributed into classes H_j $(1 \leqslant j \leqslant h)$ forming a group Γ, and that the number of classes satisfies

$$(1) \qquad 1 \leqslant h \leqslant D,$$

where D is parameter $\geqslant D_0 > 2$ which may increase indefinitely. Using a homomorphism N of $\mathfrak{G}$ into the multiplicative semigroup of real numbers $\geqslant 1$ we denote the images $N\mathfrak{a}, N\mathfrak{b}, \ldots$ (called *norms* of $\mathfrak{a}, \mathfrak{b}, \ldots$) by $a, b, \ldots$ We take for granted that for any $x \geqslant 1$

$$(2) \qquad \sum_{\substack{\mathfrak{a} \in H_j \\ a \leqslant x}} 1 = \varkappa x + O(D^{c_1} x^{1-\vartheta}), \qquad \varkappa = D^l,$$

where the constants l, c_1, ϑ do not depend on j $(0 \lesseqgtr l \ll 1; 0 \leqslant c_1 \ll 1; 0 < \vartheta \leqslant 1)$. In a previous paper (see [4]) it has been proved in particular that for $\vartheta > \frac{1}{2}$ (actually for any $\vartheta > 0$ if h is odd) in every class H_j there is a generator $\mathfrak{b}$ which in norm does not exceed $D^{O(1)}$. In the case of an even h and $\vartheta \leqslant \frac{1}{2}$ the same estimate was proved supposing that for a suitable constant $c_2 > 0$ we have

$$(3) \qquad \lim_{x \to \infty} \Big(\sum_{\substack{\mathfrak{a} \in \Gamma' \\ a \leqslant x}} \frac{1}{a} - \sum_{\substack{\mathfrak{a} \notin \Gamma' \\ a \leqslant x}} \frac{1}{a} \Big) > D^{-c_2},$$

where Γ' denotes any subgroup of the group Γ with the index 2.

In the present paper we shall use a homomorphism into the multiplicative semigroup of complex numbers

$$(4) \qquad \sqrt{a}\, e^{2\pi i \alpha} \qquad (a = N\mathfrak{a} \geqslant 1,\ 0 \leqslant \alpha < 1).$$

We suppose that there is a single $\mathfrak{a} \in \mathfrak{G}$ with $a = 1$ and that $a = 1$ implies $\mathfrak{a} = 0$. (For this particular $\mathfrak{a}$ we shall sometimes write 1.) Now we take for granted that the images (4) of the elements $\mathfrak{a} \in \mathfrak{G}$ next to (1), (2) satisfy

$$(5) \qquad \sum_{\substack{\mathfrak{a}\in H_j \\ a \leqslant x,\, 0 \leqslant \alpha < \varphi}} 1 = \varkappa\varphi x + O(D^{c_1}x^{1-\vartheta'}), \qquad 0 < \vartheta' \leqslant \vartheta\,(^1)$$

(with ϑ' independent of j) uniformly in $0 < \varphi \leqslant 1$.

Our present task is the proof of some estimate for the least norm of a generator $\mathfrak{b}$ lying in a fixed angular region

$$(6) \qquad \mathscr{A}\,\{\alpha \equiv \alpha_0 + \theta\varDelta\ (\mathrm{mod}\,1)\}, \qquad 0 \leqslant \theta < 1,\ D^{-c_0} < \varDelta \leqslant 1$$

with an arbitrarily large constant $c_0 \ll 1$. To this end we shall prove the following

THEOREM. (i) *If $\vartheta > \frac{1}{2}$, then there is a positive constant c (depending merely on $c_0, c_1, l, \vartheta, \vartheta'$) such that for any $x \geqslant 1$ and any H_j in the region*

$$(x < a < xD_1^c,\ \alpha \in \mathscr{A}), \qquad D_1 = D^{\log\log 8/\varDelta}$$

there is a generator $\mathfrak{b} \in H_j$. For an odd class number h the conclusion holds as well in the case of $\vartheta \leqslant \frac{1}{2}$.

(ii) *Let h be even and $\vartheta \leqslant \frac{1}{2}$. If (3) is true, then the conclusion of* (i) *holds (with the constant c depending also on c_2).*

For $\varDelta = 1$ the theorem reduces to that of the previous paper [4] (with $q = 1$). The result has been announced in [5].

COROLLARY. *Let $\pi(x, \mathscr{A}, H_j)$ denote the number of generators $\mathfrak{b} \in H_j$ with $\alpha \in \mathscr{A}$ and $b \leqslant x$. For appropriate constants $c_3, c_4 > 0$ and any $x > D_1^{c_3}$ we have*

$$(7) \qquad \pi(x, \mathscr{A}, H_j) > x/D_1^{c_4}\log x.$$

If $x \to \infty$ and some other conditions are satisfied, then the theorem holds for regions

$$(x < a < xD_1^{\varepsilon},\ \alpha \in \mathscr{A})$$

with arbitrarily small positive ε. This will be proved in §§ 13, 14.

(1) In order to illustrate the inequality $\vartheta' \leqslant \vartheta$ let us take, for example, the semigroup $\mathfrak{G}$ (with $h = 1$) of the ordinary complex integers $\mathfrak{a}$ with norms $a = |\mathfrak{a}|^2$, considering the integers $\mathfrak{a}$ and $\mathfrak{a}i$ as identical. Then $\varkappa = \pi/4$ and we may take $D = 4$. The number of integers $\mathfrak{a}$ in the sector

$$S\alpha_0\,\{a \leqslant x,\ \alpha \equiv \alpha_0 + \theta/3\sqrt{x}\,(\mathrm{mod}\,1)\} \qquad (0 \leqslant \theta < 1)$$

is evidently $= \sqrt{x} + O(1)$ for $\alpha_0 = 0$ and $\leqslant \sqrt{x} + O(1)$ for any other α_0. In the present instance (5) holds with $\vartheta' = \frac{1}{2}$ (cf. [7], (675)) but it is not true for $\vartheta' > \frac{1}{2}$ (since there is no $\mathfrak{a}$ in $S\alpha_0$ with a positive $\alpha_0 \leqslant 1/8\sqrt{x}$). On the other hand (2) holds with $\vartheta = \frac{2}{3}$ (see, for example, [7], (682)) and thus $\vartheta' < \vartheta$.

The theorem is of interest chiefly in the case of $D \to \infty$ and $1/\varDelta \ll 1$ (see (6)). If on the contrary $1/\varDelta \to \infty$ and $D \ll 1$, then better results can be obtained by a simpler method. I hope to return to the latter case in another paper.

An application of the present theorem for primes representable by binary quadratic forms will be given in a continuation of this paper.

The method used in this second paper is the same as that of the previous one (the density method of Yu. V. Linnik). For the proof of the main auxiliary theorems we will use the method of Turán ([10]).

Further on $B, C, c, c_3, c_4, \ldots$ denote positive constants which may depend on $l, \vartheta, \vartheta', c_0, c_1$ and c_2 (if h is even and $\vartheta \leqslant \frac{1}{2}$). Generally they retain their meaning only throughout the same paragraph.

By $\mathfrak{d}|\mathfrak{a}$ we mean that $\mathfrak{d}, \mathfrak{a} \in \mathfrak{G}$ and that there is an $\mathfrak{a}' \in \mathfrak{G}$ such that $\mathfrak{a} = \mathfrak{d}\mathfrak{a}'$. If $\mathfrak{d}$ is in norm the largest element of $\mathfrak{G}$ for which $\mathfrak{d}|\mathfrak{a}_1$ and $\mathfrak{d}|\mathfrak{a}_2$, then we write $\mathfrak{d} = (\mathfrak{a}_1, \mathfrak{a}_2)$. By (a_1, a_2) we shall denote the corresponding number in the semigroup of norms (or sometimes the interval $a_1 < x < a_2$). For the norms of $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \ldots \in \mathfrak{G}$ we shall write $a, b, d, \ldots$, respectively.

The complex variable will be denoted by $s = \sigma + it$ ($\sigma = \mathrm{re}\,s$, $t = \mathrm{im}\,s$).

The functions $\zeta(s, X)$ and their zeros near the line $\sigma = 1$

2. LEMMA 1. *Let the sequence of real numbers x_n $(n = 1, 2, \ldots, N)$ be distributed uniformly* mod 1 *with the remaining term $\ll R$, that is to say, for any $\varphi \in [0, 1]$ the number N_φ of numbers x_n with $x_n \equiv \theta\varphi\,(\mathrm{mod}\,1)$ $(0 \leqslant \theta < 1)$ satisfy $N_\varphi - \varphi N \ll R$. Then for any integer $m \neq 0$*

$$(8) \qquad \sum_{n \leqslant N} e^{2\pi i m x_n} \ll |m|R.$$

Proof. Writing

$$(9) \qquad N_\varphi - \varphi N = R(\varphi)$$

we have

$$(10) \qquad R = \max_{0 \leqslant \varphi \leqslant 1} |R(\varphi)|.$$

By (9) and Abel's identity (see [8], p. 371)

$$(11) \qquad \sum_{n \leqslant N} e^{2\pi i x_n} = -\int_0^1 \{\varphi N + R(\varphi)\}\, 2\pi i e^{2\pi i\varphi} d\varphi + N = -2\pi i \int_0^1 R(\varphi)\, e^{2\pi i \varphi} d\varphi.$$

Let $R_m(\varphi)$ (for any fixed positive integer m) denote the last term in (9) when the sequence x_n is replaced by mx_n. Then, by (11),

$$(12) \qquad \sum_{n \leqslant N} e^{2\pi i m x_n} = -2\pi i \int_0^1 R_m(\varphi)\, e^{2\pi i \varphi} d\varphi.$$

Let A_φ (for any fixed $\varphi\epsilon[0,1]$) be the number of solutions x_n of the inequality

$$mx_n \equiv \theta\varphi \ (\mathrm{mod}\, 1) \quad (0 \leqslant \theta < 1). \tag{13}$$

By the definition of $R_m(\varphi)$ we have

$$A_\varphi = \varphi N + R_m(\varphi). \tag{14}$$

Now, (13) is equivalent to $mx_n = k+\theta\varphi$ (k integer), i.e.

$$x_n = \frac{k}{m} + \theta\frac{\varphi}{m}. \tag{15}$$

The last condition is satisfied only by those x_n which lie in any of the following intervals mod1:

$$\left[0, \frac{\varphi}{m}\right), \left[\frac{1}{m}, \frac{1+\varphi}{m}\right), \ldots, \left[\frac{m-1}{m}, \frac{m-1+\varphi}{m}\right).$$

Hence, according to (9), the number of the solutions x_n of (15) is

$$\frac{\varphi}{m}N + R\left(\frac{\varphi}{m}\right) + \frac{\varphi}{m}N + R\left(\frac{1+\varphi}{m}\right) - R\left(\frac{1}{m}\right) + \ldots +$$
$$+ \frac{\varphi}{m}N + R\left(\frac{m-1+\varphi}{m}\right) - R\left(\frac{m-1}{m}\right)$$
$$= \varphi N + R\left(\frac{\varphi}{m}\right) + \sum_{1\leqslant j\leqslant m-1}\left\{R\left(\frac{j+\varphi}{m}\right) - R\left(\frac{j}{m}\right)\right\}.$$

From this and (14) we deduce

$$R_m(\varphi) = R\left(\frac{\varphi}{m}\right) + \sum_{1\leqslant j\leqslant m-1}\left\{R\left(\frac{j+\varphi}{m}\right) - R\left(\frac{j}{m}\right)\right\}.$$

Hence, by (10),

$$\max_{0\leqslant\varphi\leqslant1}|R_m(\varphi)| < 2mR.$$

Using this estimate in (12) we get (8) for a positive m. And the case of $m<0$ can be reduced to that of $m>0$.

3. Let x have a higher order of magnitude than $(D^{c_1}/\varkappa)^{1/\vartheta}$. Then by (2) and (5) the numbers

$$a = a_n \ \{n = 1, \ldots, N;\ N = \varkappa x + O(D^{c_1}x^{1-\vartheta})\}$$

corresponding to the points (4) (with $a\leqslant x$) of any class $H = H_j$ are distributed uniformly mod1 with the remaining term $R \ll D^{c_1}x^{1-\vartheta'}$. Hence, writing

$$\xi = \xi(\mathfrak{a}) = e^{2\pi i a}$$

we have by Lemma 1 for any integer $m\neq 0$

$$\sum_{\mathfrak{a}\epsilon H, a\leqslant x}\xi^m \ll |m|D^{c_1}x^{1-\vartheta'}. \tag{16}$$

Now let us introduce the function

$$\zeta(s, H, \xi^m) = \sum_{\mathfrak{a}\epsilon H}\xi(\mathfrak{a})^m a^{-s} \quad (\sigma > 1) \tag{17}$$

and the number

$$\vartheta_1 = \begin{cases} \vartheta-\eta \ (0<\eta\leqslant\frac{1}{8}\vartheta) & \text{if} \quad m = 0, \\ \frac{1}{2}\vartheta' & \text{if} \quad m\neq 0. \end{cases} \tag{18}$$

Lemma 2. *The function* (17) *is regular in the half-plane* $\delta\geqslant 1-\vartheta_1$, *except for a simple pole at* $s=1$ *with residue* $\varkappa$ *in the case of* $m=0$ *and we have uniformly in the strip* G $(1-\vartheta_1\leqslant\sigma\leqslant 2)$:

$$\begin{aligned} &\zeta(s, H, 1) - \varkappa(s-1)^{-1} \ll \eta^{-1}D^c|s|, \\ &\zeta(s, H, \xi^m) \ll D^{c_1}|ms| \qquad (m\neq 0). \end{aligned} \tag{19}$$

Proof. For $m=0$ this has been proved in [4], § 3. Further let m be a fixed integer $\neq 0$ and $f(x)$ denote the sum (16). Then in $\sigma>1$

$$\begin{aligned} &\zeta(s, H, \xi^m) = \sum_{\mathfrak{a}\epsilon H}\xi^m a^{-s} = s\int_1^\infty \frac{f(x)}{x^{s+1}}dx = s\sum_{n=1}^\infty g_n(s), \\ &g_n(s) = \int_n^{n+1}\frac{f(x)}{x^{s+1}}dx. \end{aligned} \tag{20}$$

Now we can find a number $x_0 = x_0(m) > 1$ such that

$$x_0 \geqslant (D^{c_1}/\varkappa)^{2/\vartheta}, \qquad x_0\leqslant(|m|D^{c_1}/\varkappa)^{2/\vartheta'}. \tag{21}$$

Then, by (16), for any $n\geqslant x_0$ we have in G

$$g_n(s) \ll D^{c_1}|m|\int_n^{n+1}\frac{x^{1-\vartheta'}}{x^{2-\vartheta'/2}}dx \ll D^{c_1}|m|\{n^{-\vartheta'/2}-(n+1)^{-\vartheta'/2}\}.$$

Hence by (20) $\zeta(s, H, \xi^m)$ is regular in $\sigma\geqslant 1-\vartheta_1$ and (19) holds, since by (20), (16), (2), (18) and (21) in G

$$\sum_{n\leqslant x_0}g_n(s) = \int_1^{x_0}\frac{f(x)}{x^{s+1}}dx \ll \int_1^{x_0}\frac{\varkappa x + D^{c_1}x^{1-\vartheta}}{x^{2-\vartheta_1}}dx \ll \varkappa\int_1^{x_0}\frac{x}{x^{2-\vartheta_1}}dx + D^{c_1}$$
$$\ll \varkappa x_0^{\vartheta_1} + D^{c_1} \ll \varkappa x_0^{\vartheta'/2} + D^{c_1} \ll |m|D^{c_1}.$$

4. Let χ denote the characters of the group Γ, χ_0 being principal character, and let $\chi(\mathfrak{a}) = \chi(H)$ for all $\mathfrak{a}\in H$. Write

$$X(\mathfrak{a}) = \chi(\mathfrak{a})\,\xi(\mathfrak{a})^m = \chi(\mathfrak{a})\,e^{2\pi i m a}. \tag{22}$$

Now we introduce the function

$$\zeta(s, X) = \zeta(s, \chi, \xi^m) = \sum_H \chi(H)\zeta(s, H, \xi^m) \tag{23}$$

$$= \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})\,\xi(\mathfrak{a})^m}{\mathfrak{a}^s} = \sum_{\mathfrak{a}} \frac{X(\mathfrak{a})}{\mathfrak{a}^s} \qquad (\sigma > 1).$$

Since

$$\sum_H \chi(H) = \begin{cases} h & \text{if} \quad \chi = \chi_0, \\ 0 & \text{otherwise}, \end{cases}$$

by Lemma 2 the function (23) is regular in $\sigma \geqslant 1-\vartheta_1$, except for a simple pole at $s = 1$ with residue $h\varkappa$ in the case of $X = X_0$ (when $\chi = \chi_0$ and $m = 0$). By (1), (23), (19) we have in $G\,(1-\vartheta_1 \leqslant \sigma \leqslant 2)$

$$\zeta(s, \chi, \xi^m) - e_0 h\varkappa(s-1)^{-1} \ll (\vartheta-\vartheta_1)^{-1} D^c(1+|m|)(1+|t|) \tag{24}$$

where

$$e_0 = \begin{cases} 1 & \text{if } \chi = \chi_0 \text{ and } m = 0, \\ 0 & \text{otherwise}. \end{cases}$$

$\mathfrak{G}$ being a semigroup with the generators $\mathfrak{b}$, in the half-plane $E\,(\sigma > 1)$ we have

$$\zeta(s, X) = \prod_{\mathfrak{b}} (1 - X(\mathfrak{b})\,b^{-s})^{-1}$$

whence there are no zeros of $\zeta(s, X)$ in E.

Let $\mu(\mathfrak{a}) = (-1)^\nu$ if $\mathfrak{a}$ is a product of ν distinct generators, and $= 0$ if $\mathfrak{b}^2|\mathfrak{a}$. Further let

$$\Lambda(\mathfrak{a}) = \begin{cases} \log b & \text{if} \quad \mathfrak{a} = \mathfrak{b}^n \quad (n \geqslant 1), \\ 0 & \text{otherwise}. \end{cases}$$

Using the product-form of $\zeta(s, X)$ we can prove that

$$1/\zeta(s, X) = \sum_{\mathfrak{a}} X(\mathfrak{a})\mu(\mathfrak{a})\,a^{-s},$$

(25)

$$\zeta'/\zeta(s, X) = -\sum_{\mathfrak{a}} X(\mathfrak{a})\Lambda(\mathfrak{a})\,a^{-s} \qquad (\sigma > 1).$$

Hence by (24) for any $\eta\in(0,1]$

$$1/\zeta(1+\eta+it, X) \ll h\varkappa\eta^{-1} + D^c(1+|m|)(1+|t|). \tag{26}$$

Now taking

$$\eta = 1/D(1+|m|)(1+|t_0|), \qquad s_0 = 1+\eta+it_0$$

and using (24) and (26) we can prove (cf. [4], (19)) that in $|s-s_0| \leqslant \frac{1}{4}\vartheta_1$

$$\zeta'/\zeta(s, X) + \frac{e_0}{s-1} - \sum_{|\varrho-s_0|<\vartheta_1/2} (s-\varrho)^{-1} \ll \log D(1+|m|)(1+|t_0|) \tag{27}$$

where ϱ runs through the zeros of $\zeta(s, X)$. By (27) and the arguments used in [2], § 11

$$|\zeta'/\zeta(\sigma_0, X_0)| < \tfrac{5}{4}(\sigma_0-1)^{-1}$$

where $\sigma_0 = 1 + c_3/\log D$ and c_3 is small enough. For any positive $r \leqslant 1$ we have (cf. [4], (21))

$$|\zeta'/\zeta(1+r, X_0)| \leqslant 1/r + c_4\log D.$$

Let $\nu = \nu(r, X, t_0)$ denote the number of zeros of $\zeta(s, X)$ in $|s-1-it_0| < r$. If

$$c_5/\log D(1+|m|)(1+|t_0|) \leqslant r < \vartheta_1/4 - 1/D(1+|m|)(1+|t_0|),$$

then (cf. [2], § 10)

$$\nu \ll r\log D(1+|m|)(1+|t_0|). \tag{28}$$

By the arguments of [4], § 6 (with ϑ_1 instead of ϑ) we can prove that the number of zeros of $\zeta(s, X)$ in the rectangle $(1-\vartheta_1/2 \leqslant \sigma \leqslant 1$, $|t-t_0| \leqslant \frac{1}{2})$ does not exceed $\ll \log D(1+|m|)(1+|t_0|)$.

5. Now we can repeat the arguments of [4], §§ 7-10 with $D(1+|m|)$, X instead of D, χ (and with $q = 1$). Considering that by (22) a real X implies $m = 0$ we get the following

FUNDAMENTAL LEMMA 3. *For appropriate c in the region*

$$\sigma \geqslant 1 - c/\log D(1+|m|)(1+|t|) \qquad (\geqslant 1-\vartheta'/24) \tag{29}$$

there are no zeros of $\zeta(s, X)$ with a complex $X = \chi(\mathfrak{a})\,\xi(\mathfrak{a})^m$. For at most one real X in (29) with $m = 0$, $t = 0$ there may be a simple real zero

$$\varrho' = 1 - \delta' \leqslant 1. \tag{30}$$

ϱ' (if it exists) will be called the *exceptional zero* of $\zeta(s, X)$. If the conditions of the theorem of § 1 are satisfied, then we have in (30) $\delta' > D^{-c_3}$ (see [4], § 19, Lemma 22).

Further on in (22) let m satisfy

(31) $$|m| \leqslant D^{2c_0},$$

c_0 being defined by (6). Then by the arguments used in [4], §§ 20, 21 we can prove the following

FUNDAMENTAL LEMMA 4. *Let δ' be defined by* (30). *For appropriate $A \ll 1$ and*

(32) $$\delta_0 = \min(\delta', A/\log D), \qquad \lambda_0 = A\log\frac{eA}{\delta_0 \log D} \epsilon [A, \vartheta \log D]$$

there are in $(1-\lambda_0/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant D)$ *no other zeros of the function* $\prod_{X} \zeta(s, X)$ *(with m satisfying* (31)) *than at most the exceptional zero* (30).

An upper bound for the number of generators

6. LEMMA 5. *Let*

(33) $$\mathfrak{a}_n \qquad (n = 1, 2, \ldots, N)$$

be a set of elements $\epsilon\, \mathfrak{G}$ such that for any fixed $\mathfrak{q} \epsilon \mathfrak{G}$ we have

(34) $$\sum_{\mathfrak{q}|\mathfrak{a}_n} 1 = N/f(\mathfrak{q}) + R_{\mathfrak{q}}$$

where $f(\mathfrak{q})$ is a positive function satisfying $f(\mathfrak{q}_1\mathfrak{q}_2) = f(\mathfrak{q}_1)f(\mathfrak{q}_2)$ whenever $(\mathfrak{q}_1, \mathfrak{q}_2) = 1$. Further let N_z (for any $z > 1$) denote the number of those elements $\mathfrak{a}_n$ of (33) *which are not divisible in $\mathfrak{G}$ with any generator $\mathfrak{b}$ in norm less than z. Write*

$$F(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} \mu(\mathfrak{b}) f\left(\frac{\mathfrak{a}}{\mathfrak{b}}\right), \qquad S_z = \sum_{\substack{\mathfrak{q} \\ q \leqslant z}} \frac{\mu^2(\mathfrak{q})}{F(\mathfrak{q})}, \qquad S_z(\mathfrak{a}) = \sum_{\substack{\mathfrak{q} \\ (\mathfrak{q},\mathfrak{a})=1 \\ q \leqslant z/a}} \frac{\mu^2(\mathfrak{q})}{F(\mathfrak{q})},$$

$$\lambda_{\mathfrak{a}} = \begin{cases} \mu(\mathfrak{a}) \prod_{\mathfrak{b}|\mathfrak{a}} \left(1 - 1/f(\mathfrak{b})\right)^{-1} S_z(\mathfrak{a})/S_z & \text{if } a \leqslant z, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(35) $$N_z \leqslant N/S_z + \sum_{\substack{\mathfrak{a}_1, \mathfrak{a}_2 \\ a_1 \leqslant z, a_2 \leqslant z}} |\lambda_{\mathfrak{a}_1} \lambda_{\mathfrak{a}_2} R_{\mathfrak{a}_1\mathfrak{a}_2/(\mathfrak{a}_1,\mathfrak{a}_2)}|.$$

This may be proved by the sieve method of A. Selberg ([9]). Cf. [3], § 3.

7. LEMMA 6. *Let* (33) *in the previous lemma be all the elements of any class H_j* of $\mathfrak{G}$ *for which in* (4)

$$a \leqslant x, \qquad \alpha \equiv \alpha_0 + \theta\varphi (\mathrm{mod}\, 1) \qquad (0 \leqslant \theta < 1;\ \varphi \text{ fixed},\ D^{c_1}/\varkappa x^{\vartheta'} < \varphi \leqslant 1).$$

If $z \geqslant x^{1/5}$ and $x \geqslant D^{c_3}$ (where $c_3 \ll 1$ is large enough), then the main term in (35) *does not exceed $c_4\varphi x/h\log x$.*

Proof. By (5) we have

(36) $$N = \varkappa\varphi x + O(D^{c_1}x^{1-\vartheta'}).$$

By (5) and (36) the number of elements (33) with $\mathfrak{q}|\mathfrak{a}_n$ for any $\mathfrak{q} \epsilon \mathfrak{G}$ is

$$\varkappa\varphi\frac{x}{q} + O\left(D^{c_1}\left(\frac{x}{q}\right)^{1-\vartheta'}\right) = \frac{N + O(D^{c_1}x^{1-\vartheta'})}{q} + O\left(D^{c_1}\left(\frac{x}{q}\right)^{1-\vartheta'}\right)$$
$$= \frac{N}{q} + O\left(D^{c_1}\left(\frac{x}{q}\right)^{1-\vartheta'}\right).$$

Hence (34) holds for

(37) $$f(\mathfrak{q}) = q, \qquad R_{\mathfrak{q}} \ll D^{c_1}\left(\frac{x}{q}\right)^{1-\vartheta'}$$

and thus

$$S_z = \sum_{\substack{\mathfrak{q} \\ q \leqslant z}} \frac{\mu^2(\mathfrak{q})}{q\prod_{\mathfrak{b}|\mathfrak{q}}(1-1/b)} = \sum_{\substack{\mathfrak{q} \\ q \leqslant z}} \mu^2(\mathfrak{q}) \prod_{\mathfrak{b}|\mathfrak{q}} \left(\frac{1}{b} + \frac{1}{b^2} + \ldots\right) = \sum_{\mathfrak{a}\epsilon(z)} \frac{1}{a}$$

where (z) denotes the set of elements $\mathfrak{a} \epsilon \mathfrak{G}$ such that the product of all different generators of any $\mathfrak{a}$ is in norm $\leqslant z$. Hence

$$S_z > \sum_{\substack{\mathfrak{a} \\ a \leqslant z}} \frac{1}{a} > \sum_{\substack{\mathfrak{a} \\ \sqrt{z} < a \leqslant z}} \frac{1}{a} > \int_{\sqrt{z}}^{z} \frac{\nu(y)}{y^2}\, dy$$

where $\nu(y) = \sum_{\substack{\mathfrak{a} \\ a \leqslant y}} 1$.

By (2) $\nu(g) > \frac{1}{2}h\varkappa y$ (for a sufficiently large c_3) and thus

$$S_z > \frac{1}{4}h\varkappa\log z > \frac{1}{20}h\varkappa\log x, \qquad \frac{N}{S_z} < c^4\frac{\varkappa\varphi x}{h\varkappa\log x} = \frac{c_4\varphi x}{h\log x}.$$

8. LEMMA 7. *Let W denote the remaining term in* (35) *and let in Lemma* 6

(38) $$z^{2\vartheta'} = \varphi\frac{x^{\vartheta'}}{h^3\varkappa^2D^{c_4}\log x}, \qquad c_4 = 1 + c_1 + \max(c_1, l), \qquad \varphi \geqslant x^{-\vartheta_0}$$

for any positive constant $\vartheta_0 < \vartheta'$ and for $x > D^{c_3}$ with a sufficiently large $c_3 = c_3(\vartheta_0) \ll 1$. Then

$$W < c_5\varphi x/h\log x, \qquad c_5 = c_5(\vartheta_0, c_3).$$

Proof. Since $|\lambda_{\mathfrak{a}}| \leqslant 1$ (cf. [3], (38)), by (35), (37),

$$W \ll D^{c_1} x^{1-\vartheta'} \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(\mathfrak{a}_1, \mathfrak{a}_2)}{a_1 a_2}\right)^{1-\vartheta'},$$

and thus we have to prove that

$$\sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(a_1, a_2)}{a_1 a_2}\right)^{1-\vartheta'} \ll \varphi \frac{x^{\vartheta'}}{hD^{c_1}\log x}. \tag{39}$$

From (2) we can deduce

$$\sum_{\substack{\mathfrak{a} \\ a\leqslant z}} a^{-1-\vartheta'} \ll h\varkappa z^{\vartheta'}, \quad \sum_{\substack{\mathfrak{a} \\ a\leqslant z}} a^{-1-\vartheta'} \ll D^{c_4-c_1}, \tag{40}$$

whence

$$\sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ (\mathfrak{a}_1,\mathfrak{a}_2)=1 \\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(a_1, a_2)}{a_1 a_2}\right)^{1-\vartheta'} \leqslant \Big(\sum_{\substack{\mathfrak{a} \\ a\leqslant z}} a^{-1+\vartheta'}\Big)^2 \leqslant (h\varkappa z^{\vartheta'})^2. \tag{41}$$

Writing

$$S_{\mathfrak{b}}(z) = \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ (\mathfrak{a}_1,\mathfrak{a}_2)=\mathfrak{b} \\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(a_1, a_2)}{a_1 a_2}\right)^{1-\vartheta'},$$

we have, by (41),

$$S_{\mathfrak{b}}(z) = d^{-1+\vartheta'} S_1\left(\frac{z}{d}\right) \ll d^{-1+\vartheta'} h^2 \varkappa^2 \left(\frac{z}{d}\right)^{2\vartheta'} = h^2\varkappa^2 z^{2\vartheta'} d^{-1-\vartheta'}.$$

Hence, by (40) and (38),

$$\sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ a_1\leqslant z, a_2\leqslant z}} \left(\frac{(a_1, a_2)}{a_1 a_2}\right)^{1-\vartheta'} \leqslant \sum_{\substack{\mathfrak{b} \\ d\leqslant z}} S_{\mathfrak{b}}(z) \ll (h\varkappa)^2 z^{2\vartheta'} \sum_{\substack{\mathfrak{b} \\ d\leqslant z}} d^{-1-\vartheta'}$$

$$\ll D^{c_4-c_1}(h\varkappa)^2 z^{2\vartheta'} = \varphi \frac{x^{\vartheta'}}{hD^{c_1}\log x},$$

which proves (39).

9. LEMMA 8. *Let $\pi_H(x, \varphi, a_0)$ denote the number of generators $\mathfrak{b} \in H$ with*

$$b \leqslant x, \quad \alpha \equiv \alpha_0 + \theta\varphi \ (\mathrm{mod}\, 1) \quad (0 \leqslant \theta < 1),$$

where $x^{-\vartheta_0} < \varphi \leqslant 1$ (for any positive constant $\vartheta_0 < \vartheta'$, $x > D^{c_3}$ and $c_3 = c_3(\vartheta_0)$ large enough). Then for appropriate c_4 (which does not depend on α_0)

$$\pi_H(x, \varphi, \alpha_0) < c_4 \varphi x/h \log x. \tag{42}$$

Proof. Since, by (38), $z < x^{3/5}$, it follows from Lemma 6 that all the generators larger in norm than z and satisfying the conditions of the present lemma are in the set of the N_z elements as defined in Lemma 5. Hence, by (2) and Lemmas 6, 7

$$\pi_H(x, \varphi, \alpha_0) \leqslant N_z + \pi_H(z, \varphi, \alpha_0) \leqslant N_z + \varkappa\varphi z + O(D^{c_1} z^{1-\vartheta'}) < c_4\varphi x/h \log x.$$

COROLLARY. *If $\varphi > x^{-\vartheta_0}$, $x > D^{c_3}$, then*

$$\sum_{\substack{\mathfrak{b}\in H, b\leqslant x \\ \alpha\equiv\alpha_0+\theta\varphi (\mathrm{mod}\, 1)}} \log b < c_5 \varphi x/h, \tag{43}$$

$$\sum_{\substack{\mathfrak{a}\in H, a\leqslant x \\ \alpha\equiv\alpha_0+\theta\varphi (\mathrm{mod}\, 1)}} \Lambda(\mathfrak{a}) < c_6 \varphi x/h. \tag{44}$$

Proof. The left-hand side of (43) being $< \pi_H(x, \varphi, \alpha_0)\log x$, the estimate holds by (42). By (43) and the definition of $\Lambda(\mathfrak{a})$, (44) follows from the estimate

$$\sum_{\substack{\mathfrak{b}^2\in H, b^2\leqslant x \\ 2\alpha\equiv\alpha_0+\theta\varphi (\mathrm{mod}\, 1)}} \log b + \sum_{\substack{\mathfrak{b}^3\in H, b^3\leqslant x \\ 3\alpha\equiv\alpha_0+\theta\varphi (\mathrm{mod}\, 1)}} \log b + \ldots < c_7\varphi x/h,$$

which is evident (for a sufficiently large $c_3 \ll 1$), since the number of term on the left is $< D^{c_8}\log x$, none of them exceeding $\{\varphi\varkappa\sqrt{x} + O(D^{c_1} x^{(1-\vartheta')/2})\}\log x$ (cf. [4], § 14).

A density lemma

10. Let $X \neq X_0$ and let the exponent m in (22) satisfy (31). Then, by (27) and (28), the conditions of [4], Lemma 16 (i) (with $\vartheta_0 = \frac{1}{5}\vartheta_1$) for the functions $F(s) = \zeta'/\zeta(s, X)$ are fulfilled. Further let for any real τ, for $A > 1$ and any integer $k \geqslant 2$

$$J_X(\tau, k, A) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left(\frac{e^{3As} - e^{As}}{2As}\right)^k \frac{\zeta'}{\zeta}(s+1+i\tau, X)\, ds. \tag{45}$$

LEMMA 9. *Let in (22) $|m| \leqslant M$ $(1 \leqslant M \leqslant D^{2c_0})$ and in (45) $|\tau| \leqslant D$. If $e^{kA} > D^{c_3}$, where c_3 is the constant of § 9, then*

$$\sum_X |J_X(\tau, k, A)|^2 < e^{c_4 k}\log eM. \tag{46}$$

Proof. Writing

$$R(a) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{e^{3As}-e^{As}}{2As}\right)^k e^{-s\log a}\,ds,$$

we have, by (45) and (25)

$$J_X(\tau, k, A) = \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})R(\mathfrak{a})}{\mathfrak{a}^{1+i\tau}} X(\mathfrak{a}) = \sum_{\substack{\mathfrak{a}\\ e^{kA}<a<e^{3kA}}} e_{\mathfrak{a}} X(\mathfrak{a})$$

(say), since $R(a) = 0$ outside the interval $e^{kA} < a < e^{3kA}$ (see [4], (61)). Let S denote the left-hand side of (46). Then

$$S = \sum_X \sum_{\mathfrak{a}_1,\mathfrak{a}_2\in H} e_{\mathfrak{a}_1}\bar{e}_{\mathfrak{a}_2} X(\mathfrak{a}_1)\bar{X}(\mathfrak{a}_2) \sum_{\mathfrak{a}_1,\mathfrak{a}_2}\sum_{|m|\leqslant M}\sum_{\chi} \tag{47}$$

$$= h\sum_H \Big\{\sum_{|m|\leqslant M}\sum_{\mathfrak{a}_1,\mathfrak{a}_2\in H} e_{\mathfrak{a}_1}\bar{e}_{\mathfrak{a}_2} e^{2\pi i m\alpha_1} e^{-2\pi i m\alpha_2}\Big\},$$

since

$$\sum_{\chi}\chi(H) = \begin{cases} h & \text{if } H \text{ is the principal class } H_1,\\ 0 & \text{otherwise.}\end{cases} \tag{48}$$

Now using an integer $N>1$ and considering that $e_{\mathfrak{a}} = \Lambda(\mathfrak{a})R(\mathfrak{a})\mathfrak{a}^{-1-i\tau}$ does not depend on α, we can write the sum U in brackets of (47) as follows:

$$U = \sum_{\mathfrak{a}_1\in H} e_{\mathfrak{a}_1}\Big(\sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|<1/N}} \bar{e}_{\mathfrak{a}} \sum_m e^{2\pi i m(\alpha_1-\alpha)} + \sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|\in[1/N,2/N)}} +\ldots\Big).$$

Since for any real $\alpha\in[0,1)$

$$\sum_{|m|\leqslant M} e^{2\pi i m\alpha} \ll \min\{M, 1/\min(\alpha, 1-\alpha)\}$$

(cf. [8], p. 189), we have

$$U \ll \sum_{\mathfrak{a}_1\in H}|e_{\mathfrak{a}_1}|\Big(M\sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|<1/N}}|e_{\mathfrak{a}}| + \frac{N}{1}\sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|\in[1/N,2/N)}}|e\,| + \frac{N}{2}\sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|\in[2/N,3/N)}}|e_{\mathfrak{a}}| + \ldots\Big).$$

Using (44) and the estimate

$$|R(a)| < e^{c_5 k}/A \quad \text{for} \quad e^{kA} < a < e^{3kA}$$

(see [4], (61)), we deduce

$$\sum_{\substack{\mathfrak{a}\in H\\ |\alpha-\alpha_1|\in[j/N,(j+1)/N)}} |e_{\mathfrak{a}}| \ll \frac{e^{c_5 k}}{A} \sum_{\substack{\mathfrak{a}\in H\\ e^{kA}<a<e^{3kA}\\ |\alpha-\alpha_1|\in[j/N,(j+1)/N)}} \frac{\Lambda(\mathfrak{a})}{a} \ll \frac{e^{c_5 k}}{ANh}\Big(\int_{e^{kA}}^{e^{3kA}}\frac{dx}{x}+1\Big)$$

$$\ll \frac{e^{c_5 k}}{ANh}kA = \frac{ke^{c_5 k}}{hN}.$$

Hence

$$U \ll \sum_{\mathfrak{a}_1\in H}|e_{\mathfrak{a}_1}|\frac{ke^{c_5 k}}{h}\Big(\frac{M}{N}+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{[N/2]}\Big)$$

$$\ll \frac{ke^{c_5 k}}{h}\Big(\frac{M}{N}+\log N\Big)\sum_{\mathfrak{a}\in H}|e_{\mathfrak{a}}| \ll \frac{ke^{c_5 k}}{h}\Big(\frac{M}{N}+\log N\Big)\frac{ke^{c_5 k}}{h}$$

$$= \Big(\frac{k}{h}\Big)^2 e^{2c_5 k}\Big(\frac{M}{N}+\log N\Big).$$

Taking $N = M$ we get

$$U < h^{-2}e^{c_4 k}\log eM.$$

From this and (47) follows (46).

Now let $\nu = \nu(\tau_0, \lambda)$ for any selected $\tau_0\in[-D, D]$ and let $\lambda\in[c_6, c_7\log D]$ (with appropriate c_6, c_7) denote the number of functions $\zeta(s, X)$ (with $|m|\leqslant M$; $M\leqslant D^{2c_0}$) having at least one zero $\varrho = \varrho(X)$ in the square Q $(1-\lambda/\log D\leqslant\sigma\leqslant 1,\ |t-\tau_0|\leqslant\lambda/2\log D)$. Then for at least $\nu/c_8\lambda$ functions $F(s) = \zeta'/\zeta(s, X)$ we have, by [4], (49),

$$|J(\tau_0, k, \lambda^{-1}\log D)| > e^{-c_9\lambda}$$

with the same $k = k_1 < c_8\lambda$. Hence, by (46),

$$\frac{\nu}{c_8\lambda}e^{-2c_9\lambda} < \sum_X |J_X(\tau_0, k_1, \lambda^{-1}\log D)|^2 < e^{c_4 k}\log eM,$$

whence

$$\nu < e^{c_{10}\lambda}\log eM.$$

Combining this with (28) and arguing as in [4], § 18, we can prove the following

FUNDAMENTAL LEMMA 9. *Let N_λ denote the number of zeros of the function $\prod_X\zeta(s, X)$ (with m in (22) satisfying $|m|\leqslant M$; $M\leqslant D^{2c_0}$) in the* *rectangle*

$$R_\lambda \quad (1-\lambda/\log D \leqslant \sigma \leqslant 1,\ |t| < e^{\lambda}/\log D) \qquad (0 < \lambda \leqslant \tfrac{1}{4}\vartheta' \log D).$$

Then for appropriate C (²)

(49) $$N_\lambda < e^{C\lambda} \log eM.$$

Proof of the theorem

11. LEMMA 10. *Let r denote any fixed integer $\geqslant 1$, and let $0 < \Delta < \frac{1}{2}$, $0 \leqslant a_2 - a_1 \leqslant 1-2\Delta$. There is a periodic function $f(a)$ of the real variable a with the period 1 such that* (i) $f(a) = 1$ *in* $a_1 \leqslant a \leqslant a_2$, $f(a) = 0$ *in* $a_2+\Delta \leqslant a \leqslant 1+a_1-\Delta$ *and* $0 \leqslant f(a) \leqslant 1$ *for other* a; (ii) *it has the Fourier-expansion*

$$f(a) = \sum_{m=-\infty}^{\infty} d_m e^{2\pi i m a},$$

where

(50) $$d_0 = a_2 - a_1 + \Delta, \quad d_m \ll \min(d_0, |m|^{-1}, \Delta^{-r}|m|^{-r-1}) \quad \textit{for} \quad |m| \geqslant 1.$$

This is an immediate consequence of I. M. Vinogradov's lemma ([11] I, Lemma 12). Cf. [6], p. 514.

Now let $L_{\chi,m}$ denote a broken line in the strip

$$1-\tfrac{1}{24}\vartheta' - 1/c\log^2 D(1+|t|) < \sigma < 1-\tfrac{1}{24}\vartheta'$$

(with appropriate c) such that (i) for any $s = \sigma+it \in L_{\chi,m}$ we have $\zeta'/\zeta(s, X) \ll \log^3 D(1+|t|)$ (when m in (22) satisfies (31)) and (ii) the length of the piece of $L_{\chi,m}$ between any two of its points $\sigma+it$, $\sigma'+i(t+1)$ is < 2. Write

$$g = \tfrac{1}{24}\vartheta' \ (\leqslant \tfrac{1}{24}), \qquad \sigma_1 = 1-g.$$

From the identity

$$\sum_{\mathfrak{a}} \frac{X(\mathfrak{a})\Lambda(\mathfrak{a})}{\mathfrak{a}^{\sigma_1}} \exp\left(-\frac{\log^2 \mathfrak{a}/x}{4y}\right) = i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, X)\, x^{s-\sigma_1} e^{(s-\sigma_1)^2 y}\, ds$$

$$(x > 1,\ y > 0)$$

(²) Any possible improvement of (49) would imply a corresponding improvement in the theorem of § 1. If, for example, the factor $\log eM$ could be dropped, then the theorem would be true for $D_1 = D$. Cf. the proof of (60) and (62).

(cf. [1], p. 299) and (22), (48) we deduce

(51) $$h \sum_{\mathfrak{a}\in H} \frac{\Lambda(\mathfrak{a})}{\mathfrak{a}^{\sigma_1}} \xi(\mathfrak{a})^m \exp\left(-\frac{\log^2 \mathfrak{a}/x}{4y}\right) = i\sqrt{\frac{y}{\pi}} \sum_{\chi} \bar{\chi}(H) \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, X) e^{s-\sigma_1} e^{(s-\sigma_1)^2 y}\, ds.$$

Let $\mathscr{I}$ denote the interval

$$a \equiv a_1 - \Delta + 2\theta\Delta \ (\mathrm{mod}\, 1), \qquad 0 \leqslant \theta < 1$$

with a fixed a_1 and Δ satisfying the conditions of Lemma 10. By (51), (22) and Lemma 10,

(52) $$h \sum_{\substack{\mathfrak{a}\in H \\ a\in\mathscr{I}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{a}^{\sigma_1}} \exp\left(-\frac{\log^2 \mathfrak{a}/x}{4y}\right) \geqslant h \sum_{\mathfrak{a}\in H} \frac{\Lambda(\mathfrak{a})}{\mathfrak{a}^{\sigma_1}} f(a) \exp\left(-\frac{\log^2 \mathfrak{a}/x}{4y}\right)$$
$$= h \sum_{\mathfrak{a}\in H} \frac{\Lambda(\mathfrak{a})}{\mathfrak{a}^{\sigma_1}} \exp\left(-\frac{\log^2 \mathfrak{a}/x}{4y}\right) \sum_{m=-\infty}^{\infty} d_m \xi(\mathfrak{a})^m$$
$$= h \sum_{\mathfrak{a}\in H} \sum_{|m|\leqslant M} + h \sum_{\mathfrak{a}\in H} \sum_{|m|>M} = U_1 + U_2,$$

say, where M stands for D^{2c_0}, which is the right-hand side of (31). By (51)

(53) $$U_1 = i\sqrt{\frac{y}{\pi}} \sum_{\chi} \bar{\chi}(H) \sum_{|m|\leqslant M} d_m \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, X) x^{s-\sigma_1} e^{(s-\sigma_1)^2 y}\, ds$$
$$= 2\sqrt{\pi y}\, x^g e^{g^2 y} d_0 - 2\sqrt{\pi y} \sum_{\chi} \bar{\chi}(H) \sum_{|m|\leqslant M} d_m \operatorname*{Res}_{s=\varrho_{\chi,m}} \frac{\zeta'}{\zeta}(s, X) x^{s-\sigma_1} e^{(s-\sigma_1)^2 y} +$$
$$+ i\sqrt{\frac{y}{\pi}} \sum_{\chi} \bar{\chi}(H) \sum_{|m|\leqslant M} d_m \int_{L_{\chi,m}},$$

where $\varrho_{\chi,m}$ runs through the zeros of $\zeta(s, X)$ ($X = \chi\xi^m$) on the right of $L_{\chi,m}$. Supposing $1 < y \leqslant \log^2 D$, the remaining term in (53) satisfies

$$\ll h\sqrt{y} \sum_{|m|\leqslant M} |d_m| \log^3 D \int_0^{\infty} e^{-t^2 y} \log^3(2+t)\, dt \ll h \log^3 D \log M \ll D \log^4 D,$$

by (1). Further on we shall use the notation

$$\varrho = 1-\delta+i\gamma, \qquad \varrho' = 1-\delta',$$

ϱ being a typical zero of $\prod_X \zeta(s, X)$ and ϱ' the exceptional zero of $\zeta(s, X)$ with a real exceptional character $X = \chi'$ (cf. (30)). Writing

$$(54)\qquad S = 1-E_1\chi'(H)x^{-\delta'}e^{-\delta'(2g-\delta')y}$$

(where $E_1 = 1$ if ϱ' exists, and $E_1 = 0$ otherwise) and

$$(55)\qquad S' = \sum_\chi \bar{\chi}(H) \sum_{|m|\leqslant M} d_m \sum_{\varrho_{\chi,m}(\neq\varrho')} x^{-\delta}e^{\{-\delta(2g-\delta)-\gamma^2+2i\gamma(g-\delta)\}y+i\gamma\log x},$$

we have

$$(56)\qquad U_1 = 2\sqrt{\pi y}\,x^g e^{g^2y}(d_0S-S')+O(D\log^4 D).$$

Write

$$x = D^\xi, \quad \xi \geqslant 0; \quad y = \eta\log D \quad (\eta\geqslant\eta_0>2).$$

Let I_B denote the integration B times repeated with respect to η, the range of integration being $(\eta, \eta+1)$. Then, by (56) and (55)

$$(57)\qquad \left|I_B\frac{U_1}{2\sqrt{\pi y}\,x^g e^{g^2y}}\right| \geqslant d_0I_BS-|I_BS'|-c_3D^{1-g^2\eta_0}\log^4 D,$$

$$(58)\qquad |I_BS'| \leqslant \left(\frac{2}{\log D}\right)^B \sum_\varrho |d_\varrho|x^{-\delta}\frac{\exp\{-\eta_0(g\delta+\gamma^2)\log D\}}{|g\delta+2i\gamma(g-\delta)|^B} = T_1+T_2+T_3,$$

where d_ϱ (with $\varrho = \varrho_{\chi,m}$) stands for the d_m, and T_1, T_2, T_3 denote the parts of the previous sum obtained by dissection of the region G (say) on the right of every $L_{\chi,m}$ as follows.

Let G_1 be the part G with $|t|>1$. By (58) and the estimate for the number of zeros of $\zeta(s, X)$ (mentioned at the end of § 4)

$$T_1 \ll \sum_{\varrho\epsilon G_1} d_0e^{-\eta_0\gamma^2\log D} \ll hMd_0\int_1^\infty e^{-\eta_0t^2\log D}t^2\log D\log D(1+t)\,dt$$

$$\ll hMd_0\log^2 D\int_1^\infty e^{-\eta_0t^2\log D}t^2\log(2+t)\,dt$$

$$\ll hMd_0\log^2 D\int_1^\infty e^{-\eta_0t\log D}\,dt \ll d_0D^{-\eta_0}hM\log D.$$

Let G_2 be the set of points $s = 1-\lambda/\log D+i\tau/\log D\,\epsilon G$ with $\lambda\geqslant\lambda_0$ (see Lemma 4) and $|\tau|\leqslant\tau_1 = \tau_1(\lambda) = \min(e^\lambda, \log D)$. Writing

$$\varrho = 1-\lambda/\log D+i\tau/\log D, \qquad \lambda = \lambda_\varrho,\ \tau = \tau_\varrho,$$

we have, by (58) and Lemma 9

$$(59)\qquad T_2 \ll \sum_{\varrho\epsilon G_2}|d_\varrho|e^{-\lambda\xi-g\eta_0\lambda}\lambda^{-B} \ll \sum_{\varrho\epsilon G_2}|d_\varrho|e^{-(\xi+g\eta_0)\lambda}$$

$$\ll \int_{\lambda_0}^{2g\log D}(\xi+g\eta_0)e^{-(\xi+g\eta_0)\lambda}\psi(\lambda)\,d\lambda+e^{-2g(\xi+g\eta_0)\log D}\psi(2g\log D),$$

where

$$\psi(u) = \sum_{\lambda_\varrho\leqslant u,\,|\tau_\varrho|\leqslant e^u}|d_\varrho| \qquad (c_4\leqslant u\leqslant 2g\log D).$$

Having chosen $\Delta\,\epsilon(\frac{1}{2}D^{-c_0}, \frac{1}{4}]$, let us partition all the integers m numerically $\leqslant D^{2c_0}$ into classes $(|m|\leqslant M_1 = \Delta^{-1})$ and $(|m|>M_1)$. By (49) in the sum $\psi(\lambda)$ there are $\ll e^{C\lambda}\log ex$ terms $d_\varrho = d_m\,(\varrho = \varrho_{\chi,m})$ with $|m|\leqslant x$ and by (50) (with $a_2 = a_1$) $d_m\ll\min(\Delta, |m|^{-1})$. Hence

$$(60)\qquad \psi(\lambda) \ll \Delta e^{C\lambda}\log eM_1+e^{C\lambda}\left(\int_{M_1}^M\frac{\log ex}{x^2}\,dx+\frac{\log eM}{M}\right) \ll e^{C\lambda}\Delta\log 1/\Delta.$$

From (59) and (60) we get (for a sufficiently large $\eta_0\ll 1$)

$$T_2 \ll e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0}\Delta\log 1/\Delta.$$

Let G_3 denote the remaining part of G and let $\lambda_0<\log\log D$ (otherwise there is no $\varrho\neq\varrho'$ in G_3). Then, by (58) and (60)

$$T_3 \ll \sum_{\varrho\epsilon G_3}|d_\varrho|e^{-(\xi+g\eta_0)\lambda}|\tau|^{-B} \ll e^{-(\xi+g\eta_0)\lambda_0}\sum_{\varrho\epsilon G_3}|d_\varrho|\,|\tau|^{-B}$$

$$\ll e^{-(\xi+g\eta_0)\lambda_0}\Big\{\int_{\lambda_0}^{\log\log D}e^{-B\lambda}\psi(\lambda)\,d\lambda+e^{-B\log\log D}\psi(\log\log D)\Big\}$$

$$\ll e^{-(\xi+g\eta_0)\lambda_0}e^{-(B-C)\lambda_0}\Delta\log 1/\Delta \ll e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0}\Delta\log 1/\Delta,$$

supposing that $B\geqslant C+2$ and $\eta_0\ll 1$ is large enough.

From (54) and Lemma 4 we deduce

$$(61)\qquad I_BS \geqslant 1-e^{-g\delta'\eta_0\log D} \geqslant 1-e^{-g\delta_0\eta_0\log D} \geqslant 1-e^{-(\delta_0/\Delta)\log D} \geqslant \frac{\delta_0}{2\Delta}\log D.$$

For a sufficiently large $\eta_0\ll 1$ the remaining term in (57) taken together with T_1 is in modulus $<d_0(\delta_0/8\Delta)\log D$ (since $d_0 = \Delta>\frac{1}{2}D^{-c}$

and $\delta_0 > D^{-c_5}$, by § 5), whereas by (32)

$$T_2+T_3 < c_6 e^{-(\xi+\frac{1}{2}g\eta_0)\lambda_0}\Delta\log 1/\Delta \leqslant c_6 d_0 e^{-\frac{1}{2}g\eta_0\lambda_0}\log 1/\Delta$$

$$= c_6 d_0 \exp\left(-\tfrac{1}{2}g\eta_0\Delta\log\frac{eA}{\delta_0\log D}\right)\log 1/\Delta$$

$$= d_0\left(\frac{\delta_0\log D}{A}\right)^{\frac{1}{2}g\eta_0\Delta} c_6 e^{-\frac{1}{2}g\eta_0\Delta}\log 1/\Delta < d_0\frac{\delta_0}{8A}\log D,$$

provided that

$$c_6 e^{-\frac{1}{2}g\eta_0\Delta}\log 1/\Delta < \tfrac{1}{8},$$

whence

$$\eta_0 \geqslant c_7\log\log 1/\Delta. \tag{62}$$

Let U denote the left-hand side of (57). Then, by (61) and (58)

$$U > d_0\frac{\delta_0}{4A}\log D. \tag{63}$$

12. Introducing the number

$$z = xe^{4y} = D^{\xi+4\eta},$$

we divide the sum U_0 (say) on the left of (52) up into four sums

$$U_0 = S_0 + h\sum_{\substack{\mathfrak{b}\in H\\ x<b<z,\,\mathfrak{a}\in\mathcal{J}}}\frac{\log b}{b^{\sigma_1}}\exp\left(-\frac{\log^2 b/x}{4y}\right) + S_1 + hS_2, \tag{64}$$

where

$$S_0 = h\sum_{\substack{\mathfrak{b}\in H\\ b\leqslant x,\,\mathfrak{a}\in\mathcal{J}}}\frac{\log b}{b^{\sigma_1}}\exp\left(-\frac{\log^2 b/x}{4y}\right),$$

$$S_1 = h\sum_{\substack{\mathfrak{a}\in H\\ a\geqslant z,\,\mathfrak{a}\in\mathcal{J}}}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}}\exp\left(-\frac{\log^2 a/x}{4y}\right), \tag{65}$$

$$S_2 = \sum_{\substack{\mathfrak{a}=\mathfrak{b}^n\in H,\,n\geqslant 2\\ a\leqslant z,\,\mathfrak{a}\in\mathcal{J}}}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}}\exp\left(-\frac{\log^2 a/x}{4y}\right).$$

For a sufficiently large $\eta_0 \ll 1$ and any $T > D^{c_3}$ we have, by (44),

$$\sum_{\substack{\mathfrak{a}\in H\\ D^{c_3}<a<T,\,\mathfrak{a}\in\mathcal{J}}}\frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}} \ll \Delta h^{-1}\left(\int_{D^{c_3}}^{T} t^{1-(2-g)}\,dt + T/T^{1-g}\right) \ll \Delta h^{-1}T^g, \tag{66}$$

whence

$$S_1 \ll \Delta\int_z^\infty t^g\exp\left(-\frac{\log^2 t/x}{4y}\right)\frac{\log t/x}{2yt}\,dt \tag{67}$$

$$= \Delta\int_z^\infty \exp\left(-\frac{\log^2 t/x}{4y} + g\log t\right)\left(\frac{\log t/x}{2yt} - \frac{g}{t} + \frac{g}{t}\right)dt$$

$$< 2\Delta\int_z^\infty \exp\left(-\frac{\log^2 t/x}{4y} + g\log t\right)\left(\frac{\log t/x}{2yt} - \frac{g}{t}\right)dt$$

$$= 2\Delta\exp\left(-\frac{\log^2 z/x}{4y} + g\log z\right) = 2\Delta z^g e^{-4y} = 2\Delta x^g e^{-4(1-g)y}.$$

By (65), (66) and (5)

$$S_0 < h\sum_{\substack{\mathfrak{b}\in H\\ b\leqslant x,\,\mathfrak{a}\in\mathcal{J}}}\frac{\log b}{b^{\sigma_1}} = h\sum_{\substack{\mathfrak{b}\in H\\ b<D^{c_3},\,\mathfrak{a}\in\mathcal{J}}} + h\sum_{\substack{\mathfrak{b}\in H,\,\mathfrak{a}\in\mathcal{J}\\ D^{c_3}\leqslant b\leqslant x}} \tag{68}$$

$$\ll \Delta h\varkappa D^{c_4} + \Delta x^g < \Delta e^{\frac{1}{2}g^2 y}x^g.$$

Since $\sigma_1 = 1-g$, $0 < g < \frac{1}{24}$, we have for any positive constant $g' < 1-2g$

$$\sum_{\substack{\mathfrak{b}^n\in H,\,n\geqslant 2\\ b^n\leqslant z,\,\mathfrak{a}\in\mathcal{J}}}\frac{\log b}{b^{n\sigma_1}}\exp\left(-\frac{\log^2 b/x}{4y}\right) \ll \sum_{\substack{\mathfrak{a}\in H\\ \mathfrak{a}\in\mathcal{J}}} 1/a^{1+g'} + D^{c_5}$$

$$\ll \int_1^\infty \frac{\varkappa t + D^{c_1}t^{1-\theta}}{t^{2+g'}}\,dt + D^{c_5} < D^{c_6},$$

whence for a sufficiently large $\eta_0 \ll 1$

$$hS_2 < \Delta e^{\frac{1}{2}g^2 y}x^g. \tag{69}$$

Now, by (67), (68) and (69),

$$S_0 + S_1 + hS_2 \ll \Delta x^g e^{\frac{1}{2}g^2 y}$$

and thus

$$I_B\frac{1}{2\sqrt{\pi y}\,x^g e^{g^2 y}}(S_0+S_1+hS_2) \ll D^{-\frac{1}{2}g^2\eta_0}. \tag{70}$$

Having chosen the number $\Delta\in(\frac{1}{2}D^{-c_0}, \frac{1}{4}]$, we take $M = D^{2c_0}$ and $r \geqslant 4$. Then for any m in absolute value $\geqslant M$ we have $|m|\Delta \geqslant \frac{1}{2}|m|^{1/2}$, whence by (50)

$$\sum_{|m|>M} |d_m| \ll M^{-r/2}$$

and thus in (52)

$$U_2 \ll h \sum_{\mathfrak{a} \in H} \frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}} \exp\left(-\frac{\log^2 a/x}{4y}\right) \sum_{|m|>M} |d_m|$$

$$\ll D^{-rc_0} h \sum_{\mathfrak{a} \in H} \frac{\Lambda(\mathfrak{a})}{a^{\sigma_1}} \exp\left(-\frac{\log^2 a/x}{4y}\right)$$

$$= D^{-rc_0} i \sqrt{\frac{y}{\pi}} \sum_{\chi} \bar{\chi}(H) \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s, \chi)\, x^{s-\sigma_1} e^{(s-\sigma_1)^2 y}\, ds = D^{-rc_0} \Psi$$

(say). In order to get an upper bound for Ψ we may use the same method of residues as the one we used in dealing with U_1 (except that in the present case $m = 0$, $d_0 = 1$ and there are no other coefficients d_m). Arguing as in (53)-(61) we can prove that

$$(71) \qquad I_B \frac{U_2}{2\sqrt{\pi y}\, x^g e^{g^2 y}} \ll D^{-rc_0}.$$

Let us denote by J_B the operation

$$I_B \frac{1}{2\sqrt{\pi y}\, x^g e^{g^2 y}}.$$

Then for sufficiently large r, $\eta_0 \ll 1$ we have by (63), (64), (70), (52) and (71)

$$(72) \qquad I_B \frac{1}{2\sqrt{\pi y}\, x^g e^{g^2 y}} h \sum_{\substack{\mathfrak{b} \in H, \mathfrak{a} \in \mathscr{I} \\ x<b<z}} \frac{\log b}{b^{\sigma_1}} \exp\left(-\frac{\log^2 b/x}{4y}\right)$$

$$= J_B U_0 - J_B(S_0 + S_1 + hS_2)$$

$$\geqslant J_B(U_1 + U_2) - J_B(S_0 + S_1 + hS_2) = J_B U_1 + J_B U_2 - J_B(S_0 + S_1 + hS_2)$$

$$\geqslant |J_B U_1| - |J_B U_2| - |J_B(S_0 + S_1 + hS_2)| > d_0 \frac{\delta_0}{4\Lambda} \log D - c_7 D^{-rc_0} - c_8 D^{-\frac{1}{2}g^2\eta_0}$$

$$> d_0(\delta_0/8\Lambda) \log D.$$

Since $z \leqslant x D^E$ with $E = 4(\eta_0 + B)$ and $\eta_0 = c_9 \log\log 1/\Delta$, by (62), the theorem follows. If $z > D_1^{c_{10}}$ (with appropriate c_{10}), then by (1) the left-hand side of (72) does not exceed

$$\pi(z, \mathscr{I}, H) \frac{D}{x^g} \cdot \frac{\log x}{x^{1-g}},$$

which implies (7).

Improvement of the theorem for $x \to \infty$

13. Further on we suppose that (3) holds for any $c_2 > 0$, $D > D_0(c_2)$ and that for arbitrarily small positive constants ε_1, ε' and any $D > D_0(\varepsilon')$, $x > 1$ we have

$$h\varkappa > D^{-\varepsilon'}, \qquad \sum_{\substack{\mathfrak{a} \\ a \leqslant x}} 1 \leqslant c_3(\varepsilon_1) D^{\varepsilon'} x^{1+\varepsilon_1}.$$

These conditions imply the estimate for exceptional zero $\varrho' = 1 - \delta'$:

$$(73) \qquad \delta' > D^{-\varepsilon_2} \qquad (D > D_0(\varepsilon_2))$$

with an arbitrarily small constant $\varepsilon_2 > 0$ (see [4], § 25).

We shall prove that *if* (73) *holds* (which is certainly true for an odd class number), *then there are positive constants c, c' depending merely on $c_0, c_1, l, \vartheta', \vartheta$ such that for any positive $\varepsilon \leqslant c$ and any $D > D_0(\varepsilon)$, $x \geqslant x_0 = D_1^{c' \log(c/\varepsilon)}$ (where $D_1 = D^{\log\log 8/\Delta}$) in every class H there is a generator $\mathfrak{b}$ with the image* (4) *lying in the region*

$$(x < a < x D_1^{\varepsilon},\ \mathfrak{a} \in \mathscr{A}),$$

$\mathscr{A}$ being defined by (6). *And if $x > x_0 D_1^{\varepsilon}$, then $\pi(x, \mathscr{A}, H) > x/hD_1^{3\varepsilon} \log x$.*

The proof begins as in § 11. But now we use

$$(74) \qquad \begin{gathered} y = \frac{\eta}{\nu} \log D, \qquad 2 < \eta_0 \leqslant \eta \ll \log\log 8/\Delta, \\ 1 \leqslant \nu = \nu(\xi) \leqslant \min(e^{k\xi}, \log D), \qquad k = \frac{1}{c_4 \log\log 8/\Delta}, \end{gathered}$$

where $c_4 \ll 1$ and η_0 are large enough. Considering that

$$I_B y^{-1/2} e^{-g^2 y} \ll e^{-g^2(\eta_0/\nu)\log D} \ll \begin{cases} 1 & \text{if} \quad \xi > \xi_1 = (3 + c_0)/g, \\ D^{-g^2\eta_0/\nu_1} < D^{-3-c_0} & \text{if} \quad \xi \leqslant \xi_1,\ \nu_1 = \nu(\xi_1), \end{cases}$$

we deduce that the remaining term in (57) satisfies

$$I_B y^{-1/2} e^{-g^2 y} O(x^{-g} D \log^4 D) \ll d_0 D^{-1}.$$

Let G_1 be the part of G with $|t| \geqslant \log D$. Then in (58)

$$T_1 \ll \sum_{|\gamma|\geqslant \log D} e^{-(\eta_0/\nu)\gamma^2 \log D} \ll hM \int_{\log D}^{\infty} e^{-(\eta_0/\nu)t^2 \log D} \eta_0 t^2 \log D \log D t\,dt$$

$$\ll D^{1+2c_0} \log^3 D \int_{\log D}^{\infty} e^{-(\eta_0/\nu)t^2 \log D} t^2 \log t\,dt < D^{c_5} \int_{\log D}^{\infty} e^{-(\eta_0/\nu)t^2 \log D + 3\log t}\,dt$$

$$< D^{c_5} \int_{\log D}^{\infty} e^{-c_6 t^2}\,dt < D^{c_5} \int_{\log D}^{\infty} e^{-c_6 t \log D}\,dt = \frac{D^{c_5}}{c_6 \log D} D^{-c_6 \log D},$$

whence

$$T_1 < d_0 D^{-1}.$$

In the previous definition of G_2 we now take $\tau_1(\lambda) = \min(e^{\lambda}, \log^2 D)$. The corresponding part of (58) is

$$T_2 < (2\nu)^B \sum_{\varrho\in G_2} |d_\varrho|\, e^{-\lambda\xi-(\eta_0/\nu)g\lambda} (g\lambda)^{-B} \ll \nu^B \sum_{\lambda_0\leqslant\lambda\leqslant 2g\log D} |d_\varrho|\, e^{-(\xi+g\eta_0/\nu)\lambda}$$

$$\ll \nu^B \Big\{ \int_{\lambda_0}^{2g\log D} (\xi + g\eta_0/\nu) e^{-(\xi+g\eta_0/\nu)\lambda} \psi(\lambda)\,d\lambda + e^{-(\xi+g\eta_0/\nu)2g\log D} \psi(2g\log D) \Big\}$$

$$\ll \nu^B e^{-(\xi+g\eta_0/\nu-C)\lambda_0} d_0 \log 1/\Delta < d_0 e^{-(2\xi/3+g\eta_0/\nu-C)\lambda_0} \log 1/\Delta$$

$$< \begin{cases} d_0 e^{-(\xi/3+g\eta_0/\nu)\lambda_0} \log 1/\Delta & \text{if } \xi \geqslant 3C \\ d_0 e^{-(2\xi/3+g\eta_0/\nu)\lambda_0} \log 1/\Delta & \text{if } \xi < 3C,\ \eta_0 > 2g^{-1}C\nu(3C). \end{cases}$$

In any case

$$T_2 \ll d_0 e^{-(\xi/3+g\eta_0/2\nu)\lambda_0} \log 1/\Delta.$$

Next, we have

$$T_3 \ll \nu^B \sum_{\varrho\in G_3} |d_\varrho|\, e^{-\xi\lambda_0-(\eta_0/\nu)g\lambda} |g\tau|^{-B} = \left(\frac{\nu}{g}\right)^B e^{-(\xi+g\eta_0/\nu)\lambda_0} \sum_{\varrho\in G_3} |d_\varrho|\,|\tau|^{-B}$$

$$\ll \nu^B e^{-(\xi+g\eta_0/\nu)\lambda_0} \Big\{ \int_{\lambda_0}^{2\log\log D} \psi(\lambda) e^{-B\lambda}\,d\lambda + e^{-2B\log\log D} \psi(2\log\log D) \Big\}$$

$$\ll \nu^B e^{-(\xi+g\eta_0/\nu+B-C)\lambda_0} d_0 \log 1/\Delta.$$

Hence, for a large $\eta_0 \ll 1$

$$T_3 \ll d_0 e^{-(\xi/3+g\eta_0/2\nu)\lambda_0} \log 1/\Delta.$$

This proves the following analogue of (57):

(75) $$U = \left| I_B \frac{U_1}{2\sqrt{\pi y}\, x^g e^{g^2 y}} \right| \geqslant d_0 (I_B S - c_7 e^{-(\xi/3+g\eta_0/2\nu)\lambda_0} \log 1/\Delta - c_8 D^{-1}),$$

where

$$I_B S \geqslant 1 - x^{-\delta'} e^{-g(\eta_0/\nu)\delta' \log D}.$$

Hence, by Lemma 4,

(76) $$I_B S \geqslant 1 - e^{-g(\eta_0/\nu)\delta_0 \log D} \geqslant 1 - e^{-(A\nu)^{-1}\delta_0 \log D} > \frac{\delta_0 \log D}{2A\nu}.$$

We may suppose that $k < A/3$ (see (74)). Then there is a number ξ_1 $(0 < \xi_1 \ll \log\log 8/\Delta)$ which is the least positive ξ such that

$$\tfrac{1}{3} A\xi \geqslant \begin{cases} 1, \\ k\xi + \log(4c_7 \log 1/\Delta). \end{cases}$$

If $\xi \geqslant \xi_1$, then

$$e^{-A\xi/3} \leqslant \frac{1}{4c_7 \log 1/\Delta} e^{-k\xi},$$

whence, by (74)

$$c_7 e^{-A\xi/3} \log 1/\Delta \leqslant 1/4\nu,$$

(77) $$c_7 e^{-(\xi/3+g\eta_0/2\nu)\lambda_0} \log 1/\Delta < c_7 e^{-\xi\lambda_0/3} \log 1/\Delta$$

$$= c_7 \exp\Big\{-\frac{\xi}{3} A \log \frac{eA}{\delta_0 \log D}\Big\} \log 1/\Delta = c_7 \left(\frac{\delta_0 \log D}{eA}\right)^{A\xi/3} \log 1/\Delta$$

$$= \left(\frac{\delta_0 \log D}{A}\right)^{A\xi/3} c_7 e^{-A\xi/3} \log 1/\Delta \leqslant \frac{\delta_0 \log D}{4A\nu}.$$

Taking η_0 large enough:

(78) $$\eta_0 \ll \log\log 1/\Delta,$$

we have

$$\tfrac{1}{2} g\eta_0 A e^{-k\xi_1} \geqslant \begin{cases} 1, \\ k\xi_1 + \log(4c_7 \log 1/\Delta), \end{cases}$$

whence for any $\xi \in [0, \xi_1]$

$$gA\eta_0/2\nu \geqslant 1,$$

$$e^{-gA\eta_0/2\nu} \leqslant \frac{1}{4c_7 \log 1/\Delta} e^{-k\xi_1}, \qquad c_7 e^{-gA\eta_0/2\nu} \log 1/\Delta \leqslant 1/4\nu$$

and thus (cf. (77))

$$c_7 e^{-(\xi/3+g\eta_0/2\nu)\lambda_0} \log 1/\Delta \leqslant c_7 e^{-g\eta_0\lambda_0/2\nu} \log 1/\Delta$$
$$= \left(\frac{\delta_0 \log D}{A}\right)^{g\Delta\eta_0/2\nu} c_7 e^{-g\Delta\eta_0/2\nu} \log 1/\Delta \leqslant \frac{\delta_0 \log D}{4A\nu}.$$

Hence in any case (77) holds for appropriate η_0 satisfying (78).

Since by (73)

$$c_8 D^{-1} < (\delta_0/8A\nu)\log D,$$

from (75) (76) and (77) it follows that for any $c_9 \leqslant 1/8A$

$$U > d_0(c_9/\nu)\,\delta_0 \log D. \tag{79}$$

14. Using the number $z = xe^{4y}$ we divide up the sum U_0 on the left of (52) into four partial sums as in (64). By the arguments of § 12 we prove that

$$S_0 + S_1 + hS_2 \ll \Delta x^g e^{g^2 y/2},$$

whence

$$I_B \frac{1}{2\sqrt{\pi y}\, x^g e^{g^2 y}} (S_0 + S_1 + hS_2) \ll \Delta D^{-g^2\eta_0/2\nu}.$$

Hence, by (71) and (79)

$$(80)\qquad I_B \frac{1}{2\sqrt{\pi y}\, x^g e^{g^2 y}} h \sum_{\substack{\mathfrak{b}\in H,\, \alpha\in\mathscr{I} \\ x<b<z}} \frac{\log b}{b^{\sigma_1}} \exp\left(-\frac{\log^2 b/x}{4y}\right)$$
$$\geqslant J_B U_0 - c_4 \Delta D^{-g^2\eta_0/2\nu} \geqslant |J_B U_1| - |J_B U_2| - c_4 \Delta D^{-g^2\eta_0/2\nu}$$
$$\geqslant \{\tfrac{1}{2}(c_9/\nu)\,\delta_0 \log D - c_4 D^{-g^2\eta_0/2\nu}\}\Delta.$$

Write

$$E = 4(\eta_0 + B) \leqslant c \log\log 8/\Delta, \qquad c' = (k \log\log 8/\Delta)^{-1}, \qquad D_1 = D^{\log\log 8/\Delta}.$$

For any fixed positive $\varepsilon \leqslant c$ let

$$\nu_1 = c/\varepsilon. \tag{81}$$

Then $1 \leqslant \nu_1 \leqslant \log D$, if $D > D_0(\varepsilon)$. Further on let $x \geqslant D_1^{c'\log(c/\varepsilon)}$; then $\xi \geqslant k^{-1}\log(c/\varepsilon)$, whence

$$c/\varepsilon \leqslant e^{k\xi},$$

and, by (81), (74), ν_1 satisfies the conditions imposed on ν. Writing $\varepsilon' = g^2\varepsilon/2c$, we have by (81)

$$\varepsilon' = \frac{g^2}{2\nu_1} < \frac{g^2\eta_0}{2\nu_1}. \tag{82}$$

From (73) and Lemma 4 we deduce that

$$\delta_0 > c_5(\varepsilon') D^{-\varepsilon'/2} \tag{83}$$

and, if $D > D_0(\varepsilon')$,

$$\frac{c_9 c_5(\varepsilon')}{4c_4} D^{\varepsilon'/2} > 1.$$

Hence, by (82), (83)

$$\frac{c_9}{4c_4\nu_1}\delta_0 \log D \geqslant \frac{c_9}{4c_4}\delta_0 > \frac{c_9 c_5(\varepsilon')}{4c_4} D^{\varepsilon'/2} D^{-\varepsilon'} > D^{-\varepsilon'} > D^{-g^2\eta_0/2\nu_1}.$$

This proves that

$$c_4 D^{-g^2\eta_0/2\nu_1} < \tfrac{1}{4}(c_9/\nu_1)\,\delta_0 \log D.$$

Now denoting by V the left-hand side of (80) (with ν_1 instead of ν), we have

$$V > \tfrac{1}{4}\Delta(c_9/\nu_1)\,\delta_0 \log D. \tag{84}$$

Consequently there is a generator $\mathfrak{b}\in H$ with $\alpha\in\mathscr{I}$ and

$$b \in (x, z) = (x, xe^{4y}) = (x, xD^{4\eta/\nu_1}) \subset (x, xD^{4(\eta_0+B)/\nu_1})$$
$$= (x, xD^{E/\nu_1}) \subset (x, xD_1^{c/\nu_1}) \subset (x, xD_1^{\varepsilon}).$$

(80) combined with (84) gives an lower bound for $\pi(x, \mathscr{I}, H)$. This proves the results stated, the transition from intervals $\mathscr{I}$ to $\mathscr{A}$ being a simple matter.

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Reçu par la Rédaction le 13. 1. 1964

Errata to the part I of this paper (Acta Arithm. 10(1964), pp. 137-182),

p. 165[3]: read J_χ instead of y_χ,

p. 172[3]: read $e\sum$ instead of $e\sum$,

p. 177$_3$: read $4(g-\delta')k\pi i$ instead of $4(g-\delta')gk\pi i$.

ACTA ARITHMETICA
X (1965)

Some remarks on a series of Ramanujan

by

W. Staś (Poznań)

1. In my previous papers [7], [8] I was concerned with the Ramanujan series

$$S(\beta)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(\beta/n)^2} \tag{1.1}$$

where $\mu(n)$ was the function of Möbius and β a real parameter.

G. H. Hardy and J. E. Littlewood have proved (see [1]) that

$$S(\beta)=O(\beta^{-\frac{1}{2}}),\quad \beta\to\infty, \tag{1.2}$$

is equivalent to the conjecture of Riemann.

At present we shall prove by Turán's methods the following theorem, which is stronger than my previous result (see [8]), based on Riemann's hypothesis and on the conjecture that the ζ-function has only simple zeros.

Theorem. *Suppose Riemann's conjecture. Then for $T>C$*

$$\max_{T^{1-o(1)}\leqslant\beta\leqslant T}|S(\beta)|\geqslant T^{-\frac{1}{2}-o(1)}. \tag{1.3}$$

In the proof we shall apply the method of Turán, namely we shall use the following modification ([2]) of Turán's Satz X ([11]):

Lemma 1. *Suppose that $m\geqslant 0$, $z_1,z_2,\ldots,z_N$ are complex numbers with*

$$1=|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_h|\geqslant\ldots\geqslant|z_{h_1}|\geqslant\ldots\geqslant|z_N| \tag{1.4}$$

and

$$|z_h|>2\frac{N}{N+m},\quad |z_{h_1}|<|z_h|-\frac{N}{m+N}. \tag{1.5}$$

Then there exists an integer μ with

$$m\leqslant\mu\leqslant m+N \tag{1.6}$$