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Approximations to the logarithms of certain rational numbers

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1. Introduction. In a recent paper [1] methods were introduced for investigating the accuracy with which certain algebraic numbers may be approximated by rational numbers. It is the main purpose of the present paper to deduce, using similar techniques, results concerning the accuracy with which the natural logarithms of certain rational numbers may be approximated by rational numbers, or, more generally, by algebraic numbers of bounded degree.

Results of this type were first proved by Morduchai-Boltowskoj in 1923 (see [6]) but the most precise results so far established are due to Mahler [4] and Feldman [2]. Suppose that α is an algebraic number other than 0 or 1. Then the work of Mahler leads to inequalities of the form

$$|\log\alpha - \xi| > H^{-\varkappa}$$

valid for all algebraic numbers ξ of degree n and sufficiently large height H, where $\varkappa$ is an explicit function of n, of order c^n, where c is a constant >1. For small values of n and rational α these represent the best inequalities known, but for large n very much stronger results were recently given by Feldman, indeed with $\varkappa$ of order $(n\log n)^2$.

In the present paper we shall begin by proving

THEOREM 1. *Let a, b and n be positive integers and let $\alpha = b/a$. Suppose that $\varkappa > n$ and*

$$a > \{(4\sqrt{2})^n h\}^{\varrho}, \tag{1}$$

where $h = b-a > 0$ and

$$\varrho = (n+1)(\varkappa+1)(\varkappa-n)^{-1}. \tag{2}$$

Then

$$|x_0 + x_1\log\alpha + \ldots + x_n(\log\alpha)^n| > cX^{-\varkappa} \tag{3}$$

for all integers $x_0, x_1, \ldots, x_n$, *where*

(4) $$X = \max(|x_0|, |x_1|, \ldots, |x_n|) > 0$$

and c *is given by*

(5) $$c = a^{-\lambda \log a} \quad \text{where} \quad \lambda = 50\varkappa(\varkappa+1).$$

It follows that for certain rational numbers a the inequality (3) holds with $\varkappa$ only slightly greater than n, and this is almost the best possible; for it is well known that (3) could not hold for all integers $x_0, x_1, \ldots, x_n$, not all zero, with any constant c, if $\varkappa$ were less than n.

From Theorem 1 we obtain as an immediate deduction the following

COROLLARY. *Suppose that the hypotheses of Theorem 1 hold and let* $\delta = n(\varkappa - n)$. *Then*

(i) $|\log a - \xi| > H^{-n-1-\delta}$ *for all algebraic numbers* ξ *of degree at most* n *and sufficiently large height* H.

(ii) *There are infinitely many algebraic numbers* ξ *of degree at most* n *and height* H *for which* $|\log a - \xi| < H^{-n-1+\delta}$.

The proof of Theorem 1 depends on combining a theorem of Mahler, given in [4], concerning certain polynomials in $\log x$, with a lemma of an arithmetical nature due essentially to Siegel (see [8]). The corollary is deduced by direct application of two formulae of Wirsing [9].

The condition (1) may be relaxed if we suppose that $n = 1$, $h = 1$. We prove

THEOREM 2. *For all integers* $a > 0$, p *and* $q > 0$ *we have*

(6) $$\left|\log\left(1+\frac{1}{a}\right) - \frac{p}{q}\right| > c(a) q^{-\varkappa(a)}$$

where

$$\varkappa(1) = 12\cdot 5, \quad \varkappa(2) = 7,$$

$$\varkappa(a) = 2\,\frac{\log\{4\sqrt{2}a^2/(a+1)\}}{\log\{\sqrt{2}a^3/(a+1)^2\}} \quad \text{for} \quad a \geqslant 3,$$

and

(7) $$c(1) = 10^{-10^5}, \quad c(a) = (\sqrt{2}a)^{-10^4} \quad \text{for} \quad a \geqslant 2.$$

Here $\varkappa(a)$ is decreasing and tends to 2 as a tends to infinity. For $a \geqslant 15$ we see that $\varkappa(a) < 3$ and thus we obtain, for example, the following measure of irrationality for $\log\frac{15}{16}$;

$$\left|\log\frac{15}{16} - \frac{p}{q}\right| > q^{-3}$$

for all rationals p/q with q sufficiently large.

Finally, by way of application, we give a positive lower bound for the fractional part of the sum of the series

(8) $$e^{-\theta} + e^{-2\theta} + e^{-3\theta} + \ldots,$$

where θ is any positive rational number. Clearly we need consider only the case in which the sum ζ of (8) is greater than 1. Then the result is as follows.

THEOREM 3. *Let* $\theta > 0$ *be a rational number with denominator* $q > 0$ *and suppose that*

$$\zeta = (e^{\theta} - 1)^{-1} > 1.$$

Then the fractional part of ζ *is greater than* $c(1) q^{-12\cdot 5}$ *where* $c(1)$ *is given by* (7).

I am indebted to Prof. Davenport for valuable suggestions in connection with the present work.

2. Lemmas. The following lemma is due essentially to Siegel.

LEMMA 1. *Let* n *be a positive integer and let* q_{ij} $(i, j = 0, 1, \ldots, n)$ *be* $(n+1)^2$ *integers with absolute values at most* Q *such that the matrix* (q_{ij}) *is non-singular. Suppose that* $\xi_1, \ldots, \xi_n$ *are real or complex numbers and let*

(9) $$\Phi_i = q_{i0} + q_{i1}\xi_1 + \ldots + q_{in}\xi_n$$

for $i = 0, 1, \ldots, n$. *Suppose that the* Φ_i *have absolute values at most* Φ. *If* $x_0, x_1, \ldots, x_n$ *are integers, not all zero with absolute values at most* X *and*

(10) $$\Psi = x_0 + x_1\xi_1 + \ldots + x_n\xi_n$$

then

(11) $$|\Psi| \geqslant (n!Q^n)^{-1} - n\Phi X Q^{-1}.$$

Proof. Since (q_{ij}) is non-singular, there are n of the linear forms (9) in $1, \xi_1, \ldots, \xi_n$ which together with the linear form (10) make up a linearly independent set. Without loss of generality we can take them to be the last n forms. We have

$$\begin{vmatrix} \Phi_1 & q_{11} & \ldots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \Phi_n & q_{n1} & \ldots & q_{nn} \\ \Psi & x_1 & \ldots & x_n \end{vmatrix} = \begin{vmatrix} q_{10} & q_{11} & \ldots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ q_{n0} & q_{n1} & \ldots & q_{nn} \\ x_0 & x_1 & \ldots & x_n \end{vmatrix}$$

and the determinant on the right is a non-zero integer. Expanding the determinant on the left, and estimating each term, we obtain

$$n!Q^n|\Psi| + n(n!Q^{n-1}X\Phi) \geqslant 1,$$

which gives (11).

LEMMA 2. *Let a be a real number such that $1 < a \leqslant 2$ and let m, n be positive integers. Then there exist $(n+1)^2$ polynomials $A_{ij}(x)$ $(i, j = 0, 1, \ldots, n)$ in x of degree at most m with the following properties.*

(i) *The determinant of order $(n+1)$ with $A_{ij}(a)$ in the i-th row and j-th column $(i, j = 0, 1, \ldots, n)$ is not zero.*

(ii) *Each polynomial $A_{ij}(x)$ has integer coefficients with absolute values at most*

$$n!\,2^{n-\frac{3}{2}m}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m}.$$

(iii) *The $(n+1)$ functions*

$$R_i(x) = \sum_{j=0}^{n} A_{ij}(x)(\log x)^j \quad (i = 0, 1, \ldots, n) \tag{12}$$

satisfy the inequalities

$$|R_i(a)| \leqslant n!\,2^{-\frac{3}{2}m}(e\sqrt{m})^{n+1}a^{2m+1}\{\sqrt{8}(n+1)^{-1}\log a\}^{(n+1)m}. \tag{13}$$

Proof. This is a special case of Theorem 1 of Mahler [4] (see p. 378). We have interchanged m and n and restricted the number a so that $1 < a \leqslant 2$. Then

$$n+1 > 2|\log a|$$

and thus a condition required by Mahler's Theorem is satisfied.

On the basis of Lemma 2 we introduce the following notation. Corresponding to each pair of positive integers m, n and each pair of integers a, b such that $a > 0$ and $1 < a \leqslant 2$, where $a = b/a$, we define numbers $q_{ij} = q_{ij}(m, n, a, b)$ by the equations

$$q_{ij} = a^m A_{ij}(a) \quad (i, j = 0, 1, \ldots, n), \tag{14}$$

where the $A_{ij}(x)$ are the polynomials given by Lemma 2 corresponding to m, n, a. Since the $A_{ij}(x)$ are of degree at most m it follows that the q_{ij} are integers. Also from (i) of Lemma 2 it is clear that the matrix (q_{ij}) is non-singular. We now put

$$\xi_i = (\log a)^i \quad \text{for} \quad i = 1, 2, \ldots, n \tag{15}$$

and define the numbers $\Phi_i = \Phi_i(m, n, a, b)$ $(i = 0, 1, \ldots, n)$ by (9). Then clearly, for each i,

$$\Phi_i = a^m R_i(a), \tag{16}$$

where the $R_i(x)$ are the functions given by (12) of Lemma 2. Finally we note that if the Φ_i have absolute values at most Φ, $x_0, x_1, \ldots, x_n$ are integers, not all zero, with absolute values at most X, and Ψ is given by (10) then all the hypotheses of Lemma 1 are satisfied and hence (11) holds.

3. Proof of Theorem 1 and Corollary. We note first that, from (1) and (2), $a > h$, so that a is a rational between 1 and 2 exclusive.

Let $x_0, x_1, \ldots, x_n$ be integers, not all zero, and let X be given by (4). Suppose that ξ_i is given by (15) and that Ψ is defined by (10). We proceed to prove that (3) holds, that is

$$|\Psi| > cX^{-\varkappa}. \tag{17}$$

We put

$$w = \{h(4\sqrt{2})^n(n+1)^{-1}\}^{n+1} \tag{18}$$

and suppose first that

$$X \geqslant (a/w)^{50\varkappa\log a}. \tag{19}$$

From (2) it is clear that $\varrho > n+1$ and it follows from (1) and (18) that $a > w$. Thus there is a positive integer m such that

$$(a/w)^{m-1} \leqslant n^{n+1}X < (a/w)^m. \tag{20}$$

The supposition (19) then implies that

$$m > 50\varkappa\log a. \tag{21}$$

Our next object is to calculate upper bounds for the numbers $q_{ij} = q_{ij}(m, n, a, b)$ and $\Phi_i = \Phi_i(m, n, a, b)$ defined as above. Several preliminary inequalities will be required. First we note that

$$\sum_{r=0}^{m} a^r < \sum_{r=0}^{m} 2^r < 2^{m+1}. \tag{22}$$

Secondly it is clear that $\varrho > 2$ and from (1) we obtain $a > 32h$. Hence

$$a^{2m+1} = \left(1+\frac{h}{a}\right)^{2m+1} < 2^{(2m+1)/8} < 2^{m/2}. \tag{23}$$

Thirdly, since $e^x > 1+x$ for each $x > 0$, it follows that

$$\log a < h/a. \tag{24}$$

Next we prove that

$$2^{m/2} > n^n 2^{n+1}(m+1)^{2n+1}. \tag{25}$$

From (1) and (2) we obtain

$$\log a > n(n+1)\log(4\sqrt{2}) > 3n$$

and it follows from (21) that $m > (12n)^2$. It is then easily verified that $\log(m+1) < \sqrt{m}$ and hence

$$(2n+1)\log(m+1) < 3n\sqrt{m} < \tfrac{1}{4}m < \tfrac{3}{8}m\log 2. \tag{26}$$

Also we obtain

(27) $$n\log n+(n+1)\log 2 < 2n^2 < \tfrac{1}{8}m\log 2,$$

and then (25) is deduced by adding (26) and (27). Finally we shall require the inequality

(28) $$2^m > n^n e^{n+1} m^{(n+1)/2}$$

which is clear from (25).

Now from (ii) of Lemma 2, (14) and (22) we see that the integers q_{ij} have absolute values at most

$$a^m(n!)2^{n-3m/2}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m}\sum_{r=0}^{m}a^r$$
$$\leqslant a^m n^n 2^{n-m/2+1}(m+1)^{2n+1}(4\sqrt{2})^{(n+1)m},$$

and, from (25), it follows that this is less than Q where

(29) $$Q = a^m(4\sqrt{2})^{(n+1)m}.$$

From (13), (16), (23) and (24) we deduce that the Φ_i have absolute values at most

$$a^m n^n 2^{-3m/2}(e\sqrt{m})^{n+1}2^{m/2}\{\sqrt{8}(n+1)^{-1}h/a\}^{(n+1)m}$$
$$= a^{-mn}2^{-m}n^n e^{n+1}m^{(n+1)/2}\{\sqrt{8}h(n+1)^{-1}\}^{(n+1)m},$$

and, from (28), this is less than Φ where

(30) $$\Phi = a^{-mn}\{\sqrt{8}h(n+1)^{-1}\}^{(n+1)m}.$$

We now use Lemma 1. From (11) we obtain

(31) $$|\Psi| \geqslant (nQ)^{-n}\{1-n^{n+1}\Phi XQ^{n-1}\}.$$

Since from (18), (29), (30) and the right-hand inequality of (20),

$$n^{n+1}\Phi XQ^{n-1} \leqslant (a/w)^m\Phi Q^{n-1} = 2^{-(n+1)m},$$

it follows from (31) that

(32) $$|\Psi| > \tfrac{1}{2}(nQ)^{-n}.$$

We next prove that

(33) $$(a/w)^{(m-1)\varkappa} > Q^n.$$

From (1), (2) and (18) we deduce that

$$a^{\varkappa-n} > \{h(4\sqrt{2})^n\}^{(n+1)(\varkappa+1)}$$
$$> 2\{h(4\sqrt{2})^n(n+1)^{-1}\}^{(n+1)\varkappa}(4\sqrt{2})^{n(n+1)} = 2w^\varkappa(4\sqrt{2})^{n(n+1)},$$

and hence

(34) $$a^{m(\varkappa-n)} > 2^m w^{(m-1)\varkappa}(4\sqrt{2})^{mn(n+1)}.$$

Further, from (21) we obtain

(35) $$a^\varkappa = e^{\varkappa\log a} < 2^{2\varkappa\log a} < 2^m.$$

It follows from (34) and (35) that

$$a^{m(\varkappa-n)} > a^\varkappa w^{(m-1)\varkappa}(4\sqrt{2})^{mn(n+1)},$$

and this is equivalent to (33). Then from (20) and (33) we obtain

$$(n^{n+1}X)^\varkappa > Q^n$$

so that, from (32),

(36) $$|\Psi| > CX^{-\varkappa},$$

where

(37) $$C = \tfrac{1}{2}n^{-n-(n+1)\varkappa}.$$

It is clear from (1), (2) and (37) that

$$C > (2n)^{-(n+1)(\varkappa+1)} > a^{-(\varkappa+1)\log a} > c,$$

where c is given by (5), and hence (17) certainly holds for all integers $x_0, x_1, \ldots, x_n$ such that (19) is satisfied.

Now suppose that $x_0, x_1, \ldots, x_n$ are integers, not all zero, for which (19) does not hold. Then $X < u$ where u is the integer given by

(38) $$u = [(a/w)^{50\varkappa\log a}]+1.$$

Since at least one of the $(n+1)$ integers $ux_0, ux_1, \ldots, ux_n$ has absolute value at least u, we may apply the result (36), just established, with this set of integers in place of $x_0, x_1, \ldots, x_n$ and uX in place of X. We obtain

$$u|\Psi| > C(uX)^{-\varkappa},$$

that is

(39) $$|\Psi| > Cu^{-\varkappa-1}X^{-\varkappa}.$$

Finally we show that

(40) $$Cu^{-\varkappa-1} > c.$$

From (38),

$$u^{\varkappa+1} < 2^{\varkappa+1}(a/w)^{\lambda\log a},$$

where λ is given by (5), and it follows from (1), (2) and (37) that

$$Cu^{-\varkappa-1} > (4n/w)^{-(n+1)(\varkappa+1)}a^{-\lambda\log a}.$$

It is clear from (18) that

$$w \geqslant (2\sqrt{2})^{n(n+1)} > 4n$$

and thus (40) holds as required. Then (17) follows from (39) and (40) and this completes the proof of Theorem 1.

As for the proof of the corollary, the results follow by an immediate application of two inequalities given in Wirsing [9]. With the notation of that paper, Theorem 1 implies that

$$w_n(\log a) \leqslant \varkappa.$$

Then from (3) of [9] (see p. 68) it follows that

$$w_n^*(\log a) \leqslant w_n(\log a) \leqslant \varkappa,$$

and from (7) of [9] we obtain

$$w_n^*(\log a) \geqslant w_n(\log a)/\big(w_n(\log a)-n+1\big) \geqslant \varkappa/(\varkappa-n+1).$$

However $\varkappa < n+\delta$ (for $n > 1$) and

$$\varkappa/(\varkappa-n+1) = (n^2+\delta)/(n+\delta) > n-\delta,$$

where $\delta = n(\varkappa-n)$, and, by the definition of $w_n^*(\log a)$, this proves the corollary.

4. Proof of Theorem 2. We distinguish two cases according as $a > 1$ or $a = 1$. In the first case we repeat the arguments of Theorem 1 with $n = 1$, $h = 1$ but base these arguments on stronger estimates. In the case $a = 1$ it is necessary to modify the methods of Theorem 1 and we proceed in a similar manner to Mahler [4]. Theorem 5 of [4] contains the result that (6) holds with $\varkappa(1) = 48$ and by means of suitable estimates for the numbers $q_{ij}(m, 2, 1, 2)$ and $\Phi_i(m, 2, 1, 2)$ this may be improved to $\varkappa(1) = 12\cdot 5$. The proofs follow directly on the lines indicated and we omit the details.

5. Proof of Theorem 3. Let $a = [\zeta]$. From Theorem 2 we obtain

$$\left|\log\left(1+\frac{1}{a}\right)-\theta\right| > c(a)q^{-12\cdot 5}, \tag{41}$$

where $c(a)$ is given by (7). We proceed to prove that $c(a)$ in (41) can be replaced by $c(1)a^{-1}$ and the required result then follows by application of the mean-value theorem. It is clear from the form of ζ that there is only one significant value of a, if $a \geqslant 4$, namely the integer nearest to $(\theta^{-1}-\frac{1}{2})$. Since also $c(a) > c(1)$ for $a < 32$, it suffices to consider the case $q > a \geqslant 32$. The proof now follows in a similar manner to that of Theorem 1 with $n = 1$, $h = 1$ and we again omit the details.

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