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Dirichlet convolutions and the Silverman-Toeplitz conditions

by

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In connection with his recent paper [3], L. A. Rubel asked the author about the possible characterization of classes of (complex-valued) functions $f(n)$ of a positive integral variable satisfying the following three conditions:

(I) $$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{d\mid n}f(d)=1,$$

(II) $$\sum_{n=1}^{\infty}\frac{f(n)}{n}=0,$$

(III) $$\lim_{x\to\infty}\sum_{n\leqslant x}\frac{1}{n}\Big|\sum_{k\leqslant x/n}\frac{f(k)}{k}\Big|<K.$$

The interest of these particular conditions (which are for example satisfied if $f(n)$ is the Möbius function) is that one can obtain an Abelian theorem for the Dirichlet convolution of an arbitrary function on the positive integers with such an $f(n)$. Precisely, we have

THEOREM A. *Let $h(n)$ be a function on the positive integers. Define*

$$h_f(n)=\sum_{d\mid n}f(d)h(n/d),$$

where f satisfies (I), (II), (III). *If $\lim_{n\to\infty}h(n)=L$ exists and is finite, then*

$$\sum_{n=1}^{\infty}\frac{h_f(n)}{n}=L.$$

The proof of this theorem follows exactly as Rubel's proof [3] of the case where $f(n)$ is the Möbius function. One merely observes that

$$S_m = \sum_{n=1}^{m} \frac{h_f(n)}{n}$$

may be written in the form

$$S_m = \sum_{n=1}^{\infty} C_{mn} h(n), \quad \text{where} \quad C_{mn} = \frac{1}{n} \sum_{k \leqslant m/n} \frac{f(k)}{k}$$

(all terms with $n > m$ in fact $= 0$). It is then easily verified that (I), (II), (III) ensure that the matrix (C_{mn}) satisfies the Silverman-Toeplitz conditions for regularity, whence the theorem follows.

It is perhaps slightly surprising that comparatively mild conditions on $f(n)$ imply that it satisfies (I), (II), (III). In fact we have as a partial answer to Rubel's problem, the

THEOREM. *Suppose*

$$\sum_{n \leqslant x} \frac{|g(n)|}{n} = A + O(\log^{-1-\Delta}(x)) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{g(n)}{n} = C,$$

where A and C are constants, $C \neq 0$, and Δ is a fixed positive number. Define

$$f(n) = \frac{1}{C} \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right),$$

where $\mu(d)$ is the Möbius function. Then $f(n)$ satisfies (I), (II), (III).

Proof. That $f(n)$ satisfies (I) follows from the Möbius inversion formulas.

To prove (II) since by hypothesis $\sum_{n=1}^{\infty} |g(n)|/n$ converges, $G(s) = \sum_{n=1} g(n)/n^s$, $s = \sigma + it$, is absolutely convergent for $\sigma \geqslant 1$. Since by the prime number theorem,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges for $\sigma \geqslant 1$, we have by the Dirichlet series analogue of Mertens theorem for power-series ([1], p. 63)

$$\frac{G(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu(d) g(n/d)}{n^s} = C \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges for $\sigma \geqslant 1$. Hence since $G(1) = C \neq 0$ by hypothesis, it follows on letting $s \to 1$ that $\sum_{n=1}^{\infty} f(n)/n = 0$.

Finally to prove (III), write $N(x) = \sum_{n \leqslant x} \mu(n)/n$, and

$$(1) \qquad \sum_{n \leqslant x} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{f(k)}{k} \right|$$

$$= \sum_{n \leqslant x/2} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{f(k)}{k} \right| + \sum_{x/2 < x \leqslant n} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{f(k)}{k} \right| = \sum\nolimits_1 + \sum\nolimits_2 .$$

For $\sum_2$ we have,

$$(2) \qquad \sum\nolimits_2 \leqslant \sum_{x/2 < n \leqslant x} \frac{1}{n} \sum_{k \leqslant x/n} \left| \frac{f(k)}{k} \right| = O\Big(\sum_{x/2 < n \leqslant x} \frac{1}{n} \Big) = O(1).$$

For $\sum_1$ we have

$$(3) \qquad |C| \sum\nolimits_1 = |C| \sum_{n \leqslant x/2} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{f(k)}{k} \right| = \sum_{n \leqslant x/2} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{1}{k} \sum_{d|k} \mu(d) g(k/d) \right|$$

$$= \sum_{n \leqslant x/2} \frac{1}{n} \left| \sum_{k \leqslant x/n} \sum_{d|k} \frac{\mu(d)}{d} \cdot \frac{g(k/d)}{k/d} \right| = \sum_{n \leqslant x/2} \frac{1}{n} \left| \sum_{k \leqslant x/n} \frac{g(k)}{k} N\left(\frac{x}{nk}\right) \right|$$

$$\leqslant \sum_{n \leqslant x/2} \frac{1}{n} \sum_{k \leqslant x/n} \frac{|g(k)|}{k} \left| N\left(\frac{x}{nk}\right) \right| .$$

Now by two classical results of Landau ([2], pp. 613, 528)

$$|N(y)| = O(\exp(-a\sqrt{\log y})) \quad \text{as} \quad y \to \infty$$

where a is a positive constant, and $|N(y)| \leqslant 1$ for all $y \geqslant 1$. Let δ be a fixed but otherwise arbitrary number in $(0, 1)$, to be determined later, then

$$\sum_{k \leqslant x/n} \frac{|g(k)|}{k} \left| N\left(\frac{x}{nk}\right) \right| = O\Big(\sum_{k \leqslant \exp(\log^{1-\delta}(x/n))} \frac{|g(k)|}{k} \exp\big(-a(\log(x/n) - \log k)^{1/2}\big) \Big)$$

$$+ O\Big(\sum_{\exp(\log^{1-\delta}(x/n)) < k \leqslant x/n} \frac{|g(k)|}{k} \Big)$$

$$= O\Big(\exp\big(-a(\log(x/n) - \log^{1-\delta}(x/n))^{1/2}\big)\Big) +$$

$$+ O\big(\log^{(1-\delta)(-1-\Delta)}(x/n)\big)$$

$$= O\big(\log^{(1-\delta)(-1-\Delta)}(x/n)\big)$$

by the hypothesis that

$$\sum_{n\leq x}\frac{|g(n)|}{n} = A+O(\log^{-1-\Delta}x).$$

Since δ was an arbitrary fixed number in $(0, 1)$, choosing δ so that $\delta = \Delta/(1+\Delta)-\varepsilon'$, where ε' is a fixed positive number, and $0<\varepsilon'< <\Delta/(1+\Delta)$, we get

$$(4)\qquad \sum_{k\leq x/n}\frac{|g(k)|}{k}\left|N\left(\frac{x}{nk}\right)\right| = O\left(\log^{-1-\varepsilon}(x/n)\right)\quad \text{for a fixed } \varepsilon>0.$$

Substituting (4) in (3) gives

$$(5)\qquad |C|\sum\nolimits_1 = O\Bigl(\sum_{n\leq x/2}\frac{1}{n\log^{1+\varepsilon}(x/n)}\Bigr).$$

But

$$\sum_{n\leq x/2}\frac{1}{n\log^{1+\varepsilon}(x/n)} = \sum_{n\leq x/2}\frac{1}{x(n/x)\,|\log^{1+\varepsilon}(n/x)|}$$

and this last sum converges as $x\to\infty$ to

$$\int_0^{1/2}\frac{1}{t|\log t|^{1+\varepsilon}}\,dt = \int_{\log 2}^{\infty}\frac{1}{u^{1+\varepsilon}}\,du = \frac{(\log 2)^{-\varepsilon}}{\varepsilon},$$

on setting $t=e^{-u}$.

Since ε was fixed >0, it follows that $\sum_1 = O(1)$. Substituting this and (2) in (1) proves (III).

Remarks. Taking

$$g(n) = \begin{cases}1, & n=1,\\ 0, & \text{otherwise}\end{cases}$$

gives $f(n)=\mu(n)$ and Rubel's result, taking

$$g(n) = \begin{cases}1, & n \text{ a perfect square},\\ 0, & \text{otherwise}\end{cases}$$

gives

$$f(n) = \frac{6}{\pi^2}\lambda(n),$$

where $\lambda(n)$ is Liouville's function, etc. If we drop the condition $C\neq 0$ in the statement of the theorem and define $f(n)=\sum_{d|n}\mu(d)g(n/d)$ if $C=0$, then (II) and (III) may be derived as above, but (I) fails to hold.

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References

[1] G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge Tract 18 (1915).

[2] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, with an appendix by Paul T. Bateman, Chelsea reprint.

[3] L. A. Rubel, *An Abelian theorem for number-theoretic Sums*, Acta Arith. 6 (1960), pp. 175-177; Correction, Acta Arith. 6 (1961), p. 523. Also discussed in "Report of the Institute in the Theory of Numbers (June 21 — July 17, 1959)", sponsored by the American Mathematical Society and the National Science Foundation, pp. 322-323.

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