On the divisibility of $\sigma_r(n)$

by

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I. Introduction

The divisor function $\sigma_r(n)$ is defined by

$$\sigma_r(n) = \sum_{d \leq n} d^r$$

where the sum is over all positive integral divisors of $n$; in the following pages it will be assumed that $r$ is a positive integer. The aim of this paper is to investigate a certain divisibility property of $\sigma_r(n)$.

Let $q$ be a prime and $m$ a positive integer, and assume that both are fixed and independent of $x$. Denote by $D_m(r, q; x)$ the number of positive integers $n \leq x$ for which $q^m \parallel \sigma_r(n)$, where the notation $\parallel$ means that $q^m$ divides $\sigma_r(n)$ but $q^{m+1}$ does not. In this paper an asymptotic equation for $D_m(r, q; x)$ will be established. Define $\gamma$ by $q^{\gamma} \parallel r$, and let

$$m' = [m/(r+1)]$$

and

$$k = (q-1)/(r, q-1).$$

Then the precise result to be obtained is as follows:

**Theorem 1.** (i) If $q$ and $h$ are both odd, then, as $x \to \infty$,

$$D_m(r, q; x) \sim A_1^{(m)} x.$$  

(ii) If $q$ is odd and $h$ is even, then, as $x \to \infty$,

$$D_m(r, q; x) \sim A_1^{(m)} x (\log \log x)^{\gamma}(\log x)^{-b}.$$  

(iii) As $x \to \infty$,

$$D_m(r, 2; x) \sim A_1^{(m)} x (\log \log x)^{\gamma-m-1}(\log x)^{-1}.$$  

$A_1^{(m)}$, $A_2^{(m)}$, $A_3^{(m)}$ are positive constants depending only on $r$, $q$ and $m$.

The corresponding results for the case $m = 0$ have been obtained by E. A. Rankin in a paper [1] published in 1961. The function $D_0(r, q; x)$
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right of (5). The first stage in obtaining this estimate is to express the generating function

\[
f_n(s) = \sum_{n=1}^{m} a_n(n)n^{-s} \quad (s = \sigma + it)
\]

in terms of the Riemann zeta-function and Dirichlet \( L \)-functions, and the following result is proved.

**Theorem 2.** (i) If \( q \) and \( h \) are both odd,

\[
f_n(s) = \zeta(s)g(s),
\]

where \( \zeta(s) \) is the Riemann zeta-function and \( g(s) \) is holomorphic for \( s > \frac{1}{2} \) and bounded for \( s \geq \frac{1}{2} + \delta \) \( (\delta > 0) \).

(ii) If \( q \) is odd and \( h \) is even,

\[
f_n(s) = (\zeta(s))^{1-h} \sum_{\chi \bmod q} (\log \zeta(s))^{\chi} H_n(s),
\]

where each \( H_n(s) \) \( (0 \leq \chi \leq m) \) is a function involving Dirichlet \( L \)-functions associated with non-principal characters and functions satisfying the conditions on \( g(s) \) in (i).

(iii) If \( q = 2 \),

\[
f_n(s) = \sum_{\chi \bmod 2} (\log \zeta(s))^{\chi} H_n(s),
\]

where each \( H_n(s) \) \( (0 \leq \chi \leq m) \) satisfies the conditions given in (ii).

The second stage in estimating \( D_n(r, q; z) \) entails deriving Theorem 1 from Theorem 2. Theorem 1 (i) follows immediately from Theorem 2 (i) and the Wiener-Ikehara Theorem (which is stated in Lemma 10). However another result has to be proved in order that the rest of Theorem 1 can be deduced. Let

\[
h(s) = (\zeta(s))^{-2s} (\log \zeta(s))^{\chi} H(t),
\]

where \( 0 < \beta < 1 \), \( n \) is a non-negative integer and \( H(t) \) is a product of powers of Dirichlet \( L \)-functions associated with non-principal characters, non-negative powers of the logarithm of such functions, and a function holomorphic for \( s > \frac{1}{2} \) and bounded for \( s \geq \frac{1}{2} + \delta \) \( (\delta > 0) \). Furthermore suppose that \( h(s) \) can be expressed in the form

\[
h(s) = \sum_{n=1}^{\infty} b(n)n^{-s},
\]

where \( b(n) \geq 0 \). Then:

...
Theorem 3. (i) If $0 < \beta < 1$ and $u \geq 1$, then
\[ \sum_{n=1}^{x} b(n) = \frac{H(1)}{f(1-\beta)} x(\log \log x)^{\alpha}(\log x)^{-\beta} + O\left(\frac{x(\log \log x)^{\alpha-1/2}}{(\log x)^{1/2}}\right). \]

(ii) If $0 < \beta < 1$ and $u = 0$, then
\[ \sum_{n=1}^{x} b(n) = \frac{H(1)}{f(1-\beta)} x(\log x)^{-\beta} + O\left(x(\log x)^{-\beta/2}(\log x)^{-\epsilon}\right). \]

(iii) If $\beta = 1$ and $u \geq 2$, then
\[ \sum_{n=1}^{x} b(n) = uH(1)x(\log \log x)^{\alpha}(\log x)^{-1} + O\left(x(\log \log x)^{\alpha-1/2}(\log x)^{-\epsilon}\right). \]

(iv) If $\beta = 1$ and $u = 1$, then
\[ \sum_{n=1}^{x} b(n) = H(1)x(\log \log x)^{\alpha}(\log x)^{-1} + O\left(x(\log \log x)^{\alpha-1/2}(\log x)^{-1}\right). \]

(v) If $\beta = 1$ and $u = 0$, then
\[ \sum_{n=1}^{x} b(n) = O\left(x(\log x)^{-\beta}\right). \]

A proof of part (iv) of this theorem with $H(x) = 1$ forms part of one of the proofs of the Prime Number Theorem; Rankin [1] applied part (ii) of this result with $\beta = 1/\lambda$, and Watson’s paper [2] includes the proof of a similar result with $\lambda$ replaced by $\psi(\lambda)$. However, although some cases of this theorem are already known, to the author’s knowledge the statement and proof of the general result have not previously appeared in print.

Theorem 1 (ii) and (iii) will be deduced from Theorems 2 and 3 in part V of this paper. By the method of the following pages one can prove results analogous to Theorem 1 for the functions $d(n)$ and $\varphi(n)$, where $d(n)$ is the number of divisors of $n$ and $\varphi(n)$ is Euler’s function. The results which can be obtained in this way will be stated in part V. The proofs of Theorem 2 (i) and (ii) and Theorem 1 (i) are contained in part II. In order to simplify the details of the proof of Theorem 2, the case $\beta = 1$, stated in part (iii), is proved separately in part III although the method used is essentially the same as that contained in part II. Part IV contains the proof of Theorem 3.

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II. Proof of Theorem 2 (i) and (ii) and Theorem 1 (i)

The main object of part II is to prove Theorem 2 when $q$ is an odd prime; hence we shall assume throughout part II that $q \neq 2$. Our aim is to find an expression for
\[ f_{m}(q) = \sum_{n=1}^{\infty} a_{n}(n)n^{-s}, \]

to do this we first find (in Lemma 5) the positive integers $a$ for which $a_{n}(p^{r}) = 1$, where $p$ is a prime, and, since we shall see that $a_{n}(n)$ is not multiplicative, the next step, given in Lemma 8, is to express $a_{n}(m)$ in terms of $a_{r}(p^{r})$ ($0 \leq r \leq m$). It will then be shown that the required result follows.

In § 4 we shall deduce Theorem 1 (i) from Theorem 2 (i) and the Wiener-Ikehara Theorem.

1. A preliminary result. Let $p$ denote a prime. The purpose of this section is to obtain an expression for the order of $p^{r}$ modulo powers of $q$, and hence to prove Lemma 4, which will be needed later. Let $g$ be a primitive root (mod $q^{r}$) for all positive integers $r$. Then every prime $p$, $p \neq q$, satisfies a congruence relation of the form
\[ q^{m} \equiv g^{\varphi(m)} (\text{mod } q^{r}) \quad \text{where } 0 \leq \varphi(m) \leq q^{r-1}(q-1). \]

If $r_{2} > r_{1}$, $q^{r_{2}} \equiv g^{\varphi(r_{2})} (\text{mod } q^{r_{1}})$ and hence
\[ c_{p}(r_{2}) = c_{p}(r_{1}) (\text{mod } q^{r_{1}}). \]

Define $c_{p}(r)$ to be the highest power of $q$ dividing $c_{p}(r)$, so that $q^{c_{p}(r)} \nmid c_{p}(r)$ where, clearly, $0 \leq c_{p}(r) \leq r-1$. From (7) it follows that
\[ c_{p}(r_{1}) = \min\{c_{p}(r_{2}), r_{1}-1\}, \]

together with
\[ c_{p}(r_{1}+1) = c_{p}(r_{1}) \quad \text{or} \quad c_{p}(r_{2}) + 1. \]

Lemma 1. If $r > 1$ and $q \mid c_{p}(r)$, then $q \mid c_{p}(2)$ and
\[ c_{p}(2) = q^{c_{p}(r)}. \]

where $c_{p} = c_{p}(1)$. If $r > 1$ and $q^{c_{p}(r)} \nmid c_{p}(r)$, then
\[ c_{p}(r) = q^{c_{p}(r)}. \]

Proof. If we put $r_{1} = 2$ and $r_{2} = r$ in (8), we see that if $q \mid c_{p}(r)$, then $q \mid c_{p}(2)$. On putting $r_{1} = 1$ and $r_{2} = 2$ in (7) and using the inequality in (6), we obtain
\[ c_{p}(2) = c_{p} + q(q-1) \quad \text{where } 0 \leq q \leq q-1. \]
If \( q \mid q \), it follows that \( q \mid (c_q - u) \); since \(|c_q - u| < q\), we have \( u = c_q \). This gives (10).

Suppose now that \( r > 2 \) and \( q^{-1} \parallel c_q(r) \). Then \( c_q(r) = r - 1 \), and \( r - 1 \leq r \leq r - 2 \), and it follows from (8) that \( c_q(r - 1) = r - 2 \), \( c_q(r - 2) = r - 3 \), ..., \( c_q(2) = 1 \).

If \( 3 \leq i \leq r \), we have from (7) that

\[
c_q(i) = c_q(i - 1) + u_i q^{-2} (q - 1) \quad \text{where} \quad 0 \leq u_i \leq q - 1.
\]

If \( c_q(i - 1) = q^{-1} c_q \) and \( c_q(i - 1) \parallel c_q(i) \), it follows that \( u_i = c_q \) and \( c_q(i) = q^{-1} c_q \); this is true for \( i = 3, 4, \ldots, r \). Since \( q \mid c_q(r) \), (10) holds so that, when \( i = 3, c_q(i - 1) = q^{-2} c_q \); hence if \( r > 2 \) and \( q^{-1} \parallel c_q(r) \), then

\[
c_q(r) = q^{-1} c_q.
\]

We recall that \( q^t \parallel r \) and that \( h = (q - 1)/(r, q - 1) \). Define \( t = t(p) \) by

\[
q^t \parallel (r - 1);
\]

we shall assume now that \( r > t \). Our next lemma gives an expression for the order of \( p^t \) (mod \( q^t \)) when \( t \geq 1 \), and Lemma 3 gives a corresponding expression valid for \( t > 0 \). We adopt the convention that the order of \( p^t \) (mod \( q^t \)) is 1; if \( r < t \), then the order of \( p^t \) (mod \( q^t \)) is not defined.

If \( r > t \), then clearly the order of \( p^t \) (mod \( q^t \)) must exceed 1.

**Lemma 2.** If \( r > t \) and \( t \geq 1 \), then the order of \( p^t \) (mod \( q^t \)) is \( q^t \).

This result is proved by LeVeque [3] in Theorem 4.6, and will be deduced from Lemma 3.

**Lemma 3.** The order of \( p^t \) (mod \( q^t \)) is

\[
\lambda_p(r) h(h, c_q),
\]

where

\[
\lambda_p(r) = \begin{cases} 
q^{-1-t-p} & \text{if } r - 1 - \gamma - c_q(r) > 0, \\
1 & \text{if } r - 1 - \gamma - c_q(r) \leq 0.
\end{cases}
\]

**Proof.** We shall use (6). The order of \( q \) (mod \( q^t \)) by definition of a primitive root. Hence the order of \( (q^t) \) (mod \( q^t \)), \( h(r) \) say, is given by

\[
h(r) = \psi(q^t) = \frac{q^t - 1}{(q^t, q - 1)} = \frac{q^t - 1 - \gamma}{h} \frac{q - 1}{r - 1 - \gamma} \quad \text{if } r - 1 - \gamma > 0,
\]

\[
\text{and } h(r) = \frac{q - 1}{r - 1 - \gamma} \quad \text{if } r - 1 - \gamma \leq 0.
\]

It follows that, if \( r - 1 - \gamma > 0 \), the order of \( q^{p^t} \) (mod \( q^t \)), that is the order of \( p^t \) (mod \( q^t \)), is equal to

\[
\frac{h(r)}{(h(r), c_q(r))} = \frac{q^{-1-r}}{(q^{-1-r} c_q(r), h, c_q)} = \frac{h}{(h, c_q)}.
\]

provided that \( r - 1 - \gamma - c_q(r) > 0 \); the fact that \( h, c_q(r) = h, c_q \), which is used in the last step, follows from (7) on putting \( r = 1 \) and \( r = 1 \), since \( h \equiv (q - 1) \). If \( r - 1 - \gamma = 0 \), replace \( q^{r-1} \) by 1, and if \( r - 1 - \gamma - c_q(r) \leq 0 \), replace \( q^{r-1} \) by \( q^{-1} \) by 1, and thus in either of these cases the order of \( p^t \) (mod \( q^t \)) is \( h/(h, c_q) \). This completes the proof of the lemma.

We observe that, by (9), \( \lambda_p(r) = q^{p_r}(\gamma) \) or \( \lambda_p(r) \) according as \( c_q(r) = c_q(r) \) or \( c_q(r) + 1 \). It is not immediately evident that Lemmas 2 and 3 are equivalent if \( t > 1 \), so we shall now deduce Lemma 2 from Lemma 3. The order of \( p^t \) (mod \( q^t \)) is 1, and so \( h/(h, c_q) = 1 \). If \( t > 1 \), the order of \( p^t \) (mod \( q^t \)) is \( \lambda_p(r) = q^{-1} \) by Lemma 2, and \( \lambda_p(r) > 1 \). Hence, since \( \lambda_p(r + 1) > 1 \) and \( \lambda_p(r + 1) = 1 \), \( \lambda_p(r) = 1 \) if \( r = 1 \) and \( \lambda_p(r) = q^t \) by the remark at the beginning of this paragraph, so that \( \lambda_p(r + 1) = 1 \). On putting \( r = 1 \) and \( r = r \) in (8), we obtain \( c_q(r) = c_q(r) \). Hence, since \( c_q(r) = 1 \) so that \( r - 1 - \gamma - c_q(r) = 0 \), \( q^t \) (mod \( q^t \)) is \( q^{-1} \).

We define \( \mu_p(r) \) (mod \( q^{-1} \)) by Lemma 2 and 3.

\[
\mu_p(r) = \begin{cases} 
\lambda_p(r) h/(h, c_q) & \text{if } p' \neq 1 \text{ (mod } q) \text{,} \\
q^{-1} & \text{if } p' = 1 \text{ (mod } q) \text{,}
\end{cases}
\]

where \( \mu_p(r) = \mu_p(1) \). We observe that \( \mu_p(r) \geq \mu_p \geq 2 \) always, and that \( \mu_p(r) \geq \mu_p(r) \geq 3 \) if \( p' = 1 \) (mod \( q) \).

**Lemma 4.** If \( h \) is even, then \( \mu_p(r) = 2 \) and \( \mu_p(r) > 2 \). Then \( \mu_p(r) > 2 \). Clearly if \( \mu_p(r) = 2 \), then \( \mu_p = 2 \), and \( \lambda_p(r) = 1 \). Since \( \mu_p = h/(h, c_q) \), \( \mu_p = 2 \) if and only if \( c_q \) is an odd multiple of \( h \); this is so when \( c_q = \frac{1}{2} (2u - 1) \) where \( 1 \leq u \leq (r, q - 1) \), the bounds for the integer \( u \) following since \( 1 \leq c_q \leq q - 1 \), so that

\[
\frac{1}{2} (\frac{1}{2} + 1) \leq u \leq \frac{1}{2} (2(q - 1) - 1) = (r, q - 1) - 1.
\]

Thus there are exactly \( (r, q - 1) \) values of \( c_q \) which are such that \( \mu_p = 2 \), and hence \( c_q \geq 2 \) if and only if \( p \) is congruent to one of \( r, q - 1 \) elements of a reduced residue system (mod \( q) \).

We now find the number of values of \( c_q(r + 1) \), corresponding to a given value of \( c_q \), for which \( \mu_p(r + 1) = \mu_p(r) = \mu_p \). Clearly \( \mu_p(r) = q^t \) but \( \lambda_p(r + 1) > \lambda_p(r + 1) = 1 \) thus

\[
(r + 1) - 1 - \gamma - c_q(r + 1) = 1 \quad \text{and} \quad r - 1 - \gamma - c_q(r) = 0,
\]
giving \( e_r(r+1) = e_r(r) = r - 1 - \gamma \), provided \( r \geq \gamma + 1 \). By (8)

\[
e_r(r-\gamma) = \min(e_r(r), r - 1 - \gamma) = r - 1 - \gamma,
\]

and hence \( e_r(r+1) = e_r(r) = \ldots = e_r(r-\gamma) = r - 1 - \gamma \). Therefore \( q^{r-1-r} \parallel c_r(r-\gamma) \), and, by (11), \( c_r(r-\gamma) = q^{r-1}c_r \); thus to each \( c_r \) there corresponds exactly one \( c_r(r-\gamma) \). Now, by (7)

\[
c_r(r+1) = c_r(r-\gamma) + wq^{r-1}(q-1) \quad \text{where} \quad 0 \leq w < q^{r+1},
\]

so that

\[
c_r(r+1) = q^{r-1}(c_r + w(q-1)).
\]

Hence, if \( q^{r-1} \parallel c_r(r+1) \), \( q \parallel (c_r - u) \). This means that \( u \) can take any value from 0 and \( q^{r+1} - 1 \) except

\[c_r, c_r + q, \ldots, c_r + (q^{r-1} - 1)q,\]

and so, and hence \( c_r(r+1) \), can take \( q^{r+1} - q^r = \phi(q^{r+1}) \) values for each given value of \( c_r \).

It follows that \( \mu_\phi(r+1) = q \mu_\phi(r) = 2q \phi \) if and only if \( p \) is congruent to one of \( \phi(q^{r+1}), q(r-1) \) elements of a reduced residue system (mod \( q^{r+1} \)) provided \( r \geq 1 + 1 \). If \( r < \gamma + 1 \), we observe that \( \mu_\phi(r+1) = \mu_\phi(r) = 0 \) for all \( p \), so that \( p \) satisfies the required conditions. This completes the proof of the lemma.

2. The evaluation of \( \sum_{n=1}^{\infty} a_\gamma(p^n) \). We have already defined

\[a_\gamma(n) = \begin{cases} 1 & \text{if } q^{\gamma} \parallel a_\gamma(n), \\ 0 & \text{otherwise} \end{cases}\]

for \( r \geq 1 \); we define also \( a_\gamma(n) = 1 \) or 0 according as \( q \) does not divide or divides \( a_\gamma(n) \). Clearly the definitions imply that \( a_\gamma(1) = 1 \) and \( a_\gamma(1) = 0 \) for \( r \geq 1 \). For convenience we shall frequently write \( a(n) \) for \( a_\gamma(n) \). The results of this section and the next which involve \( a(n) \), but not \( a_\gamma(n) \) for \( r \geq 1 \), are all proved by Bankin [1]; Lemmas 5 and 6, parts (i) and (ii), and Lemma 7 are proved in the first part of §2 of his paper.

The next Lemma enables us to determine the form of \( a \) when \( a_\gamma(p^n) = 1, r \geq 0 \).

**Lemma 5.** (i) If \( p \neq q, a(p^n) = 1 \) if and only if \( a \neq 0\mu_\gamma - 1 \) for any integer \( u \).

(ii) \( a(q^n) = 1 \) for all \( a \).

(iii) If \( r \geq 1, p \neq q \) and \( \mu_\gamma(r+1) = q \mu_\gamma(r) \), then \( a(p^n) = 1 \) if and only if \( a = u\mu_\gamma(r) - 1 \) where \( (u, q) = 1 \).

(iv) If \( r \geq 1 \) and either \( q \) or \( \mu_\gamma(r+1) = \mu_\gamma(r) \), then \( a(p^n) = 0 \) for all \( a \).

**Proof.** We have

\[a(p^n) = 1 + p^n + p^{2n} + \ldots + p^{rn} = (p^{rn+1} - 1)/(p^{n} - 1),\]

and \( q^n \parallel (p^{n} - 1) \) where \( t > 0 \). For any \( r > 0 \), \( q^n \parallel a(p^n) \) implies that \( q^{rn} \parallel (p^{rn+1} - 1) \), and this occurs if and only if the order of \( p \) (mod \( q^{rn+1} \)), which is \( \mu_\gamma(r) \) by definition, divides \( r+1 \) but the order of \( p \) (mod \( q^{rn+1} \)), which is \( \mu_\gamma(r+1) \), does not. (We recall that the order of \( p \) (mod \( q^{rn+1} \)) is 1, and we use this convention also when \( t = 0 \).)

(i) If \( p \neq q, a(p^n) = 1 \) if and only if \( \mu_\gamma(r+1) \neq (r+1) \), which gives the result.

(ii) \( a(q^n) = 1 \) (mod \( q^n \)), and hence the result follows.

(iii) If the given conditions are satisfied, then from above \( a(p^n) = 1 \) if and only if

\[\mu_\gamma(r) \neq (r+1) \quad \text{but} \quad \mu_\gamma(r+1) = (r+1).
\]

Since \( \mu_\gamma(r+1) = q \mu_\gamma(r) \), the result follows.

(iv) This part is an immediate consequence of the proof of (ii) if \( p = q \) and of (iii) if \( \mu_\gamma(r+1) = \mu_\gamma(r) \).

**Lemma 6.** (i) If \( p \neq q \),

\[\sum_{n=0}^{\infty} a_\gamma(p^n) p^{-n} = (1 - p^{-r(\text{mod} q^{rn+1})})(1 - p^{-r(\text{mod} q^{rn+1})}).\]

(ii) \( \sum_{n=0}^{\infty} a(p^n) q^{-n} = (1 - q^{-1}).\)

(iii) If \( r \neq 1 \), \( p \neq q \) and \( \mu_\gamma(r+1) = q \mu_\gamma(r) \), then

\[\sum_{n=0}^{\infty} a_\gamma(p^n) p^{-n} = (1 - p^{-r(\text{mod} q^{rn+1})})(1 - p^{-r(\text{mod} q^{rn+1})})(1 - p^{-r(\text{mod} q^{rn+1})}).\]

(iv) If \( r \neq 1 \) and either \( p = q \) or \( \mu_\gamma(r+1) = \mu_\gamma(r) \), then

\[\sum_{n=0}^{\infty} a_\gamma(p^n) p^{-n} = 0.
\]

**Proof.** This Lemma follows from the previous one. For example, to prove (iii) we have, if the given conditions hold, that

\[\sum_{n=0}^{\infty} a_\gamma(p^n) p^{-n} = \sum_{n=0}^{\infty} p^{-n(\text{mod} q^{rn+1})} p^{n} \sum_{n=0}^{\infty} p^{-n(\text{mod} q^{rn+1})} - \sum_{n=0}^{\infty} p^{-n(\text{mod} q^{rn+1})},\]

and we obtain the result on summing these two geometric series.
3. The generating functions. Since \( \sigma(n) \) is multiplicative, we can write
\[
\sigma(n) = \prod_{p^a \mid n} \sigma(p^a),
\]
where the product is over all distinct primes dividing \( n \). From this it follows that \( \sigma(n) \) is multiplicative; for \( q \not\mid \sigma(n) \) if and only if \( q \not\mid \sigma(p^a) \) for every \( p^a \mid n \). Hence
\[
a(n) = \prod_{p^a \mid n} a(p^a).
\]

Let
\[
f(s) = \sum_{n=1}^\infty a(n) n^{-s}.
\]
then we have

**Lemma 7.** \( f(s) = \zeta(s) \prod_{p^a} \left( 1 - p^{-c p^{a-1}s} \right)/\left( 1 - p^{-mp^a} \right) \).

Proof. Since \( a(n) \) is multiplicative, we have by Lemma 6 (i) and (ii) that
\[
f(s) = \sum_{n=1}^\infty a(n) n^{-s} = \prod_{p^a} \left( \sum_{m=1}^\infty a(p^a) p^{-ms} \right)
\]
\[
= (1 - q^{-1})^{-1} \prod_{p^a} \left( 1 - p^{-c p^{a-1}s} \right)/\left( 1 - p^{-mp^a} \right) = \zeta(s) \prod_{p^a} \left( 1 - p^{-c p^{a-1}s} \right)/\left( 1 - p^{-mp^a} \right).
\]

However, although \( a(n) \) is multiplicative, \( a_n(n) \) is not; for \( q^n = \sigma(p^a) \) does not hold if \( q^n = \sigma(p^a) \) unless \( n = p^a \). Nevertheless we can obtain an expression for \( a_n(n) \) in terms of \( a(n) \) and \( a(p^a) \), where \( n_1 \mid n \), \( p^a \mid n \) and \( k \leq m \). In the following lemma we assume that \( p^a \mid n \) for all \( i \) (with or without a suffix), and that two primes \( p \) with different suffixes are distinct. Let \( R_k \) denote a set \( r_1, r_2, \ldots, r_k \) of positive integers, with \( 1 \leq r_1 \leq r_2 \leq \ldots \leq r_k \), and let \( R(m) \) represent the collection of sets \( R_k \) whose members satisfy \( r_1 + r_2 + \ldots + r_k = m \), where \( k \) takes all possible values; clearly \( 1 \leq k \leq m \). Let \( \Psi_k \) denote an ordered set of distinct primes \( p_1, p_2, \ldots, p_k \), and \( \Psi_k \) those of \( \Psi_k \) that denote the ordered set of primes \( p_1, p_2, \ldots, p_k \).

**Lemma 8.** If \( m \geq 1 \),
\[
a_n(n) = \sum_{R \in R(m)} \left( \sigma(R) \right)^{-1} \sum_{k=1}^m a_k(p_1^a) a_k(p_2^a) \ldots a_k(p_k^a) a(p_1^{a_1} \ldots p_k^{a_k}),
\]
where (i) the set \( R_k \) ranges over all sets belonging to \( R(m) \), (ii) each \( k \) does not exceed the number of primes dividing \( n \), (iii) the sum over \( \Psi_k \) represents the sum over all sets \( \Psi_k \) consisting of the distinct primes dividing \( n \), and (iv) \( \sigma(R_k) \) is defined below.

Proof. Since \( \sigma(n) \) is multiplicative, we may write
\[
\sigma(n) = \prod_{p^a \mid n} \sigma(p^a),
\]
where \( q^a = \sigma(p^a) \) if \( j = 1, 2, \ldots, k \), and \( q \not\mid \sigma(p^a) \) if \( j \not\mid \sigma(p^a) \); then we have \( a_k(p_j^a) = 1 \) if \( j = 1, 2, \ldots, k \), and \( a(p_1^{a_1} \ldots p_k^{a_k}) = 1 \). It follows that \( \sigma(R_k) \) if \( a_k(n) = 1 \) if and only if \( r_1 + \ldots + r_k = m \), and that there will be only one set \( R_k \in R(m) \), which we denote by \( R_k \), and certain sets \( \Psi_k \), which we denote by \( \Psi_k \), for which this holds.

Consider now the expression
\[
M(R_k, \Psi_k) = a_k(p_1^{a_1} \ldots p_k^{a_k}) a(p_1^{a_1} \ldots p_k^{a_k}).
\]
From above \( a_k(n) = 1 \) if and only if \( M(R_k, \Psi_k) = M(R_k, \Psi_k) = 1 \). We shall now calculate the number \( \sigma(R_k) \) of sets \( \Psi_k \) for which, fixed \( R_k \), leave \( M(R_k, \Psi_k) \) essentially unaltered, that is, unaltered except for a rearrangement in the order of its factors. Suppose that \( R_k \) has \( r \) distinct elements \( r_1', \ldots, r_r' \), where \( r_1' < r_2' < \ldots < r_r' \), occurring \( l_1, l_2, \ldots, l_r \) times respectively in the set; clearly \( l_1 + \ldots + l_r = k \). If we rearrange the primes of the set \( \Psi_k \), amongst themselves (and there are \( 1 \) ! = \( 1 \) difference arrangements of \( \Psi_k \), \( M(R_k, \Psi_k) \) will remain essentially unaltered (since every member of \( \Psi_k \) is associated with \( r_j' \) in \( M(R_k, \Psi_k) \)); similarly if we rearrange the members of the set \( \Psi_k \), amongst themselves, \( M(R_k, \Psi_k) \) will remain essentially unaltered. Hence we can arrange the members of the set \( \Psi_k \), amongst themselves, \( M(R_k, \Psi_k) \), without essentially altering \( M(R_k, \Psi_k) \). However, if we alter the order of the members of \( \Psi_k \), in any way other than those mentioned above, or if we replace \( R_k \) by another set \( R_k \) (possibly having a different number of members) and/or replace \( \Psi_k \) by another set \( \Psi_k \), we shall essentially alter \( M(R_k, \Psi_k) \).

If \( a_k(n) = 1 = R_k \in R(m) \), it follows from above that \( M(R_k, \Psi_k) = 0 \) unless \( R_k = \Psi_k \) and \( \Psi_k \) is one of the \( \sigma(R_k) \) sets \( \Psi_k \) if \( a_k(n) = 0 \), \( M(R_k, \Psi_k) = 0 \) for all \( R_k \in R(m) \) and all \( \Psi_k \). Hence
\[
\sum_{R_k \in R(m)} \sum_{\Psi_k} M(R_k, \Psi_k) = \sigma(R_k) a_k(n) \text{ if } a_k(n) = 1,
\]
\[
0 \text{ if } a_k(n) = 0.
\]

Unless \( a_k(n) = 1 \) and \( R_k = \Psi_k \), the inner sum on the left is zero (for at least one factor of each term \( M(R_k, \Psi_k) = 0 \)), hence the result follows.
We are now in a position to find the generating function

$$f_m(s) = \sum_{n=1}^{\infty} a_n(n)n^{-s}$$

for \( m \geq 1 \). Let \( S(r) \) be the set of all primes \( p \) for which \( \mu_p(r+1) = q \mu_p(r) \). Then we have

**Lemma 9.**

$$f_m(s) = f(s) \sum_{p \in S(r)} \left( q^s R_0 \right)^{-1} \sum_{r_1 \leq \cdots \leq r_k} P[\mu_1, \mu_2(r_1); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k)] \times$$

$$\times P[\mu_1, \mu_2(r_1); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k)]$$

where

$$P(p, \mu_p(r); s) = \frac{(1 - q^{-s})(1 - p^{-s})(1 - p^{-s})(1 - p^{-s})(1 - p^{-s})}{(1 - q^{-s})(1 - q^{-s})(1 - q^{-s})(1 - q^{-s})(1 - q^{-s})},$$

and where the sum over \( p_i (i = 1, 2, \ldots, k) \) is over all primes \( p \in S(r) \) except \( q, p_1, p_2, \ldots, p_{k-1} \).

**Proof.** By Lemma 8,

(13) \( f_m(s) = \sum_{n=1}^{\infty} \left( \sum_{\mu \in S(r)} \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} M(R_0, \mu \mu) \right) n^{-s} \)

$$= \sum_{\mu \in S(r)} \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} a_1(p_1^{n_1}) \cdots a_k(p_k^{n_k}) \times$$

$$\ldots a_{k-1}(p_{k-1}^{n_{k-1}}) \times \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} a_1(p_1^{n_1}) \cdots a_k(p_k^{n_k}) \times$$

$$\ldots a_{k-1}(p_{k-1}^{n_{k-1}}) \times \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} a_1(p_1^{n_1}) \cdots a_k(p_k^{n_k}) \times$$

$$\ldots a_{k-1}(p_{k-1}^{n_{k-1}}) \times \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} a_1(p_1^{n_1}) \cdots a_k(p_k^{n_k}) \times$$

$$\ldots a_{k-1}(p_{k-1}^{n_{k-1}}) \times \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} a_1(p_1^{n_1}) \cdots a_k(p_k^{n_k}) \times$$

where the sum over \( p_i (i = 1, 2, \ldots, k) \) is over all primes except \( p_1, \ldots, p_{k-1} \).

By Lemma 6 (i) and (ii), we have if \( p_i \neq q \), \( i = 1, 2, \ldots, k \), that

(14) \[ \sum_{n=1}^{\infty} a_n(n)n^{-s} = \prod_{p \neq q, p \in S(r)} \left( \sum_{n=0}^{\infty} a(p^n)p^{-ns} \right) \]

$$= f(s) \prod_{p \neq q, p \in S(r)} \left( \sum_{n=0}^{\infty} a(p^n)p^{-ns} \right) \times$$

$$f(s) \prod_{p \neq q, p \in S(r)} \left( 1 - p^{-s} \right) \left( 1 - p^{-s} \right) \left( 1 - p^{-s} \right) \left( 1 - p^{-s} \right) \left( 1 - p^{-s} \right).$$

We do not need to consider the above sum with any \( p_i \) equal to \( q \); for if \( p_i = q \), \( a_q(q^n) = a_q(q^n) = 0 \) for all \( q \) by Lemma 5 (iv) and the corresponding term on the right of (13) is zero. If we substitute (14) in (13) and use Lemma 6 (iii) and (iv), we obtain the result of the lemma.

The following considerations may help to make the form of the above result seem logical. We may write

(15) \[ f_m(s) = \sum_{\mu \in S(r)} \left( q^s R_0 \right)^{-1} \sum_{\mu \in S(r)} P[\mu_1, \mu_2(r_1); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k)] \times$$

$$\times P[\mu_1, \mu_2(r_1); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k)],$$

where the error term is of smaller order of magnitude than the first term, as will become apparent in § 5 when \( m' \geq 1 \), unless \( f_m(s)/f(s) = O(1) \) for \( s > \frac{1}{2} \) which is so in § 4 and in § 5 when \( m' = 0 \). It can easily be shown that the main term on the right of (15) is the coefficient of \( a^n \) in the expansion of

$$\exp \left[ \sum_{r=1}^{\infty} \sum_{\mu \in S(r)} P[\mu_1, \mu_2(r_1); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k); \ldots, \mu_{k-1}, \mu_k(r_k)] \right].$$

**4. Proofs of Theorems 1 (i) and 2 (i).** In this section we shall assume that \( k \) is odd. It follows from (12) that \( \mu_p \) cannot be even, so that \( \mu_p \geq 3 \), and hence \( \mu_p \geq 3 \) for all \( r \geq 1 \). From the definition of \( P[p, \mu_p(r); s] \),

$$|P[p, \mu_p(r); s]| \leq \frac{(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})}{(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})(1 + p^{-s})},$$

$$\leq Q(s)p^{-5s},$$

where \( Q(s) \), a function of \( s = \text{Re} s \) only, is obtained by using the inequalities \( p \geq 2 \), \( \mu_p \geq 3 \), \( \mu_p \geq 3 \). Hence

(16) \[ \left| \sum_{p \in S(r)} P[p, \mu_p(r); s] \right| \leq \sum_{p} |P[p, \mu_p(r); s]| \times$$

$$\leq Q(s) \sum_{p} p^{-5s} \leq O(s) \sum_{p} p^{-3s}$$

which is convergent for \( s > \frac{1}{2} \) thus the sum on the left is absolutely convergent for \( s > \frac{1}{2} \).
Since $\mu_p \gg 3$, the infinite product in the expression for $f(s)$, given in Lemma 7, is also absolutely convergent for $s > \frac{1}{2}$. Hence it follows from Lemmas 7 and 9 and above that

$$f_m(s) = \zeta(s)g(0),$$

where $g(s)$ is holomorphic for $s > \frac{1}{2}$ and bounded for $s \geq \frac{1}{2} + \delta$ ($\delta > 0$). This completes the proof of Theorem 2 (i).

We now show that Theorem 1 (i) follows from Theorem 2 (i) and the Wiener-Ikehara Theorem we state in

**Lemma 10.** If $\Phi(r)$ is a non-negative, non-decreasing function in $0 \leq r < \infty$ such that the integral

$$F(s) = \int_0^\infty e^{-r}\Phi(r)dr$$

converges for $s > 1$, and if for some constant $B$ and some function $G(t)$, where $t = \text{Im} s$,

$$\lim_{t \to \infty} \left\{ F(s) - \frac{B}{s - 1} \right\} = G(t)$$

uniformly in every finite interval $-a \leq t \leq a$, then

$$\lim_{t \to \infty} \frac{\Phi(r)}{e^{\pi r}}e^{-\pi r} = B.$$

This is given in § 17 of Chapter V of Widder [4]. To deduce the result from this, let

$$sF(s) = f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s}, \quad \Phi(r) = S(e^r) = \sum_{n=1}^{\infty} a_m(n), \quad B = g(1);$$

then in order to prove Theorem 1 (i) we need to estimate $S(s)$, for

$$S(s) = \sum_{n=1}^{\infty} a_m(n) = D_m(r, q; \sigma).$$

Clearly $f_m(s)$ is holomorphic for $s > 1$, so that

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s} = \sum_{n=1}^{\infty} (S(n) - S(n-1))n^{-s} = \int_0^\infty y^{-s}S(y)\frac{dy}{y}.$$

$$= \int_0^\infty e^{-s}\Phi(r)dr = \int_0^\infty e^{-s}\Phi(r)dr + \int_0^\infty e^{-s}\Phi(r)dr = \lim_{t \to \infty} \frac{\Phi(r)}{e^{\pi r}}e^{-\pi r} = B.$$

converges for $s > 1$. Since $\zeta(s) - (s-1)^{-1}$ is holomorphic for $s > 0$ (see Lemma 13 (i)) and $g(s)$ is holomorphic for $s > \frac{1}{2}$, it follows that $f_m(s)e^{-s}$ is holomorphic for $s > \frac{1}{2}$, so that

$$\lim_{s \to 1^+} \left\{ f_m(s)e^{-s} - g(1)(s-1)^{-1} \right\} = G(t)$$

uniformly in every finite interval $-a \leq t \leq a$. Hence the conditions of Lemma 10 are satisfied and an application of it yields

$$\lim_{r \to \infty} \Phi(r)e^{-\pi r} = g(1),$$

whence

$$\lim_{r \to \infty} S(e^r) = g(1).$$

Thus as $r \to \infty$,

$$S(r) = \sum_{n=1}^{\infty} a_m(n) = D_m(r, q; \sigma) \sim g(1)e^{r}$$

which is Theorem 1 (i).

**5.** Proof of Theorem 2 (ii). We assume first that $k$ is even and

$$m' = [m/(y+1)] \geq 1.$$ Then it follows from the proof of Lemma 4 that for any positive integer $r$ there exist primes $p$ for which $\mu_p(r) = 2$. For such a prime $p$ we have by the definition of $P(p, \mu_p(r); s)$ that

$$P(p, \mu_p(r); s) = \frac{1}{1 - p^{-k-1-s}} = \frac{1}{1 - p^{-k-2}}.$$ Let $S_k(r)$ be the set of all primes $p$ which satisfy $p \leq S(r), p \geq q$, $\mu_p(r) = 2$. We recall that $p \leq S(r)$ if $\mu_p(r) = 2$. Then

$$\sum_{n \leq \sigma_p-1} P(p, \mu_p(r); s) = \sum_{p \leq S_k(r)} p^{-s} + \psi_k(s),$$

where the runs on the left and right are non-empty if and only if $n_k > y+1$. By Lemma 4 and, by the arguments used at the beginning of § 4, $\psi_k(s)$ is holomorphic for $s > \frac{1}{2}$ and bounded for $s > \frac{1}{2} + \delta$ for any $\delta > 0$.

**Lemma 11.** Assume that $r \geq y+1$. Let $b_1, b_2, \ldots, b_n$, where $\kappa = \phi(q^{n+1})/(q-1)$, be the distinct elements of a reduced residue system (mod $q^{n+1}$) which occur in the proof of Lemma 4, and let $\chi$ be a character and $\chi_1$ the principal character (mod $q^{n+1}$). If $L(s, \chi)$ is the Dirichlet L-series associated with the character $\chi$, and $G(s, \chi)$ is a certain function which is holomorphic for $s > \frac{1}{2}$ and bounded for $s > \frac{1}{2} + \delta$ for any $\delta > 0$, then

$$\sum_{n \leq \sigma_p-1} p^{-s} = (r, q-1)g^{-s}(\log(z) + \log(1 - q^{-s})) +$$

$$+ q^{-s}(q-1)\sum_{n=1}^{\infty} \sum_{\chi} \frac{\log L(s, \chi)}{\chi(b_1)} + \sum_{\chi} G(s, \chi)$$

where the sum over $\chi$ is over all characters $\chi$ (mod $q^{n+1}$) except, when indicated, $\zeta_0$. 

On the divisibility of $a_n(s)$
Proof. By Lemma 4,

\[
\sum_{\chi \mod q^{1+\delta}} \frac{1}{\chi(b)} = \sum_{\chi \mod q^{1+\delta}} \frac{\chi(b)}{\chi(b)} = \sum_{\chi \mod q^{1+\delta}} \frac{\chi(p)}{\chi(b)} = \zeta(p)^{-1},
\]

and the \( b_i \) can be determined from the proof of Lemma 4. It is well known that

\[
\psi \left( q^{1+\delta} \right) \sum_{\chi \mod q^{1+\delta}} \frac{1}{\chi(b)} = \sum_{\chi \mod q^{1+\delta}} \frac{\chi(p)}{\chi(b)} \psi \left( q^{1+\delta} \right),
\]

where the sum over \( \chi \) is over all characters \( \chi \mod q^{1+\delta} \), and that

\[
\sum_{\chi \mod q^{1+\delta}} \frac{\chi(p)}{\chi(b)} = \log L(s, \chi) - \sum_{\chi \mod q^{1+\delta}} \frac{1}{\chi} \frac{\chi(p)}{\chi(b)} = \log L(s, \chi) + G(s, \chi)
\]

(say), where \( G(s, \chi) \) satisfies the conditions given in the statement of the lemma; these results appear, for example, in §13 and §14 of Hasse [15]. Since \( L(s, \chi) = (1 - \zeta(s))^{\chi(s)} \), it follows that

\[
\sum_{\chi \mod q^{1+\delta}} \frac{\chi(p)}{\chi(b)} = \left( q^{1+\delta} \right)^{\gamma - 1} \left( \log \left( 1 - \zeta(s) \right) \right) + \sum_{\chi \mod q^{1+\delta}} \frac{\log L(s, \chi)}{\chi(b)} + \sum_{\chi \mod q^{1+\delta}} G(s, \chi).
\]

and hence the result of the lemma follows from this and (19).

Lemma 12. Let \( \gamma \) be a primitive root \( \mod q \) and let \( \chi(n) \) be the character defined by

\[
\chi(n) = e(\beta n) \quad \text{for} \quad n = \gamma \quad \text{mod} \quad q,
\]

where \( e(z) = \exp(2\pi iz) \). Then

\[
f(c) = \zeta(s) \left( F(s) \right)^{\beta} \psi(c),
\]

where \( \psi(c) \) is holomorphic for \( c > \frac{1}{2} \) and bounded for \( c > \frac{1}{2} + \delta \) for any \( \delta > 0 \), and where

\[
F(s) = \prod_{\chi \mod q^{1+\delta}} \left( L(s, \chi) \right)^{\beta} \zeta(s, \chi) = \prod_{\chi \mod q^{1+\delta}} \left( 1 - p^{-s} \right)^{-1} \left( \log \zeta(s) \right)^{-1} \left( \log \zeta(s) \right)^{-1}.
\]

With the exception of the last representation for \( F(s) \), this lemma is proved by Rankin [1] in the paragraphs leading up to equation (14) of his paper. To prove the last part, we observe that, since \( \gamma \) is a character \( \mod q \),

\[
F(s) = \prod_{\chi \mod q^{1+\delta}} \left( L(s, \chi) \right)^{\beta} \zeta(s, \chi) = \prod_{\chi \mod q^{1+\delta}} \left( 1 - p^{-s} \right)^{-1} \left( \log \zeta(s) \right)^{-1} \left( \log \zeta(s) \right)^{-1}.
\]

We are now able to complete the proof of Theorem 2 (ii) when \( m' \geq 1 \).

From (18) and Lemmas 9 and 11 we obtain

\[
f_m(c) = f(c) \sum_{\gamma \mod q^{1+\delta}} \left( \log \left( 1 - \zeta(s) \right) \right) + \sum_{\gamma \mod q^{1+\delta}} G(s, \chi) \psi(c)
\]

where \( \chi(c) \) is a character \( \mod q^{1+\delta} \) and \( \gamma \) represents a set of positive integers \( r_1, r_2, \ldots, r_k \) satisfying \( \gamma + 1 < \gamma_1 < \gamma_2 < \cdots < \gamma_k \) (so that the set of all \( \gamma \) is a subset of the set of all \( \gamma \)). Clearly the term on the right containing the highest power of \( \log \zeta(s) \) will occur when the product contains its maximum number of terms which implies that \( k \) takes its maximum value. Now \( k \) will be greatest when the \( r_k \) are as near to the value \( \gamma + 1 \) as possible, and hence the maximum value of \( k \) is

\[
\left[ \frac{m}{\gamma + 1} \right] = m';
\]

in this case \( r_i = \gamma + 1 + r_i \), \( i = 1, 2, \ldots, m' \), where

\[
0 < r_i < m - m' (\gamma + 1) < \gamma + 1 \quad \text{and} \quad \sum_{i=1}^{m'} r_i = m - m' (\gamma + 1).
\]

Now \( \gamma \) represents a set of the form \( \gamma + 1 + r_1 + r_2, \gamma + 1 + r_1 + r_2, \ldots, \gamma + 1 + r_m \), and the number of sets \( \gamma \) is \( \psi(m, \gamma) \), where \( \psi(m, \gamma) \) is the number of unrestricted partitions of \( m - m' (\gamma + 1) \) into at most \( m' \) parts. It follows that the term on the right of (21) becomes the highest power of \( \log \zeta(s) \) is

\[
f_m(c) \psi(m, \gamma) \gamma (\gamma - 1)^{m' - m} \psi(m, \gamma) \psi(m, \gamma)
\]

where

\[
\psi(m, \gamma) = \sum_{\gamma \mod q^{1+\delta}} \left( \log \left( 1 - \zeta(s) \right) \right)^{-1},
\]
the sum having \( \zeta(m, \gamma) \) terms. The remaining terms will be of the form

\[
 f(s)(\log \zeta(s))^{\nu} \prod_{\nu=1}^{\nu} \left[ \log L(s, \chi^{(n)}) \right] \eta(s),
\]

where \( 0 \leq u < m' \), \( 0 \leq u < m' - u \) and \( r_1 < m - u \gamma + 1 \), where the \( r_1 \) are not necessarily all distinct, and \( \chi^{(n)} \) is a non-principal character (mod \( q^{1+n} \)), and where \( \eta(s) \), a function of \( s \) and the characters occurring in (21), is holomorphic for \( \sigma > \frac{1}{2} \) and bounded for \( \sigma \geq \frac{1}{2} + \delta \) for any \( \delta > 0 \). From Lemma 12 and (21) to (23), it follows that

\[
 f_m(s) = (\zeta(s))^{1-1/k} \sum_{m' \leq m} \log \zeta(s)^{2H_{m'}(s)},
\]

where \( H_{m'}(s) \), \( 0 \leq u < m' \), satisfies the conditions of Theorem 2 (ii), and \( H_{m'}(s) \) can be obtained from (22) and Lemma 12.

In order to complete the proof of Theorem 2 (ii), we now assume that \( m' = 0 \), and as before that \( k \) is even. Since \( m' = 0 \), \( m \leq \gamma \) so that \( \gamma^s \) is analytic for \( s > 1/2 \). If \( r \leq m \leq \gamma \), then by (12) and Lemma 4,

\[
 \mu_1(r) = 2 \quad \text{and} \quad \mu_1(r+1) = \frac{q}{\mu(p(r))}
\]

cannot both hold; for if \( p' \equiv 1 \pmod{m} \), \( \mu_1(r) = \mu_2(p') \), and if \( p' \equiv 1 \pmod{m} \), \( \mu_1(r) = \mu_1(p') \geq q \geq 3 \). Hence, for \( r_1 \leq m \),

\[
 \sum_{1 \leq i \leq m', 1 \leq j \leq n} P(p_i, \mu(p_i); \xi_i, \xi_j)
\]

is absolutely convergent for \( \sigma > 1/2 \) by the arguments which lead to (16). From Lemma 9 it follows that

\[
 f_m(s) = f(s) \eta(s),
\]

where \( \eta(s) \) is holomorphic for \( \sigma > \frac{1}{2} \) and bounded for \( \sigma \geq \frac{1}{2} + \delta \) for any \( \delta > 0 \). Since \( k \) is even there exist \( \sigma \) for which \( \mu_\sigma = 2 \), and hence by Lemma 12

\[
 f_m(s) = (\zeta(s))^{1-1/k} H_{m'}(s),
\]

where \( H_{m'}(s) \) satisfies the conditions of Theorem 2 (ii).

III. Proof of Theorem 2 (iii)

We have proved Theorem 2 when \( g \) is an odd prime, and we shall now sketch the proof of part (iii) for which we assume that \( g = 2 \). Where possible we shall refer the reader to part II of the paper; to facilitate this, when a lemma or equation has to be restated or a section replaced, it will be given the same number followed by *'. The main differences occur in the first section (owing to the peculiarities of the prime 2), and sections 4 and 5 are replaced by a new section.

1. A preliminary result. If \( r \geq 3 \), every odd prime \( p \) satisfies

\[
 p = \pm 5^{\nu+1}, \quad \text{where} \quad 1 \leq c_p(r) < 2^{r-2},
\]

where we take the + or − sign according as \( p = 1 \) or 3 (mod 4). Equations (7) and (8) become

\[
 c_p(r_1) = c_p(r_1), \quad \text{mod} \quad 2^{r-2},
\]

and equation (9) still holds; Lemma 1 is not relevant when \( g = 1 \).

When \( g = 2 \), \( h = 1 \). We define \( t \) as before by \( 2 \equiv (g^r-1) \). Furthermore if \( p = 3 \) (mod 4) we define \( t = t(p) \) by

\[
 2^t \equiv (-p^r-1) \mod 2,
\]

then \( t = t \) when \( p \) is even, but \( t \geq 2 \) and \( t = 1 \) when \( p \) is odd. Clearly \( t \geq 1 \) always. We assume that \( r \geq t \) and \( r \geq 3 \).

Lemma 3'. The order of \( p' \mod 2 \) is \( \lambda_p(r) \), where

\[
 \lambda_p(r) = 2^r \quad \text{if} \quad r \geq 2 - \gamma - c_p(r) \geq 0,
\]

\[
 1 \quad \text{if} \quad r \geq 2 - \gamma - c_p(r) < 0
\]

except when \( p \) is odd, \( p = 3 \) (mod 4) and \( p' = -1 \) (mod 2), in which case

\[
 \lambda_p(r) = 2.
\]

Proof. When \( p = 1 \) (mod 4), the proof is similar to the proof of Lemma 3. Hence suppose that \( p = 3 \) (mod 4), so that \( -p = 1 \) (mod 4). If \( r \geq t' \), the order of \( (-p)^r \mod 2 \) is \( 2^{r-2} \equiv -1 \). Thus, since \( (-1)^{\nu+1} = -1 \), the order of \( p' \mod 2 \) is \( \lambda_p(r) \). If \( r < t' \), \( (-p)^r = 1 \) (mod 2), and hence \( p' = 1 \) (mod 2). We are assuming that \( r \geq t' \) and \( r = t' \) when \( p \) is even, and so the order of \( p' \mod 2 \) is \( \lambda_p(r) \). However when \( r \) is odd, \( p' = -1 \) (mod 2) and the order of \( p' \mod 2 \) is 2.

When \( t \geq 3 \), so that \( 8 \equiv (g^r-1) \), Lemma 3' gives us a simpler expression for the order of \( p' \mod 2 \) than Lemma 3'; (the fact that these expressions are equivalent follows as in part II). We can also simplify Lemma 3' when \( t = 1 \) or 2 as we show in the following
Corollary. If \( t = 2 \) or \( t = 1 \) and \( p = 3 \) (mod 8), then the order of \( p^r \) (mod 2) is \( 2^{t-r} \). If \( t = 1 \) and \( p = 7 \) (mod 8), then the order of \( p^r \) (mod 2), that is \( \lambda_p(r) \), is given by
\[
\lambda_p(r) = 2^{t-r} \quad \text{if} \quad 0 \leq r < t \quad \text{and} \quad \lambda_p(r) = 2^{-r} \quad \text{if} \quad r \geq t.
\]

Proof. If \( t = 1 \) or \( t = 2 \), then \( r \) must be odd, so that \( r = 0 \); if \( r \) is even, \( p^r = 1 \) (mod 8) and \( t \geq 3 \). If \( t = 2 \), so that \( p \equiv (p^r-1) \equiv 2 \) (mod 8), then \( p = 5 \) (mod 8), and if \( t = 1 \) and \( p = 3 \) (mod 8), then \( p = -5 \) (mod 8); in either case \( \sigma_p(3) = 1 \), \( \sigma_p(3) = 0 \), and \( \lambda_p(3) = 2 \). By (7)
\[
\sigma_p(r) = \sigma_p(3) (mod 2)
\]
for \( r \geq 3 \), and hence \( \sigma_p(r) \) is odd, so that \( \sigma_p(0) = 0 \). Thus by Lemma 3', \( \lambda_p(r) = 2^{-r} \).

If \( t = 1 \), then \( p = 3 \) (mod 4), and the only case left to consider is \( p = 7 \) (mod 8). Since \( r \) is odd and \( 2^t \equiv (p^r+1) \equiv p^r \) (mod 8), and hence \( p = -5 \equiv 5 \) (mod 2), giving \( \sigma_p(t) = 2^{-t} \) and \( \sigma_p(t) = 2^{t-2} \). By the lemma, \( \lambda_p(r) = 2 \) for \( 3 \leq r < t \). The argument used to deduce Lemma 2 from Lemma 3 can be used to show that \( \lambda_p(t+1) = 2 \), so that \( \lambda_p(t+1) = \sigma_p(t) = 2 \) and hence if \( t > r > \), \( \sigma_p(r) = \sigma_p(t) = 2 \). From Lemma 3', it follows that
\[
\lambda_p(r) = 2^{-r} \quad \text{for} \quad r \geq t.
\]

We again define \( \mu_p(r) \) to be the order of \( p^r \) (mod \( 2^t \)); then from above
\[
\mu_p(r) = \begin{cases} 
2^t & \text{if} \quad r \geq t > 2, \\
2^{-1} & \text{if} \quad r > 2, \, t = 1 \quad \text{and} \quad p \equiv 3 \, (\text{mod} \, 8), \\
\lambda_p(r+1) & \text{if} \quad r > 2, \, t = 1 \quad \text{and} \quad p \equiv 7 \, (\text{mod} \, 8),
\end{cases}
\]
where \( \lambda_p(r+1) \) is given by the Corollary. Note that if \( t > 2 \), \( \mu_p = 2 \) and if \( t = 1 \), \( \mu_p(2) = 2 \); for completeness we define \( \mu_p = 2 \) when \( t = 1 \).

Lemma 4'. If \( t \geq 2 \), then \( \mu_p = 2 \) and for all \( r \geq 1 \), \( \mu_p(r+1) = 2 \mu_p(r) \).

If \( t = 1 \) and \( p = 3 \) (mod 8), then \( \mu_p(2) = 2 \) and for all \( r \geq 2 \), \( \mu_p(r+1) = 2 \mu_p(r) \).

If \( t = 1 \) and \( p = 7 \) (mod 8), then \( r \leq 2 < r < t \), \( \mu_p(r) = 2 \) and for all \( r \geq 1 \), \( \mu_p(r+1) = 2 \mu_p(r) \).

This lemma follows from the definition of \( \mu_p(r) \). We observe that when \( t = 1 \) and \( p = 7 \) (mod 8), \( p = 2^t-1 \) (mod \( 2^t \)) and \( t \geq 3 \).

2'. The evaluation of \( \sum_{k=1}^{\infty} \sigma_k(p^r) \). Lemmas 5 and 6, parts (i) and (ii), hold without modification when \( q = 2 \). The proof of Lemma 5 (iii) and (iv) is valid in most cases. However if \( p \equiv (p^r-1), \sigma_k(p^r) = 0 \) for all \( k > r \);

for \( r \) is odd and \( p \equiv 3 \) (mod 4) and so \( \mu_p = \mu_p(2) = 2 \), and result follows from the proof of Lemma 5. Hence we have

Lemma 5'. If \( p \neq 2 \) and \( p \geq 3 \) when \( 2 \parallel (p^r-1) \) and \( r \geq 1 \) otherwise, and if \( \mu_p(r+1) = 2 \mu_p(r) \), then \( \sigma_k(p^r) = 1 \) if and only if \( k = \mu_k(p^r-1) \).

(iv) If \( r \geq 1 \) and \( p \equiv 2 \), \( p \equiv 1 \) when \( 2 \parallel (p^r-1) \) or \( \mu_p(r+1) = \mu_p(r) \), then \( \sigma_k(p^r) = 0 \) for all \( k > r \).

Lemma 6'. If \( p \neq 2 \), \( p \geq 3 \) when \( 2 \parallel (p^r-1) \) and \( r \geq 1 \) otherwise, and if \( \mu_p(r+1) = 2 \mu_p(r) \), then
\[
\sum_{k=1}^{\infty} \sigma_k(p^r) = \frac{2^t}{(1-2^t)(1-p^{-2^t})}.
\]

(iv) If \( r \geq 1 \) and \( p \equiv 2 \) or \( p \equiv 1 \) when \( 2 \parallel (p^r-1) \) or \( \mu_p(r+1) = \mu_p(r) \), then \( \sigma_k(p^r) = 0 \) for all \( k > r \).

3'. The generating functions.

Lemma 7'.
\[
f(s) = (1+2s)/z(2s).
\]

This follows from Lemma 7 since \( \mu_p = 2 \). Lemma 8 continues to hold when \( q = 2 \).

Lemma 9'.
\[
f_m(s) = f(s) \sum_{E \in \text{all}} \prod_{i=1}^{m} \left( \begin{pmatrix} E_i \end{pmatrix} \right)^{-1} \sum_{k=1} \left( P_{k, \mu_k(E_i)} s \right) \prod_{i=1}^{m} \left( P_{k, \mu_k(E_i)} s \right) \prod_{k=1}^{m} \sum_{k=1} f_m(s) \prod_{k=1}^{m} \left( P_{k, \mu_k(E_i)} s \right)
\]
where
\[
P_{k, \mu_k(E_i)} s = (1-p^{-2s})^{-1} = (1-p^{-2s})^{-1} \prod_{k=1}^{m} \left( P_{k, \mu_k(E_i)} s \right)
\]
and where the sum over \( p_i \) (\( i = 1, 2, \ldots, k \)) is over all primes \( p_i \) (\( r_i \)) except \( 2, p_1, p_2, \ldots, p_{r-1} \) and, when \( r \) is odd and \( r_i = 1 \), those \( p_i \) satisfying \( p = 3 \) (mod 4).

The notation is the same as in Lemma 9. The necessity of the additional condition needed when \( r \) is odd and in the sums involving \( r_i = 1 \) arises from Lemma 6' (iv); for \( r \) is odd and \( p \equiv 3 \) (mod 4), \( 2 \parallel (p^r-1) \).
5'. Proof of Theorem 2 (iii). From Lemma 4' we observe that
\[ \mu_p(r) = \begin{cases} 2 & \text{if } r = 1, \\ 2\mu_p(r) & \text{if } r > 1, \end{cases} \]
do not both hold unless (i) \( r \) is even and \( r = 1 \), (ii) \( r \) is odd, \( r = 1 \) and \( p = 1 \pmod{4} \), (iii) \( r \) is odd, \( r = 2 \) and \( p = 3 \pmod{8} \) or (iv) \( r \) is odd, \( r > 3 \) and \( p = 2^r - 1 \pmod{2^{r+1}} \); thus for every odd prime \( p \) there is exactly one value of \( r \) for which \( p \nmid \delta_r(r) \). It follows that
\[ (24) \sum_{p \leq T} p^{-s} = \begin{cases} \sum_{p \leq T} p^{-s} & \text{if } v \text{ is even and } r = 1, \\ \sum_{p > T} p^{-s} & \text{if } v \text{ is odd and } r > 1. \end{cases} \]
As in § 5 of part II,
\[ (25') \sum_{p \leq T} p^{-s} = \sum_{p \leq T} p^{-s} + \sum_{p > T} p^{-s}, \]
where \( \psi_+(s) \) is holomorphic for \( \sigma > \frac{1}{2} \) and bounded for \( \sigma > \frac{1}{2} + \delta \) for any \( \delta > 0 \); see from (24) that the sum on the right of (25') may be empty, but it follows from Lemma 4' that the sum on the left of (25') is never empty.

Suppose that \( v \) is even. It is well known that
\[ \sum_{p \leq T} p^{-s} = \log \zeta(s) - \sum_{\mu(p) = 1} \frac{1}{p} \sum_{n = 1}^\infty \frac{1}{n^{s+1}} - 2^{1-s} \log \zeta(s) + \mathcal{O}(s) \]
(say), where \( \mathcal{O}(s) \) is holomorphic for \( \sigma > \frac{1}{2} \) and bounded for \( \sigma > \frac{1}{2} + \delta \) for any \( \delta > 0 \). Hence from (24), (18') and Lemmas 7' and 9' we obtain
\[ f_m(s) = (1 + 2^{-s}) \zeta(2s) \sum_{i = 1}^m \left\{ \log \zeta(s) + \mathcal{O}(s) + \psi_k(s) \right\} \]
where the set of \( R_n^0 \) is that subset of the set of \( R_n \) for which \( r_i = 1 \) for \( 1 \leq i < j \), and \( r_i > 1 \) for \( j + 1 \leq i \leq k. \) It follows that
\[ f_m(s) = \sum_{i = 1}^m \left\{ \log \zeta(s) \right\}^s H_m(s), \]
where \( H_m(s) = \frac{1}{m!} (1 + 2^{-s}) \zeta(2s) \), and where \( H_m(s) \) satisfies the conditions of Theorem 2 (iii).

Suppose now that \( v \) is odd. Then by (20)
\[ (20') \sum_{p \nmid \delta_r(r)} p^{-s} = 2^{-s} \left\{ \log \zeta(s) + \log(1 - 2^{-s}) \sum_{\nu = 1}^\infty \frac{1}{\nu} \left\{ \log L(s, \chi_{2^\nu}) \right\} \right\} + \sum_{\nu = 1}^\infty \frac{1}{\nu} \left\{ \log L(s, \chi_{2^\nu}) \right\} + \mathcal{O}(s) \]
where \( \chi \) runs through all characters \( \pmod{2^{r+1}} \) except when indicated, \( \chi \neq \chi_2 \). Hence from (24), (18') and (20') and Lemmas 7' and 9',
\[ f_m(s) = (1 + 2^{-s}) \zeta(2s) \sum_{i = 1}^m \left\{ \log \zeta(s) + \log(1 - 2^{-s}) \sum_{\nu = 1}^\infty \frac{1}{\nu} \left\{ \log L(s, \chi_{2^\nu}) \right\} \right\} \]
where \( \chi_{2^0} \) is a character \( \pmod{2^{r+1}} \). The maximum value of \( h \) is \( m \) and when \( k = m, r_2 = r_3 = \ldots = r_m = 0 \); thus the highest power of \( \log \zeta(s) \) appearing on the right is \( (\log \zeta(s))^m \). Hence
\[ f_m(s) = \sum_{i = 1}^m (\log \zeta(s))^s H_m(s), \]
where \( H_m(s) = \frac{1}{m!} 2^{-m}(1 - 2^{-s}) \zeta(2s) \), and \( H_m(s) \) satisfies the conditions of Theorem 2 (iii).

IV. Proof of Theorem 3

In part I we defined \( h(s) \) to be a function which can be expressed both as an infinite sum of the form
\[ h(s) = \sum_{n = 1}^\infty b(n)n^{-s}, \]
where \( b(n) \geq 0 \), and as a product of the form
\[ h(s) = \left\{ \zeta(s) \right\}^{1-s} \left\{ \log \zeta(s) \right\}^s H(s), \]
where \( 0 < \beta < 1 \), \( u \) is a non-negative integer, and \( H(s) \) is a product of powers of Dirichlet L-functions associated with non-principal characters, non-negative powers of the logarithms of such functions, and a function holomorphic for \( \sigma > \frac{1}{2} \) and bounded for \( \sigma > \frac{1}{2} + \delta \) (\( \delta > 0 \)). More precisely we can write \( H(s) \) in the form
\[ H(s) = \prod_{\nu = 1}^\infty \left\{ \log L(s, \chi_{2^\nu}) \right\}^{1-s} \prod_{\nu = 1}^\infty \left\{ L(s, \chi_{2^\nu}) \right\}^{s} \psi_0(s), \]
where the $a_i, i = 1, 2, \ldots, A$, are non-negative integers, the $w_i, i = 1, 2, \ldots, A$, are positive numbers, the $\mathcal{W}_i, i = 1, 2, \ldots, A$, are positive numbers, where the $\psi_j^2$, for $j = 1$ and $\mathcal{W}_i$, are non-principal characters (mod $\mathcal{W}_i$) and where $\varphi(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2} + \delta$ ($\delta > 0$).

The aim of part IV is to obtain an estimate for $\sum_{n=1}^{x} b(n) \log(n/n)$. The method used to do this follows in principle the corresponding part of one of the methods used to prove the Prime Number Theorem (given, for example, in Landau [6]). Briefly, we integrate the function $\pi^2 b(n) s^{-2}$ round a certain contour $\mathcal{C}$ inside and on which $h(s)$ is holomorphic in order to obtain, in Lemma 20, an estimate for

$$\sum_{n=1}^{x} b(n) \log(n/n).$$

The required result is deduced from this.

1. Preliminary lemmas. In the next two lemmas we shall state some properties of $\zeta(s)$ and of $L(s, \chi)$ which we shall need in order to determine the behaviour of $h(s)$. In these lemmas $c_1, c_2, \ldots$ denote positive constants, and in Lemma 14 these constants depend on the character $\chi$ occurring in the lemma.

**Lemma 13.** (i) $\zeta(s) - (s-1)^{-1}$ is holomorphic for $\sigma > 0$.

(ii) There exist $c_1$ such that $\zeta(s) \neq 0$ for $\sigma \geq 1 - c_1(\log |t|)^{-3/2}$, $|t| \geq 3$, and for $\sigma \geq 1 - c_1(\log|t|)^{-3}$, $|t| \leq 3$.

(iii) There exist $c_2$ such that

$$|\zeta(s)| < c_2 \log |t|$$

for $\sigma \geq 1 - (\log |t|)^{-3}$, $|t| \geq 3$, and $c_1$ such that

$$|\log \zeta(s)| < c_1(\log |t|)^3$$

for $\sigma \geq 1 - c_1(\log |t|)^{-3}$, $|t| \geq 3$.

(iv) There exist $c_4, c_3$ and $c_5$ such that

$$|\zeta(s)| < c_4 \quad \text{and} \quad |\log \zeta(s)| < c_5$$

for $1 - c_1(\log 3)^{-3} \leq \sigma \leq 1 - c_1 < 1$, $|t| \leq 3$.

The properties given in parts (i), (ii) and (iii) of the lemma are contained in § 42 to § 48 and § 64 of Landau [5]; part (iv) is an immediate consequence of the rest of the lemma.

**Lemma 14.** Let $\chi$ be a non-principal character (mod $\mathcal{W}$); then:

(i) $L(s, \chi)$ is holomorphic for $\sigma > 0$.

(ii) There exist $c_1, c_2, c_3$ and $c_5$ such that

$$|L(s, \chi)| < c_5 \log |t|$$

for $\sigma \geq 1 - (\log |t|)^{-1}$, $|t| \geq 3$, and

$$|L(s, \chi)| > c_1 \log |t|^{-1} \quad \text{and} \quad [\log L(s, \chi)] < c_5(\log |t|)^{-1}$$

for $\sigma \geq 1 - c_5(\log |t|)^{-2}$, $|t| \geq 3$.

(iii) There exist $c_4, c_1$ and $c_6$ such that

$$0 < c_1 < |L(s, \chi)| < c_2 \quad \text{and} \quad \log L(s, \chi) < c_3,$$

for $1 - c_5(\log 3)^{-3} \leq \sigma \leq 1$, $|t| \leq 3$.

With the exception of the bound for $\log L(s, \chi)$, the properties given in parts (i) and (ii) of the lemma are contained in § 114 to § 117 of Landau [6]. The bound for $\log L(s, \chi)$ can be deduced from that of $|L(s, \chi)|$ (which is $c_6 \log |t|^{-1}$, as is given in § 117 of Landau [6]) in the same way as the bound for $|\log \zeta(s)|$ is deduced from that of $|\zeta(s)|$ in § 94 of Landau [6]. Part (iii) is an immediate consequence of the rest of the lemma. We observe that the lower bound for $|L(s, \chi)|$ implies that $L(s, \chi) \neq 0$. It is known that the powers of $\log |t|$ appearing in Lemmas 13 and 14 may be replaced by numerically smaller powers, but no advantage would be gained by using this development in this paper.

The next lemma follows immediately from Lemmas 13 and 14 and the definition of $h(s)$. We observe that, if $\sigma \geq \frac{1}{2} + \delta$ ($\delta > 0$), $|\psi(s)| < c_5$ since $\varphi(s)$ is bounded. For suitable positive constants $d_1, d_2, d_3, d_4$, we have

**Lemma 15.** (i) The function $h(s)$ is holomorphic for $\sigma > 1 - d_3(\log |t|)^{-2}$, $|t| \geq 3$ and for $\sigma > 1 - d_1(\log |t|)^{-2}$, $|t| \leq 3$ except for a singularity at $s = 1$.

(ii) $|h(s)| < d_2(\log |t|)^{2}$ for $\sigma > 1 - d_2(\log |t|)^{-2}$, $|t| \geq 3$, where $d_1 > 0$, and $|h(s)| < d_4$ for $\sigma = 1 - d_3(\log 3)^{-3}$, $|t| \leq 3$.

It follows from Lemmas 13 and 14 and the definition of $h(s)$ that we may take

$$k = (1 - \beta) + 9u + \sum_{c=1}^{n} \mathcal{C}' + \sum_{c=1}^{n} \mathcal{C} \mathcal{W}_i + \sum_{c=1}^{n} 5 [\mathcal{W}_i],$$

and that the constants $d_1$ and $d_3$ are products of the constants $c$. The constant $d_1$ must be chosen so that all parts of Lemma 13 and, for all characters $\chi$ appearing in the definition of $h(s)$, all parts of Lemma 14 are applicable in the corresponding regions of Lemma 15.
LEMMMA 16. If \(|s-1| \leq d_1 \log(3)^{-3}\),
\[
h(s)s^{-1} - H(1)(s-1)\frac{s^{-1}}{(s-1)^{\beta - 1}} \sum_{j=1}^{\infty} \omega_j (s-1)^{j-1},
\]
where the \(\omega_j\) are constants, and \(\sum_{j=1}^{\infty} \omega_j (s-1)^{j-1}\) is convergent for each \(j\).

Proof. By Lemma 13 (i), \((s-1)\zeta(s)\) is holomorphic for \(\sigma > 0\), and
\[
\lim_{s \to 1^+} (s-1)\zeta(s) = 1;
\]
also, by Lemma 13 (ii), it is certainly true that
\[
\zeta(s) \neq 0
\]
if \(|s-1| \leq d_1 \log(3)^{-3}\). Hence \(K(s) = \log(\zeta(s-1))\) is holomorphic when \(|s-1| \leq d_1 \log(3)^{-3}\), and
\[
\lim_{s \to 1^+} K(s) = 0.
\]
Now
\[
h(s)s^{-1} = (\zeta(s) - K(s))s^{-1},
\]
where \(H(s)s^{-1}\) is holomorphic and bounded when \(|s-1| \leq d_1 \log(3)^{-3}\). By Lemma 14. From above we may write
\[
h(s)s^{-1} = (s-1)^{j-1} \sum_{j=1}^{\infty} \omega_j (s-1)^{j-1},
\]
where \(H(s)s^{-1}\) is holomorphic when \(|s-1| \leq d_1 \log(3)^{-3}\), and hence it can be expanded as a (convergent) power series in the form
\[
\sum_{j=1}^{\infty} \omega_j (s-1)^{j-1}.
\]
From (25) and (26) we have that
\[
\omega_j = \lim_{s \to 1^+} (s-1)\zeta(s) - K(s))s^{-1} = 0
\]
for all \(j > 1\), and that
\[
\omega_0 = \lim_{s \to 1^+} ((s-1)\zeta(s)) - K(s))s^{-1} = H(1).
\]

Hence
\[
h(s)s^{-1} = H(1)(s-1)^{\beta - 1} \sum_{j=1}^{\infty} \omega_j (s-1)^{j-1},
\]
which gives the result of the lemma.

2. An estimate for \(\sum_{n=1}^{\infty} b(n) \log(\sigma/n)\).

LEMMA 17.
\[
\frac{1}{2\pi i} \int_{1-\delta}^{1+\delta} y^{s-1} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \log y & \text{if } y > 1. \end{cases}
\]

This is proved in § 49 of Landau [6].

LEMMA 18.
\[
\sum_{n=1}^{N} b(n) \log(\sigma/n) = \frac{1}{2\pi i} \int_{1-\delta}^{1+\delta} y^{s-1} ds.
\]

Proof.
\[
\frac{1}{2\pi i} \int_{1-\delta}^{1+\delta} y^{s-1} ds = \frac{1}{2\pi i} \int_{1-\delta}^{1+\delta} y^{s-1} \sum_{n=1}^{N} b(n) \log(\sigma/n) ds
\]
\[
= \frac{1}{2\pi i} \sum_{n=1}^{N} b(n) \int_{1-\delta}^{1+\delta} (\sigma/n)^{-1} s^{-1} ds = \sum_{n=1}^{N} b(n) \log(\sigma/n)
\]
by Lemma 17.

Our next aim is to estimate the integral appearing on the right in Lemma 18. To do this we cut the complex plane along the real axis from the point \(\sigma = 1\) to the left. Let \(\Gamma\) be the contour \(\Gamma = AABCDEE\), where the vertices above the real axis are defined by \(A = 2 + \delta \alpha^2\), \(B = 1 - d_1 \log(\alpha)^{-3} + \delta \alpha^2\), \(C = 1 - d_1 \log(3)^{-3} + 3 \delta \), \(D = 1 - d_1 \log(3)^{-3} + 3 \delta\), \(E = 1 - \delta\) for a small, positive \(\delta\) (which will tend to zero later), and where \(A, B, C, D, E\) are the complex conjugates of \(A, B, C, D, E\), curves joining neighboring vertices are straight lines except for \(BC\) which is the curve \(\sigma = 1 - d_1 \log(3)^{-3} \delta \) \((\sigma > 0 \Rightarrow \delta > 3)\). The image in the real axis of \(BC\) and \(EE\) which is the circle \(|s-1| = \delta\). The constant \(d_1\) has been chosen so that \(h(s)\) is holomorphic in the region bounded by \(\Gamma\); this follows from Lemma 15 (i). Hence by Cauchy's Theorem
\[
\int_{\Gamma} y^{s-1} ds = 0,
\]

so that
\begin{equation}
\int_{DA} x^\sigma h(s)s^{-2}ds = \int_{ABCD} x^\sigma h(s)s^{-2}ds.
\end{equation}

Let \( \Sigma_1(\sigma) = \sum_{n=1}^\infty b(n)\log(\sigma/n) \).

**Lemma 19.**
\[
\Sigma_1(\sigma) = -\frac{1}{2\pi i} \left( \int_{ABCD} + \int_{DA} \right) x^\sigma h(s)s^{-2}ds + O(xe^{-\log\sigma})^{1/3}.
\]

Proof. From Lemma 18 and (27) we obtain
\begin{equation}
\Sigma_1(\sigma) = -\frac{1}{2\pi i} \left( \int_{ABCD} + \int_{DA} \right) x^\sigma h(s)s^{-2}ds = \frac{1}{2\pi i} \left( \int_{ABCD} + \int_{DA} \right) x^\sigma h(s)s^{-2}ds.
\end{equation}

We now show that all the integrals on the right except those along DE and \( \overline{DE} \) are sufficiently small in absolute value to be included in the error term of the lemma.

(i) By Lemma 15 (ii),
\[
\left| \int_{1+\epsilon}^{2+\epsilon} \int_{1+\epsilon}^{2+\epsilon} x^\sigma h(s)s^{-2}ds dt \right| < \int_{1+\epsilon}^{2+\epsilon} x^\sigma h(s)ds \int_{1+\epsilon}^{2+\epsilon} s^{-2}dt < \int_{1+\epsilon}^{2+\epsilon} d_n(\log t)\log t^{-1}dt = O(x^\sigma)
\]
for any small \( \epsilon > 0 \).

(ii) Since \( |AB| < 2 \), we have by Lemma 15 (ii) that
\[
\left| \int_{AB} x^\sigma h(s)s^{-2}ds \right| < \int_{1+\epsilon}^{2+\epsilon} d_n(\log t)\log t^{-1}dt = O(x^{2+\epsilon})
\]
for any small \( \epsilon > 0 \).

(iii) On \( BC, \sigma = 1 - d_n(\log t)^{-1} \) and \( x^\sigma \gg t \geq 3 \); hence by Lemma 15 (ii),
\[
\left| \int_{BC} x^\sigma h(s)s^{-2}ds \right| < \int_{1+\epsilon}^{2+\epsilon} \left[ x^\sigma - d_n(\log t)^{-1} \right] d_n(\log t)^{-1}dt = O(x^\sigma)
\]
for any small \( \epsilon > 0 \).

(iv) By Lemma 15 (ii),
\[
\left| \int_{CD} x^\sigma h(s)s^{-2}ds \right| = O(x^{1-d_n(\log t)^{-1}}).
\]

(v) By Lemma 16
\[
\left| \int_{DE} x^\sigma h(s)s^{-2}ds \right| = O(x^{1-d_n(\log t)^{-1}}).
\]
since \( \beta > 0 \),
\[
\lim_{x \to 4} \{ x^\sigma \log(\delta)^{\delta^2} \} = 0,
\]
and thus
\[
\lim_{x \to 1} \left\{ \int_{\epsilon<\delta} x^\sigma h(s)s^{-2}ds \right\} = 0.
\]

By symmetry the bounds for the integrals along curves in the lower half plane are the same as the bounds for the corresponding integral (i), (ii), (iii) or (iv) in the upper half plane. The result of the lemma now follows from (28) and (i) to (v).

**Lemma 20.** (i) If \( 0 < \beta < 1 \) and \( u \geq 1 \), then
\[
\Sigma_1(\sigma) = \frac{H(1)}{\Gamma(1-\beta)} x(\log x)^{1/3}(\log x)^{-1/3} + O(\sigma(\log x)^{-1}(\log x)^{-1}).
\]

(ii) If \( 0 < \beta < 1 \) and \( u = 0 \), then
\[
\Sigma_1(\sigma) = \frac{H(1)}{\Gamma(1-\beta)} x(\log x)^{1/3}(\log x)^{-1/3} + O(\sigma(\log x)^{-1}(\log x)^{-1}).
\]

(iii) If \( \beta = 1 \) and \( u \geq 2 \), then
\[
\Sigma_1(\sigma) = uH(1)x(\log x)^{1/3}(\log x)^{-1/3} + O(\sigma(\log x)^{-1}(\log x)^{-1}).
\]

(iv) If \( \beta = 1 \) and \( u = 1 \), then
\[
\Sigma_1(\sigma) = H(1)x(\log x)^{1/3} + O(\sigma(\log x)^{-1})(\log x)^{-1}).
\]
If \( \beta = 1 \) and \( u = 0 \), then
\[
\Sigma_i(s) = O(\varphi(\log z)^{-\eta}).
\]

Proof. Let \( \theta = 1 - \delta_i(\log 3)^{-\eta} \). Suppose first that \( \beta \) and \( u \) satisfy the conditions of (i), (ii) or (iii). If \( \theta \leq s \leq 1 \), then by Lemma 16
\[
|s^f h(s)^{e^{-1}} - H(1)s^f (-\log (s-1)^e - \log s)| = O\left[ x^\delta \sum_{n=1}^{\infty} |(\log(s-1))^{e^{-1}}(s-1)^{e-1}| \right]
\]
\[
= O\left[ x^\delta \sum_{n=1}^{\infty} |(\log(s-1))^{e^{-1}}(s-1)^{e-1}| \right]
\]
\[
= O\left( x^\delta \sum_{n=1}^{\infty} |(\log(s-1))^{e^{-1}}(s-1)^{e-1}| \right)
\]

since \( \sum_{n=1}^{\infty} (s-1)^{e-1} \) is convergent. When \( \theta \leq s \leq 1 \), \((\log(s-1))^{e^{-1}}(s-1)^{e-1} \) = \( O(1) \) since \( \beta > 0 \); hence
\[
\int_\theta^1 s^f \sum_{n=1}^{\infty} |(\log(s-1))^{e^{-1}}(s-1)^{e-1}| ds = O\left( \int_\theta^1 s^f ds \right) = O\left( \varphi(\log z)^{-\eta} \right).
\]

Hence
\[
\int_{E^2} + \int_{E^2} s^f h(s)^{e^{-1}} ds = \int_{E^2} H(1)s^f (-\log (s-1)^e - \log s) ds +
\]
\[
- \int_{E^2} H(1)s^f (-\log (s-1)^e - \log s) ds + O\left( \varphi(\log z)^{-\eta} \right),
\]

where \( s^e \) and \( s^f \) indicate the upper edge and lower edge respectively of the cut. Since \( (s^e - 1) = (1-s^e) e^{-\eta} \) and \( (s - 1) = (1-s) e^{-\eta} \), it follows that
\[
\int_{E^2} + \int_{E^2} s^f h(s)^{e^{-1}} ds = \int_{E^2} H(1)s^f (-\log (1-s)^e - \log s) ds +
\]
\[
- \int_{E^2} H(1)s^f (-\log (1-s)^e - \log s) ds + O\left( \varphi(\log z)^{-\eta} \right)
\]

Assume now that the conditions of (i) are satisfied, so that \( 0 < \beta < 1 \)
and \( u \geq 1 \), and consider the integral
\[
I = \int_{E^2} s^f (1-s)^{e^{-1}} (-\log(1-s))^e ds
\]

where \( 0 < k < u \). On using the substitution \( s = 1 - \frac{\varphi}{\log z} \), we obtain
\[
I = \varphi(\log z)^{-\eta} \int_0^\infty e^{-\eta \eta^{-1}} (\log \log s - \log \varphi) \eta \eta^{-1} ds
\]
\[
= \varphi(\log z)^{-\eta} \sum_{n=1}^{\infty} \left[ (-1)^n \right] (\log \log s)^{\eta^{-2}} \eta^{-1} \int_0^\infty e^{-\eta \eta^{-1}} (\log \varphi) \eta \eta^{-1} ds.
\]

Now
\[
\int_0^\infty e^{-\eta \eta^{-1}} (\log \varphi) \eta \eta^{-1} \eta \eta^{-1} = \int_0^\infty e^{-\eta \eta^{-1}} (\log \varphi) \eta \eta^{-1} \eta \eta^{-1} + O\left( \varphi(\log z)^{\eta^{-2} + \eta^{-1}} \right)
\]

for any \( s \) satisfying \( 0 < s < 1 - \beta \). The integral on the right is absolutely convergent for all \( r_1 \), and in particular when \( \tau = 0 \) its value is \( I(\beta) \). Hence
\[
I = I(\beta) \varphi(\log \log s)^{\eta^{-2}} (\log \varphi)^{-\eta} + O\left( \varphi(\log \log s)^{\eta^{-1}} (\log \varphi)^{-\eta} \right)
\]

unless \( k = 0 \), in which case the error term is \( O\left( \varphi(\log \log s)^{-\eta} \right) \). Hence by (29)
\[
\int_{E^2} + \int_{E^2} s^f h(s)^{e^{-1}} ds
\]
\[
= \int_{E^2} H(1)s^f (1-s)^{e^{-1}} (-\log(1-s)^e - \log s) ds + O\left( \varphi(\log z)^{-\eta} \right)
\]

since \( I(\beta) \Gamma(1-\beta) = \pi(1-\beta) \). Part (i) of the lemma now follows from Lemma 19.

Similarly in case (ii), when \( 0 < \beta < 1 \) and \( u = 0 \), the integral \( I \) is given by
\[
I = \int_0^1 s^f (1-s)^{e^{-1}} ds = I(\beta) \varphi(\log \log s)^{\eta^{-2}} (\log \varphi)^{-\eta} + O\left( \varphi(\log \log s)^{-\eta} \right).
\]

As above it follows that
\[
\int_{E^2} + \int_{E^2} s^f h(s)^{e^{-1}} ds = -2\varphi \int_{E^2} H(1) (1-s)^{\eta^{-2}} - \varphi(\log \log s)^{\eta^{-2}} - \varphi(\log \log s)^{-\eta}
\]

and on using Lemma 19 we obtain the result of Lemma 20 (ii).
We turn now to case (iii), so that \( \beta = 1 \) and \( u \geq 2 \). Then (29) becomes

\[
\int_{D} + \int_{D} \alpha(s) \alpha^{\prime}(s) \, ds
\]

\[
= H(1) \sum_{n \in \mathbb{N}} (\alpha(n) - (n\alpha)^{\prime}) \int_{1}^{\infty} \alpha(s) (-\log(1-s))^{u-2} \, ds + O(\log x)^{-1};
\]

we observe that the term corresponding to \( m = 0 \) is zero. Now \( \Gamma(1) = 1 \), and hence by (30)

\[
I = \int_{1}^{\infty} \alpha(s) (-\log(1-s))^{\beta} \, ds
\]

\[
= \alpha^{\prime}(\log x)^{\beta}(\log x)^{-1} + O(\alpha(\log x)^{-1})(\log x)^{-1}
\]

unless \( k = 0 \), in which case

\[
I = (\alpha - \alpha^{\prime}) \log x.
\]

Hence

\[
\int_{D} + \int_{D} \alpha(s) \alpha^{\prime}(s) \, ds
\]

\[
= -2\pi i a H(1) \alpha(\log x)^{-1} \log x^{-1} + O(\alpha(\log x)^{-1})(\log x)^{-1},
\]

and the result of part (iii) follows from Lemma 19.

Assume now that the conditions of case (iv) hold, so that \( \beta = 1 \) and \( u = 1 \). In this case the function to be integrated round \( \Gamma \) is the product of the holomorphic function \( H(s) \) and \( \alpha(s) \log x^{\gamma}(s) \), the function integrated round a similar contour in one proof (given in § 64 of Landau [5]) of the Prime Number Theorem. Proceeding as above, we obtain from Lemma 16

\[
|\alpha^{\prime}(s) - H(1)\alpha^{\prime}(-s-1)| = O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

Hence, since

\[
\int_{1}^{\infty} \alpha(s) \log x^{-1} \log x \, ds = O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

which is proved by using the substitution \( s = 1 - \frac{\alpha}{\log x} \),

\[
\left( \int_{D} + \int_{D} \alpha(s) \alpha^{\prime}(s) \, ds = \int_{1}^{\infty} H(1) \alpha^{\prime}(-s-1) \, ds + \right.
\]

\[
- \int_{1}^{\infty} H(1) \alpha^{\prime}(-s-1) \, ds + O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

\[
= -2\pi i H(1) \alpha(\log x)^{-1} + O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

The result of (iv) now follows from Lemma 19.

For part (v) of the lemma, \( \beta = 1 \) and \( u = 0 \) and so \( h(s) = H(s) \).

Hence \( h(s) \) is holomorphic inside the contour \( A B C D D C B \), where the complex plane is no longer cut so that \( \Gamma = D \), and where the rest of the contour is the same shape as the corresponding part of \( \Gamma \). Using the results of Lemma 19 and integrating round this contour we obtain

\[
\Sigma_{1}(s) = O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

which is (v). This completes the proof of Lemma 20.

3. Proof of Theorem 3. Let \( \Sigma_{1}(s) = \sum_{n \leq x} b(n) \). Then we have

**Lemma 21.** Suppose that

\[
\Sigma_{1}(s) \leq B \alpha^{\prime}(\log x)^{-1}(\log x)^{-1} + O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

where \( B, \alpha, \beta, \gamma, \beta_{1}, \gamma_{1} \) are non-negative constants and \( B \neq 0, \alpha \neq 0 \), and where either \( \gamma_{1} < \gamma \) and \( \beta \leq \beta_{1} < \beta + 2 \) or \( \gamma_{1} \geq \gamma \) and \( \beta < \beta_{1} \leq \beta + 2 \).

Then

\[
\Sigma_{1}(s) = B \alpha^{\prime}(\log x)^{-1}(\log x)^{-1} + O(\alpha(\log x)^{-1})(\log x)^{-1})
\]

Proof. Let \( \delta = \delta(\alpha) = o(1) \) be a positive function of \( x \) to be chosen later, and suppose that \( \alpha(1+\delta) \) is an integer. Then, since

\[
\log x(1+\delta) = C + O(\delta) \]

and \( \log x(1+\delta) = C + O(\delta) \),

\[
\Sigma_{1}(s(1+\delta)) = B(1+\delta) \alpha^{\prime}(\log x)^{-1}(\log x)^{-1}
\]

\[
\times \left[ 1 + O(\delta(\log x)^{-1}) \right] + O(\delta(\log x)^{-1})
\]

and

\[
\Sigma_{1}(s(1+\delta)) = B \alpha^{\prime}(\log x)^{-1}(\log x)^{-1}
\]

By definition

\[
\Sigma_{1}(s(1+\delta)) = \Sigma_{1}(s) \leq B \alpha^{\prime}(\log x)^{-1}(\log x)^{-1}
\]

The second sum is not negative. Similarly

\[
\Sigma_{1}(s(1+\delta)) = \Sigma_{1}(s) = \log(1+\delta) \sum_{n \leq x} b(n) \log x(1+\delta)/n
\]

\[
\leq \log(1+\delta) \Sigma_{1}(s)
\]

since the second sum is not positive.
By (31), (32) and (33)

\[ \sum_2 \sigma(x) = \left[ \sum_2 \sigma(1 + \delta) - \sum_1 \sigma(x) \right] \log(1 + \delta) \]

\[ = \left[ \sum_2 \sigma(1 + \delta) - \sum_1 \sigma(x) \right] (1 + O(\delta)) \delta^{-1} \]

\[ = Bx^\alpha \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\delta) + O(\log x)^{-\gamma} \right] + \]

\[ + O(\log x)^{1+\gamma-\delta^2} (1 + \log x)^{-\gamma^2+\gamma x - \delta^4}. \]

By (31), (32) and (34)

\[ \sum_2 (1 + \delta) \sum_1 \sigma(x) \log(1 + \delta) \]

\[ = Bx^\alpha \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\delta) + O(\log x)^{-\gamma} \right] + \]

\[ + O(\log x)^{1+\gamma-\delta^2} (1 + \log x)^{-\gamma^2+\gamma x - \delta^4}. \]

If we replace \( x \) by \( x/(1 + \delta) \) in (36), we obtain

\[ \sum_2 (x) \geq Bx^\alpha \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\delta) + O(\log x)^{-\gamma} \right] + \]

\[ + O(\log x)^{1+\gamma-\delta^2} (1 + \log x)^{-\gamma^2+\gamma x - \delta^4}. \]

We now choose \( \delta \) so that all the error terms of (35) and (37) are of a smaller order of magnitude than the first term; since \( \beta \leq \beta_1 \leq \beta + 2 \), we can take \( \delta = \omega^{-1} [\omega'] \), where

\[ \delta' = (\log x)^{1+\gamma-\delta} (1 + \log x)^{-\delta^2}, \]

and then the error terms of (35) and (37) are

\[ O(\omega^2 \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\log x)^{-\gamma} \right] + \]

\[ + O(\log x)^{1+\gamma-\delta^2} (1 + \log x)^{-\gamma^2+\gamma x - \delta^4}). \]

Hence from (35) and (37) it follows that

\[ \sum_2 (x) = Bx^\alpha \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\log x)^{1+\gamma-\delta^2} (1 + \log x)^{-\gamma^2+\gamma x - \delta^4} \right], \]

which is the result of the lemma.

We observe that if \( \beta_1 > \beta + 2 \), the result of the lemma holds provided that we replace the error term by

\[ O(\omega^2 \log(x)^\beta \log(x)^{-\gamma} \left[ a + O(\log x)^{-\gamma} \right]. \]

**Corollary.** If \( \sum_1 (x) = O(\omega [\log x]^{-\gamma}) \), then \( \sum_2 (x) = O(\omega [\log x]^{-\gamma}). \)

**Proof.** By the above method, we can show that

\[ \sum_2 (x) = O(\omega [\log x]^{-\gamma}) = O(\omega [\log x]^{-\gamma}) \]

if we choose \( \delta = \omega^{-1} [\omega'] \) where \( \delta' = (\log x)^{-\gamma}. \)

We can now deduce Theorem 3 from Lemmas 20 and 21. If we take

(i) \( \alpha = 1, \gamma = u - 1 \geq 1, \)

(ii) \( \alpha = 1, \gamma = u - 1 = 0, \)

(iii) \( \alpha = 1, \gamma = u - 1 \geq 1, \gamma_1 = u - 2, \beta = \beta_1 = 1, \)

(iv) \( \alpha = 1, \gamma = u - 1 = 0, \gamma_1 = 1, \beta = 1, \beta_1 = 2 \)

in Lemma 21, the estimates for \( \sum_1 (x) \) appearing in the statement of this lemma being those given in the corresponding part of Theorem 20, then we obtain, in turn, the first four parts of Theorem 3. We obtain the last part of Theorem 3 from Lemma 20 (v) and the Corollary to Lemma 21.

**V. Some deductions**

1. **Proof of Theorem 1 (ii) and (iii).** If \( q \) is odd and \( \lambda \) is even, then by Theorem 2 (ii),

\[ f_m(x) = \left( \sum_{\nu=1}^{m} \omega(x) \right)^{1-\lambda} \sum_{\nu=1}^{m} \omega(x) H_{m}(x), \]

where each \( H_{m}(x) \) is a sum of functions satisfying the conditions imposed on \( H_{m}(x) \) in Theorem 3. Hence, if \( m' \geq 1 \), we have from Theorem 3 (i) and (ii) (with \( \beta = 1/\lambda < 1 \)) that

\[ D_m(v, q; x) = \sum_{\nu=1}^{m} \omega(x) \omega H_{m}(1) \]

\[ = \frac{H_{m}(1)}{1 - \frac{1}{\lambda}} \omega(x) [\log x]^\nu \left( \log x \right)^{-\nu^2 + \nu x - \delta^4}. \]

where the constant \( H_{m}(1) \) can be obtained from (22) and Lemma 12. Similarly if \( m' = 0 \) we have from Theorem 3 (ii) that

\[ D_m(v, q; x) = \frac{H_{m}(1)}{1 - \frac{1}{\lambda}} \omega(x) [\log x]^\nu \left( \log x \right)^{-\nu^2 + \nu x - \delta^4}. \]

where \( H_{m}(1) \) may be obtained from the end of \( \S \) 5 and Lemma 12. This proves Theorem 1 (ii).

If \( q = 2 \), then by Theorem 2 (iii),

\[ f_m(x) = \sum_{\nu=1}^{m} \omega(x) H_{m}(x), \]

where each \( H_{m}(x) \) is a sum of functions satisfying the conditions imposed on \( H_{m}(x) \) in Theorem 3. Hence if \( m \geq 2 \) we have from Theorem 3 (i), (iv) and (v) that

\[ D_m(v, 2; x) = m H_{m}(1) \omega(x) [\log x]^\nu \left( \log x \right)^{-\nu^2 + \nu x - \delta^4}, \]

and if \( m = 1 \) we have from Theorem 3 (iv) and (v) that

\[ D_1(v, 2; x) = H_{1}(1) \omega(x) [\log x]^\nu \left( \log x \right)^{-\nu^2 + \nu x - \delta^4}. \]
in either case

\[ H_m(1) = \begin{cases} \frac{1}{m!} \left(1+2^{-1}\right)(2) = \frac{1}{4} \quad & \text{if } v \text{ is even}, \\ \frac{1}{2^{m-1}} \left(1+2^{-1}\right)(2) = \frac{1}{4} \quad & \text{if } v \text{ is odd}. \end{cases} \]

The result of Theorem 1 (iii) now follows.

2. An asymptotic expression for \( N(v, q^m; x) \). We have already seen in equation (3) that

\[ N(v, q^m; x) = \sum_{r=1}^{\frac{x}{q^m}} D_r(v, q; x), \]

and that \( D_r(v, q; x) = N(v, q; x) \) is given by (2). Let \( I = [(m-1)/(\gamma+1)] \) and assume that \( m > 2 \); then we have

**Corollary 1.** As \( x \to \infty \)

\[ B_{m}^{\text{new}} = \begin{cases} B_{m}^{\text{old}} & \text{if } q \text{ and } h \text{ are both odd}, \\ B_{m}^{\text{old}}(\log\log x)^{\gamma} & \text{if } q \text{ is odd and } h \text{ is even}, \\ B_{m}^{\text{old}}(\log\log x)^{\gamma} & \text{if } q = 2, \\ B_{m}^{\text{old}}(\log\log x)^{\gamma-1} & \text{if } q \text{ is even and } h \text{ is odd}, \\ B_{m}^{\text{old}}(\log\log x)^{\gamma} & \text{if } q \text{ is odd and } h \text{ is even}. \end{cases} \]

where

\[ B_{m}^{\text{old}} = \sum_{r=1}^{\frac{x}{q^m}} A_r, \quad B_{m}^{\text{old}} = \sum_{r=0}^{\frac{x}{q^m}} A_r, \quad B_{m}^{\text{new}} = A_{m+1}. \]

Proof. If \( q = 2 \), or if \( q \) is odd, \( h \) is even and \( r \equiv 1 \) (so that \( \gamma = 0 \)), then from (38) and Theorem 1 (ii) and (iii), we obtain

\[ N(v, q^m; x) \sim D_{m+1}(v, q; x), \]

and the result follows in these cases. If \( q \) and \( h \) are both odd, the result follows immediately from (38) and Theorem 1 (i). Finally suppose that \( q \) is odd and \( h \) is even and \( q \mid v \) (so that \( \gamma > 1 \)). Then it is clear from (38) and Theorem 1 (ii) that

\[ N(v, q^m; x) \sim \sum_{r=1}^{\frac{x}{q^m}} A_r(\log\log x)^{\gamma(r+1)}(\log x)^{-1}. \]

Since \( \max_{1 \leq r < m-1} \frac{r}{r+1} = \frac{m-2}{m-1} = \frac{m-1}{m} \), and since \( \left[ \frac{r}{r+1} \right] = \frac{m-1}{m} \) when \( \frac{r}{r+1} < r < m-1 \), the result of the Corollary follows.

3. Some results for \( N(v, k; x) \). In this section we assume that \( k \) is divisible by at least two distinct primes and we shall deduce some estimates for \( N(v, k; x) \). Some results in this direction have already been obtained by Rankin in § 4 of his paper [12] and these results are improvements on Watson’s estimate for \( N(v, k; x) \), stated in (4).
Corollary 3. (i) If \( h_r \) is odd for at least two integers \( r \) satisfying 
\( 1 \leq r \leq t \), then 
\[
C_t \leq \lim_{x \to \infty} \frac{1}{x} \sum_{r=1}^{t} N(r, \varphi(x); \varphi(x)) \leq C_2,
\]
where \( C_t \) and \( C_2 \) are positive constants and \( C_t \neq C_2 \).
(ii) If all the \( h_r \) are even, and if the relations 
\[
h_r = 1 \quad \text{and} \quad \left( \sum_{r=1}^{t} \frac{1}{(p_r + 1)} \right) = \mu
\]
hold simultaneously for at least two integers \( r \) satisfying 
\( 1 \leq r \leq t \), then 
\[
C_t \leq \lim_{x \to \infty} \frac{1}{x} \sum_{r=1}^{t} N(r, \varphi(x); \varphi(x))^{1/(2r)} \leq C_4,
\]
where \( C_t \) and \( C_4 \) are positive constants and \( C_t \neq C_4 \).

Proof. (i) Suppose that \( h_r \) is odd when \( r = r_i, i = 1, 2, \ldots, j \), where 
\( 2 \leq j \leq t \), and \( h_r \) is even otherwise. Then by Corollary 1, as \( x \to \infty \),
\[
\sum_{r=1}^{t} \frac{N(r, \varphi(x); \varphi(x))}{x} \sim \sum_{r=1}^{t} \frac{B(r, \varphi(x))}{x} = C_3 x,
\]
and
\[
\max_{1 \leq r \leq t} N(r, \varphi(x); \varphi(x)) \sim \max_{1 \leq r \leq t} B(r, \varphi(x)) = C_4 x;
\]
clearly \( C_t \neq C_4 \). The result now follows from (39).

The proof of (ii) is similar.

4. Results for \( d(n) \) and \( \varphi(n) \). We have said that the proof of Theorem 1 can be adapted to prove results analogous to Theorem 1 for the functions \( d(n) \) and \( \varphi(n) \). In this section we state these results, and we consider \( d(n) \) first.

If \( n = \prod p_s \), then 
\[
d(n) = \prod_s (a_s + 1) \text{.}
\]
Define
\[
b_s(n) = \begin{cases} 1 & \text{if } q \mid d(n), \\ 0 & \text{otherwise,} \end{cases} \quad b_r(n) = \begin{cases} 1 & \text{if } q' \mid d(n), \\ 0 & \text{otherwise,} \end{cases}
\]
for \( r \geq 1 \), where \( q \) is a prime. Let
\[
D_n(0, q; x) = \sum_{n=1}^{x} b_n(qn)
\]
for \( m \geq 0 \), and
\[
g_n(x) = \sum_{n=1}^{x} b_n(qn)n^{-r}.
\]
Then we have

Theorem 4. (i) If \( q \neq 2 \) and \( m \geq 0 \),
\[
g_n(x) = \zeta(q)x^{1/q} \%
\]
where \( \varphi(x) \) is holomorphic for \( x > 1 \) and bounded for \( x \geq 1 + \delta \) \((\delta > 0)\).

Hence
\[
D_n(0, q; x) \sim \varphi(x).
\]
(ii) If \( q = 2 \) and \( m > 0 \),
\[
g_n(x) = \sum_{n=1}^{x} \left( \log x \right)^{m+1} \varphi_n(x),
\]
where \( \varphi_n(x) \) is holomorphic for \( x > 1 \) and bounded for \( x \geq 1 + \delta \) \((\delta > 0)\),
and
\[
\varphi_n(x) = \frac{1}{m!} \zeta(2x) \%
\]
Hence
\[
D_n(0, 2; x) \sim \frac{1}{(m-1)!} \left( \log x \right)^{m-1} \varphi(x).
\]
If \( q = 2 \) and \( m = 0 \),
\[
g_0(x) = \zeta(2x),
\]
and hence
\[
D_0(0, 2; x) \sim x^{1/2}.
\]
We observe that \( D_n(0, q; x) \) represents the number of positive integers \( n \leq x \) for which \( q \mid d(n) \). If \( q \neq 2 \), from the proof of Theorem 4 it can be deduced that
\[
D_n(0, q; x) \sim \frac{\zeta(q)}{\zeta(q-1)} x.
\]
This result is an immediate consequence of a result proved by L. G. Satoh in a paper \cite{7} published in 1943. In fact from Satoh's result it follows that
\[
D_n(0, q; x) = \frac{\zeta(q)}{\zeta(q-1)} x + O(x^{1/q-1}\log x).
\]
We turn now to \( \varphi(x) \). If \( n = \prod p_s \), \( \varphi(x) = \prod p_s^{1/(p_s-1)} \). Define
\[
\varphi_m(n) = \begin{cases} 1 & \text{if } q \mid \varphi(n), \\ 0 & \text{otherwise,} \end{cases} \quad \varphi_r(n) = \begin{cases} 1 & \text{if } q' \mid \varphi(n), \\ 0 & \text{otherwise,} \end{cases}
\]
for \( r \geq 1 \), where \( q \) is a prime. Let
\[
D_n^*(q; x) = \sum_{n=1}^{x} \varphi_m(n)
\]
for \( m \gg 0 \), and

\[
  h_m(s) = \sum_{n=1}^{m} a_m(n)n^{-s}.
\]

Then we have

**Theorem 5.** (i) If \( q \neq 2 \) and \( m \gg 0 \),

\[
  h_m(s) = (\zeta(s))^{1/(q-1)} \sum_{u \leq m} (\log \zeta(s))^{u} H_u(s),
\]

where each \( H_u(s) \) \((0 \leq u \leq m)\) satisfies the conditions given in Theorem 2 (ii). Hence

\[
  D_m^*(q; x) \sim \frac{H_m(1)}{\Gamma(1-1/(q-1))} x(\log \log x)^m (\log x)^{-1/(q-1)}.
\]

(ii) If \( q = 2 \) and \( m > 0 \),

\[
  h_m(s) = \sum_{u=1}^{m} (\log \zeta(s))^{u} H_u(s),
\]

where \( H_u(s) = \frac{1}{m!} 2^{-m} (1 + 2^{-u}) \) and each \( H_u(s) \) \((0 \leq u \leq m)\) satisfies the conditions given in Theorem 2 (ii). Hence

\[
  D_m^*(2; x) \sim \frac{3}{2} \frac{2^{-m}}{(m-1)!} x(\log \log x)^{m-1} (\log x)^{-1}.
\]

If \( q = 2 \) and \( m = 0 \),

\[
  h_0(s) = 1 + 2^{-s}
\]

and hence for all \( s \geq 2 \),

\[
  D_m^*(2; x) \sim 1.
\]

The proofs of Theorems 4 and 5 are more straightforward than the proof of Theorem 1. From the definitions we can immediately obtain the result analogous to Lemma 5, and no section analogous to §1 of part II is necessary; Lemma 8 can again be used. The generating functions \( g_m(s) \) and \( h_m(s) \) can be obtained by following the method of part II. No variation of the method of proof is required to obtain \( g_m(s) \). To obtain \( h_m(s) \) the only additional fact required in the proof is an estimate of

\[
  \prod_{p \equiv 1 \mod{a}} (1 - p^{-s}),
\]

and otherwise the method of proof is unaltered. Clearly we may use Theorem 3 or the Wiener-Ikehara Theorem, whichever is applicable, to deduce the asymptotic equations for \( D_m(0, q; x) \) and \( D_m^*(q; x) \) from \( g_m(s) \) and \( h_m(s) \) respectively.

**References**


