Singular moduli (4).

By

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In this paper I give some arithmetical developments of the theoretical researches on singular moduli and class-invariants due to Kronecker and Dedekind. A full account of these researches is to be found in Weber's Algebra [18].

In the ordinary notation of elliptic functions write

\[ q = e^{i\pi\tau}, \quad \left| q \right| < 1 \]

\[ f(n) = q^{-\frac{1}{6}} \prod_{m=1}^{\infty} (1 + q^{2m}), \quad f_1(n) = q^{-\frac{1}{6}} \prod_{m=1}^{\infty} (1 - q^{2m}), \]

\[ f_2(n) = 2 \prod_{m=1}^{\infty} \left( 1 + q^{2m} \right) = 2 \prod_{m=1}^{\infty} \left( 1 - q^{4m-2} \right), \]

so that

\[ f_2(n) = f_1(n) + f_1(-n), \]

\[ f(n) f_1(n) f_3(n) = \sqrt{2}; \]

and let \( f(n) \) be Dedekind's invariant such that \( f^{24}(n), -f^{24}(n), -f^{24}(-n) \) are the roots of the equation

\[ (x - 16)^2 - x f(n) = 0. \]

Let \( n \) be an integer of the form \( 8m - 1 \); and let the number of genera of classes of quadratic forms of negative determinant \( -n \) be \( N \), the number of classes in each genus being \( h \); and let \( Nh = h \). In the sequel this Gaussian class-number is indicated by appending \( (G. N. h) \) to each value of \( n \). If

\[ a_r x^2 + b_r x y + c_r y^2 \quad (r = 1, 2, \ldots, h) \]

is a complete set of quadratic forms for which \( b_r^2 - 4a_r c_r = -n \), and if the \( h \) values of \( \tau \) (with positive imaginary parts) which satisfy the equations

\[ a_r \tau^2 + b_r \tau + c_r = 0 \]

are called \( \tau_1, \tau_2, \ldots, \tau_h \), it is known that the equation

\[ \prod_{\tau = 1}^{h} \left[ x - f(\tau) \right] = 0 \]

(where the left-hand side is multiplied out) is an Abelian equation in \( x \) of degree \( h \) with integral coefficients. Hence also the equation

\[ \prod_{\tau = 1}^{h} \left[ x - f^{24}(\tau) \right] \left[ x + f^{24}(\tau) \right] \left[ x + f^{24}(\tau) \right] = 0, \]

that is to say

\[ \prod_{\tau = 1}^{h} \left[ x - f^{24}(\tau) \right] \left[ x + f^{24}(\tau) \right] \left[ x + f^{24}(\tau) \right] = 0, \]

when multiplied out, has integral coefficients. Since one of the \( \tau \) is a root of the equation \( \tau^2 + \tau + 2m = 0 \), and is consequently equal to

\[ \frac{1 + \sqrt{1 + 8m}}{2}, \]

and since \( f(n) a_1^{\frac{1}{12}} \left( \frac{1 + \sqrt{1 + 8m}}{2} \right) = c_1^{\frac{1}{12}} \sqrt{2} \), it follows that one of the roots of the above equation of degree \( 3h \) is \( 2^{12} f^{24}(\tau) \sqrt{\frac{1}{2}} \).

To avoid the occurrence of superfluous square-roots, I shall subsequently write

\[ f(\sqrt{-n}) \sqrt{2} = F_n; \]

and, when no confusion can arise, I shall omit the suffix and write simply \( F \) for \( F_n \).

It is shown by Weber [18], p. 473, that, from the equation of degree \( 3h \) with roots \( f^{24}(\tau), -f^{24}(\tau), -f^{24}(\tau) \), it is possible to extract an equation (with integral coefficients), of degree \( h \) in the class-invariant \( F_n \), when \( n \) is not a multiple of 3; when \( n \) is a multiple of 3, the corresponding class-invariant is \( F_n^2 \).

In this paper I give a catalogue of the 75 of these equations for
which $n$ has the values 7, 15, 23, ..., 599 respectively \(^{3}\). The paper may therefore be regarded as an extension of Greenhill \([2]\) in that paper the values of $n$ of the form under consideration go only as far as 95 (87 being omitted). Whereas Greenhill’s results were mainly obtained by the use of modular equations, my own methods of constructing new equations are purely arithmetical. My procedure is to compute for each value of $n$ a complete set of values of $f^*_{2n}(c)$, $-f^*_{2n}(c)$, $-f^*_{2n}(t)$, and then to go through the somewhat laborious task of making the proper selection of the twenty-four (or eighth) roots of these numbers in such a way as to obtain an equation of degree $k$ with integral coefficients, one of whose roots is $F_0$ (or $F_2$). The details of the manner in which the selection is effected are described rather more fully by Watson \([12]\), where the same process is carried out for another set of values of $n$.

For the 75 values of $n$ discussed in this paper, $N$ always has one of the values I, II, IV; in 28 cases $N$ is I, in 42 cases it is II, and in the remaining 5 cases it is IV. Since the paper just cited deals with the set of equations for values of $n$ for which $N$ is I and $k$ is 1, 3, 5, ..., 15, there is a certain amount of overlapping between the two papers; for the 21 values of $n$ common to the two papers, I give here the equations satisfied by $F_0$ only, without repeating their solutions in terms of radicals; for the 56 values of $n$ which are not discussed by Watson \([12]\), in addition to giving the equations, I reduce them as much as seems conveniently feasible.

When $n$ is of the form $ab$, where $b$ is a prime and $a$ is either a prime or a power of a prime ($a \neq b$), $N$ is II. I then give an equation satisfied by $F$ which is of degree $k$ and which has quadratic irrationalities in its coefficients; the equation whose coefficients are the conjugate irrationalities has either $F_{ab}$ or $-F_{ab}$ for one of its roots, with exceptions when $k$ is even and $F_{ab}$ or $-F_{ab}$ satisfies the same equation as $F_{ab}$.

When $N$ is II, the following abbreviations will be used throughout:

$$
F_{ab} = F_{0}^{a}, \quad F_{ab} = F_{0}^{b}, \quad F_{ab} = F_{0}^{a} + F_{0}^{b}, \quad F_{ab} = F_{0}^{a} - F_{0}^{b}, \quad F_{ab} = F_{0}^{a}, \quad F_{ab} = F_{0}^{b},
$$

\(^{3}\) I originally intended to go only to 399, but the number of interesting class-invariants in the fifth and sixth centuries was so great that I decided to go to the limit to which this paper extends.

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Then $\alpha$ (or $a^2$) and $\beta$ (or $\gamma$) satisfy equations of degree $k$ with integral coefficients, and, since

$$
2 F_{ab}^{2} = \alpha \left( \beta + \sqrt{\beta^{2} - 4} \right),
$$

it is evident that $F_{0}$ is determinate when $\alpha$ and $\beta$ have been determined.

I show how to obtain $\alpha$ and $\beta$ (or $\gamma$) when $k$ has no prime factors other than 2 or 3. In other cases, in addition to the equations for $\alpha$ and $\beta$ (or $\gamma$), I give equations of degree $k$ satisfied by $S_{1}$ (or $S_{0}$) and $D_{1}$ (or $D_{0}$); of the latter pairs of equations, one member has all its coefficients integers, the other member has integers and quadratic surds as alternate coefficients.

The values of $n$ for which $N$ is IV are so few that abbreviations of the type just explained are unnecessary for them.

When $k$ is a multiple of 3, I have usually given the results in terms of Berwick’s cubic irrationalities; I quote the equations which define these irrationalities from the Table which was computed by Berwick and published by Mathews \([5]\). To each cubic equation given I append the value of its discriminant $\Delta$.

After these preliminary explanations I give the catalogue of equations, arranged in order of magnitude of values of $n$, not in order of magnitude of values of $k$ as was more convenient in my previous paper.

$n = 7$. \((\text{G. I. 1})\).

The result

$$
F = 1
$$

is due to Joubert \([4]\).

$n = 15$. \((\text{G. II. 1})\).

The equation satisfied by $F_{1}$ is

$$
F^{3} - F^{3} - 1 = 0,
$$

which, by adjunction of $\sqrt[4]{5}$, reduces to

$$
F^{3} - 1 + \frac{\sqrt[4]{5}}{2} = 0,
$$

a result due to Joubert \([4]\).

Here we have

$$
\alpha = 1, \quad \gamma = 3, \quad D_{4} = 1.
$$
n = 23. (G. I. 3).

The equation satisfied by $F_{23}$, given by Weber [13] and Greenhill [1], is

$$F^3 - \bullet - F - 1 = 0. \quad (\Delta = -23)$$

n = 31. (G. I. 3).

The equation satisfied by $F_{31}$, given by Weber [13] and Greenhill [1], is

$$F^3 - E^3 = 1 = 0. \quad (\Delta = -31)$$

n = 39. (G. II. 2).

The equation satisfied by $F_{39}$, due to Joubert [4], is

$$F^{13} - 3 F^8 - 4 F^4 - 2 F - 1 = 0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$\left[ F - \frac{3 + \sqrt{13}}{2} \right]^2 = \frac{23 + 7 \sqrt{13}}{6}.$$

Here we have

$$\alpha = \frac{3 + \sqrt{13}}{2}, \quad \beta = \frac{1 + \sqrt{13}}{2}. \quad (D_3 = 1).$$

n = 47. (G. I. 5).

The equation satisfied by $F_{47}$, due to Weber [13] and Greenhill [1], is

$$F^2 - F^3 - 2 F - 1 = 0,$$

references to solutions of this equation are given by Watson [12].

n = 53. (G. II. 2).

The equation satisfied by $F_{53}$ is

$$F^2 - 2 F + 1 = 0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$\left[ F - \frac{1 + \sqrt{5}}{2} \right] = \frac{3 + 2 \sqrt{5}}{4}.$$

9) I use the Weierstrassian symbol $\bullet$ to indicate that a term is absent from an equation.

These results are given by Weber [13], and also they are attributed to Russell by Greenhill [2].

Here

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{3 + \sqrt{5}}{2}, \quad D_3 = 1.$$  

n = 63. (G. II. 2).

The equation satisfied by $F_{63}$, due to Joubert [4], is

$$F^{13} - 8 F^8 + \bullet + F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{21}$, reduces to

$$\left[ F^3 - \frac{4 + \sqrt{21}}{2} \right] = \frac{27 + 6 \sqrt{21}}{4}.$$

results equivalent to these are given by Weber [13] and [15].

Here we have

$$\alpha = \frac{5 + \sqrt{21}}{2}, \quad \beta = \frac{1 + \sqrt{21}}{2}, \quad S_4 = 4 + \sqrt{21}.$$

n = 71. (G. I. 7).

The equation satisfied by $F_{71}$, due to Russell [9] and Weber [13], is

$$F^2 - 2 F^9 - F + 1 = 0,$$

its solution is given by Watson [12].

n = 79. (G. I. 5).

The equation satisfied by $F_{79}$, due to Russell [9], is

$$F^9 - 2 F^2 - F^8 - F^3 + F^2 + F - 1 = 0,$$

its solution is given by Watson [12].

n = 87. (G. II. 3).

The equation satisfied by $F_{87}$ is

$$F^{13} - 3 F^{15} - 11 F^8 - 4 F^4 - 4 F^6 - F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{29}$, reduces to

$$F^9 = \frac{13 + 3 \sqrt{29}}{2} F^1 + (6 + \sqrt{29}) F^3 - \frac{5 + \sqrt{29}}{2} = 0.$$
Results equivalent to these are given by Hanna [3].
Here \(a, \tau\) and \(D_1\) are given in terms of Berwick's cubic irrationality
\[97 - 2\,9^3 + 3\,9 - 3 = 0 \quad (\Lambda = -87)\]
by the formulae
\[a = 97 - 3 - 9 + 8, \quad D_1 = 97 - 3 - 9 + 9.\]

Further we have
\[3\,F^2 = \frac{13 + 3\sqrt[29]{2} + 2(11 + 2\sqrt[29]{2})}{11 + 2\sqrt[29]{2}} + \frac{1 + 11 + 2\sqrt[29]{2}}{2} \sqrt{\frac{3 + \sqrt[29]{2}}{2}}.\]

\(n = 95.\) (G. II. 4).

The equation satisfied by \(F_{95}\) is
\[F_3 - 2\,F^2 - 2\,F^3 + 2\,F^3 - F^4 + \cdots + F - 1 = 0;\]
this equation is given by Russell [9]; Greenhill [2], who gives its solution by radicals (with some errors in signs), attributes it to G.B. Mathews.

By adjunction of \(\sqrt[2]{5}\) we have
\[\left[ F^2 - \frac{1 + \sqrt[2]{5}}{2} F - \frac{1 + \sqrt[2]{5}}{4} \right]^2 = \left( 2\sqrt[2]{5} - 1 \right) \left[ \frac{1 + \sqrt[2]{5}}{4} \right]^2.\]

Here we have
\[a = \frac{1 + \sqrt[2]{5}}{4} \left[ 1 + \frac{\sqrt[2]{5}}{2} \right],\]
\[\beta = \frac{5 - \sqrt[2]{5}}{4} + \frac{3 + \sqrt[2]{5}}{4} \sqrt{\frac{2}{\sqrt[2]{5} - 1}};\]
also
\[\beta = 2\,a^2 - 3\,a^2 - 2\,a - 1;\]
and
\[D_1 = -a^3 + a^3 + a + 1 = \frac{1 + \sqrt[2]{5}}{2}.\]

\(n = 103.\) (G. II. 5).

The equation satisfied by \(F_{103}\), due to Russell [9], is
\[F_3 + 3\,F^3 - 3\,F^3 - 2\,F - 1 = 0;\]
its solution is given by Watson [12].

\(n = 111.\) (G. II. 4).

The equation satisfied by \(F_{n_{111}}\) is
\[F^3 - 2\,F^2 - 2\,F^3 - F^4 + 3\,F^3 - 1 = 0;\]
it is given (with an error in sign) by Hanna [3]. By adjunction of \(\sqrt[3]{37},\)
it is expressible in the form
\[\left[ F^2 - \frac{1}{2} \pm \frac{3\sqrt[3]{37}}{2} F + \frac{31 + 5\sqrt[3]{37}}{2} \right]^2 = \frac{1 + \sqrt[3]{37}}{4} \left[ 29 - \frac{29 + 5\sqrt[3]{37}}{2} F + \frac{43 + 7\sqrt[3]{37}}{2} \right]^2.\]

Here we have
\[a = \frac{1}{2} + \frac{3 + \sqrt[3]{37}}{2} \sqrt{\frac{137 - 5}{8}};\]
\[\beta = \frac{1}{2} + \frac{7 + \sqrt[3]{37}}{2} \sqrt{\frac{137 - 5}{8}},\]
also
\[a^3 = \beta^3 + \beta^3 - \beta + 1, \quad D_1 = \beta^3 - 2\beta + 2.\]

\(n = 119.\) (G. II. 5).

The equation satisfied by \(F_{119}\) is
\[F_3 - 4\,F^3 - 9\,F^3 - 7\,F^3 + 5\,F^4 - 4\,F^3 + 2\,F^2 - F + 1 = 0,\]
which, by adjunction of \(\sqrt{17},\) reduces to
\[F^3 - 2\,F^2 - 2\,F^3 - 3\,F^2 - \sqrt{\frac{17 - 1}{2} - F^2 - 1} = 0.\]

The equations satisfied by \(a, \beta, S_1\) and \(D_1\) are
\[a^3 - 3\,a^3 + a^3 + \cdots + 1 = 0,\]
\[\beta^3 - 3\,\beta^3 - 5\,\beta^3 + 7\,\beta^3 - 12\,\beta - 13 = 0,\]
\[S_1^3 = 4\,S_1^3 + 2\,S_1^3 - 3\,S_1^3 - 6\,S_1 - 7 = 0,\]
\[D_1^3 - 8\,D_1 = (D_1^3 + 1)\sqrt{17}.\]
The equations satisfied by $a, b, D_1$, and $S_1$ are
\[ a^5 - 3a^4 + 9a^3 - 16a^2 - 19a - 1 = 0, \]
\[ b^3 - 9b^2 - 16b^2 - 19b - 11 = 0, \]
\[ D_1^3 - 6D_1^2 + 15D_1 - 24 - 26D_1 - 13 = 0, \]
\[ S_1^3 + 17S_1^2 + 14S_1 = (2S_1^3 + 6S_1^2 + 111)^{1/3}. \]

The equations satisfied by $F_{117}$, constructed by Hanna [3], is
\[ F_{117} = 3F_3^3 - 3F_3^2 - 3F_3 = 3F_3 - F = 1 = 0; \]
its solution is given by Watson [12].

The equations satisfied by $F_{115}$, constructed by Hanna [3], is
\[ F_{115} = 3F_3^3 - 3F_3 - 3F_3 = 3F_3 - F = 1 = 0; \]
which, by adjunction of $\sqrt[3]{13}$, reduces to
\[ F_{115} = 47F_3^3 - 145F_3^2 - 196F_3 - 219 - 121F_3^3 - 63F_3^2 - 7F_3 - F = 1 = 0, \]

The equations satisfied by $n = 159$. (G. II. 3).

The equation satisfied by $F_{110}$ is
\[ n = 159. \]

The equations satisfied by $a, b, D_1$, and $S_1$ are
\[ a^5 - 3a^4 + 9a^3 - 16a^2 - 19a - 1 = 0, \]
\[ b^3 - 9b^2 - 16b^2 - 19b - 11 = 0, \]
\[ D_1^3 - 6D_1^2 + 15D_1 - 24 - 26D_1 - 13 = 0, \]
\[ S_1^3 + 17S_1^2 + 14S_1 = (2S_1^3 + 6S_1^2 + 111)^{1/3}. \]

The equations satisfied by $F_{117}$, constructed by Hanna [3], is
\[ F_{117} = 3F_3^3 - 3F_3^2 - 3F_3 = 3F_3 - F = 1 = 0; \]
its solution is given by Watson [12].

The equations satisfied by $F_{115}$ is
\[ F_{115} = 3F_3^3 - 3F_3 - 3F_3 = 3F_3 - F = 1 = 0; \]
which, by adjunction of $\sqrt[3]{13}$, reduces to
\[ F_{115} = 47F_3^3 - 145F_3^2 - 196F_3 - 219 - 121F_3^3 - 63F_3^2 - 7F_3 - F = 1 = 0, \]

The equations satisfied by $n = 159$. (G. II. 3).

The equation satisfied by $F_{110}$ is
\[ n = 159. \]

The equations satisfied by $a, b, D_1$, and $S_1$ are
\[ a^5 - 3a^4 + 9a^3 - 16a^2 - 19a - 1 = 0, \]
\[ b^3 - 9b^2 - 16b^2 - 19b - 11 = 0, \]
\[ D_1^3 - 6D_1^2 + 15D_1 - 24 - 26D_1 - 13 = 0, \]
\[ S_1^3 + 17S_1^2 + 14S_1 = (2S_1^3 + 6S_1^2 + 111)^{1/3}. \]
n = 175. (G II. 3).

The equation satisfied by $F_{175}$, constructed by Weber [13], is

$$F^2 - 4F^3 + 9F^4 - 5 = 0;$$

by adjunction of $\sqrt{5}$, this reduces to

$$F^2 - (2 + \sqrt{5})F + 1 = 0.$$  

The corresponding reduction for $F_{175}^2$ has been effected by Mitra [6].

In terms of Berwick's cubic irrationality

$$\alpha = \beta + 2, \quad \beta = \theta^3 + 3,$$

$$S_3 = \theta^3 + 9 + 2 + \frac{2}{\sqrt{5}},$$

$$2F_{175} = \theta - 2 + \frac{2}{\sqrt{5}}.$$  

n = 183. (G II. 4).

The equation satisfied by $F_{183}$ is

$$F^3 - 71F^3 - 53F^3 - 98F^3 + 10F^3 + 11F^3 - 1 = 0,$$

which, by adjunction of $\sqrt{61}$, is expressible in the form

$$\left[\frac{F^3 - 71 + 9\sqrt{61}/8 + 149 + 19\sqrt{61}}{4}\right]^3.$$

Here we have

$$\alpha^3 - 149a^3 + 6a^4 + 8a^5 - 1 = 0,$$

so that

$$\beta^3 - 9\beta^2 - 3\beta + 9 = 0,$$

so that

$$\alpha^3 = \frac{149 + 19\sqrt{61}}{4} + \frac{55 + 7\sqrt{61}}{4},$$

$$\beta = \frac{1 + \sqrt{61}}{4} + \frac{\sqrt{2 + \sqrt{61}}}{8}.$$  

also

$$\alpha = 2\beta^3 + 5\beta^3 - 5, \quad D_3 = \beta^3 + 2\beta^2 - 1.$$  

n = 191. (G I. 13).

The equation satisfied by $F_{191}$, constructed by Hanna [3], is

$$F^3 - 6F^3 + 10F^3 - 16F^3 + 22F^3 - 19F^3 + 11F^3 - 5F^3$$

$$- 3F^3 + 5F^3 - 4F^3 + \alpha + 2F^3 - 1 = 0.$$  

I have not solved this equation.

n = 199. (G I. 9).

The equation satisfied by $F_{199}$ is

$$F^3 - 5F^3 + 3F^3 - 3F^3 - 3F^3 - 3F^3 - F^3 - F^3 - 1 = 0.$$  

it is solved by Watson [12].

n = 207. (G II. 3).

The equation satisfied by $F_{207}$ is

$$F^3 - 102F^3 + 151F^3 - 103F^3 - 46F^3 - 11F^3 + 1 = 0,$$

which, by adjunction of $\sqrt{69}$, reduces to

$$F^3 - (51 + 6\sqrt{69})F^3 - (17 + 2\sqrt{69})F^3 - \frac{25 + 3\sqrt{69}}{2} = 0.$$  

Hence

$$3F^3 = 51 + 6\sqrt{69} + (50\sqrt{3} + 18\sqrt{23})\sqrt{\frac{3\sqrt{3} - \sqrt{23}}{2}}.$$

It is convenient to work with the cubic for $F_{207}$, namely

$$\theta^3 - 9\theta^2 - 3\theta + 9 = 0,$$

it is found that

$$\alpha = \theta^3 + 9\theta^2 + 8\theta^3 - 1 = 0,$$

$$\gamma = 36\theta^3 + 45\theta^3 + 26,$$

$$S_3 = 24\theta^3 + 31\theta^3 + 18.$$  

$$2F_{207}^3 = 24\theta^3 + 31\theta^3 + 18 + \frac{198\theta^3 + 255\theta^3 + 144}{\sqrt{166}}.$$  

$$\sqrt{3\sqrt{3} - \sqrt{23}}.$$
we have

\[ F^{15} - (21 9^{1} + 3 9^{0} + 30) F^{4} - (51 9^{3} + 9 0^{1} + 70) F^{5} 
+ (22 9^{0} + 4 0 + 30) F^{1} + (29 9^{0} + 5 0 + 40) F^{6} = 0. \]

Further we have

\[ F_{31} F_{73} F_{37} = F_{297} F_{211} = \theta^{*} = 0 + 0 + 1, \]
\[ (F_{273} F_{277})^{2} -(F_{73} F_{211})^{2} = 18 \theta^{9} + 3 \theta + 25 \]
\[ F_{297} F_{273} F_{321} = \theta^{*} = 0 + 1, \]
\[ F_{297} F_{73} F_{231} F_{211} = \theta^{*} = 0 + 1. \]

\[ n = 239. \quad (G. \text{ I. } 15). \]

The equation satisfied by \( F_{239} \), constructed by Hanna [3], is

\[ F^{15} - 6 F^{14} + 2 F^{13} + 8 F^{12} - 4 F^{11} - 27 F^{10} + 13 F^{9} + 15 F^{8} - 4 F^{7} 
- 20 F^{6} + 13 F^{5} + 5 F^{4} - 4 F^{3} - 4 F^{2} + 4 F - 1 = 0; \]
its solution is given by Watson [12].

\[ n = 247. \quad (G. \text{ II. } 3). \]

The equation satisfied by \( F_{247} \) is

\[ F^{4} - 4 F^{3} - 7 F^{2} - 3 F - 1 = 0, \]
which, by adjunction of \( \sqrt[3]{13} \), reduces to

\[ F^{3} - (2 + \sqrt[3]{13}) F^{2} + F - \frac{3 + \sqrt[3]{13}}{2} = 0. \]

In terms of Berwick's cubic irrationality

we have

\[ \theta^{1} = 3 \theta^{1} + 4 \theta + 3 = 0, \quad (\Delta = -247) \]
\[ \theta^{1} = 3 \theta^{1} + 4 \theta + 3 = 0, \quad (\Delta = -247) \]
\[ 2 F_{221} = \theta^{1} + 0 + 2 + \frac{5 \theta^{2} + 7 \theta + 14}{\sqrt[3]{13}}. \]

\[ n = 255. \text{ (G. IV. 3).} \]

The equation satisfied by \( F_{255} \) is
\[
F^{26} = 186 F^{23} - 194 F^{20} + 839 F^{17} - 702 F^{14} + 1012 F^{11} + 912 F^{8} + 513 F^{5} - 221 F^{2} + 66 F - 11 F^{1} + 1 = 0,
\]
which, by adjunction of \( i \sqrt{5} \) and \( i \sqrt{17} \), reduces to
\[
4 F^{3} - (186 + 84 i \sqrt{5} + 46 i \sqrt{17} + 20 i \sqrt{85}) F^{1} - (11 + 3 i \sqrt{5} + i \sqrt{17} + i \sqrt{85}) F^{2} - (36 + 18 i \sqrt{5} + 10 i \sqrt{17} + 4 i \sqrt{85}) = 0.
\]
In terms of Berwick's cubic irrationality
\[
q^{3} - q^{2} - q - 3 = 0, \quad (\Delta = -255)
\]
we have
\[
F^{12} = (27 q^{2} + 24 q + 45) F^{9} - (98 q^{3} + 84 q + 157) F^{6} - (31 q^{2} + 27 q + 50) F^{3} + (59 q^{3} + 51 q + 95) = 0.
\]

Further we have
\[
(F_{135} F_{135})^{3} + (F_{135} F_{135})^{2} = 80 q^{3} + 69 q + 129,
\]
and
\[
F_{135} F_{135} + F_{135} F_{135} = q^{3} + q - 2,
\]
and
\[
F_{135} F_{135} + F_{135} F_{135} = q^{3} + q - 1,
\]
and
\[
F_{135} F_{135} + F_{135} F_{135} = q^{3} + q + 2.
\]

\[ n = 263. \text{ (G. I. 13).} \]

The equation satisfied by \( F_{263} \), constructed by Watson [12], is
\[
F^{15} = -8 F^{12} + 16 F^{11} - 27 F^{10} + 38 F^{9} - 36 F^{8} + 22 F^{7} - 12 F^{6} + 13 F^{5} - 19 F^{4} + 21 F^{3} - 15 F^{2} + 6 F - 1 = 0.
\]
I have not solved this equation.

\[ n = 271. \text{ (G. I. 11).} \]

The equation satisfied by \( F_{271} \), constructed by Watson [12], is
\[
F^{11} = 5 F^{10} - 6 F^{9} - 5 F^{8} + 3 F^{7} + 6 F^{6} + 3 F^{5} - 3 F^{4} - 3 F^{3} - F^{2} - q^{3} = 0.
\]
I have not solved this equation.

\[ n = 279. \text{ (G. II. 6).} \]

The equation satisfied by \( F_{279} \) is
\[
F^{26} = 248 F^{23} - 384 F^{20} + 508 F^{17} - 110 F^{14} + 264 F^{11} + 259 F^{8} + 3 F^{5} - 27 F^{2} + 23 F^{1} + 31 F^{0} - 6 F^{3} + 1 = 0,
\]
which, by adjunction of \( i \sqrt{93} \), reduces to
\[
F^{3} = (124 + 13 i \sqrt{93}) F^{1} - \frac{43 + 5 i \sqrt{93} F^{1}}{2} - \frac{205 + 21 i \sqrt{93} F^{1}}{2} - \frac{87 + 9 i \sqrt{93} F^{1} + 29 + 3 i \sqrt{93}}{2} = 0.
\]
It is simpler to work with the cubic for \( F_{279} \), namely
\[
q^{3} - q^{2} - q - 1 = 0, \quad (\Delta = -279)
\]
than with Berwick's cubic irrationality
\[
q^{3} - q^{2} + 4 q - 3 = 0; \quad (\Delta = -279)
\]

It is found that
\[
2 q^{3} = 4 q^{3} + q + 6 i \sqrt{9} + 3, \quad 2 q^{3} = 3 q^{3} + 4 i + (q^{3} - q + 4) i \sqrt{9} + 3, \quad S_{5} = 6 q^{3} + q^{3} i \sqrt{9} + 3;
\]

It is also to be noticed that
\[
27 q^{3} - 9 q - 6 = i \sqrt{93} (9 q^{3} + 3),
\]

\[ n = 287. \text{ (G. II. 7).} \]

The equation satisfied by \( F_{287} \) is
\[
F^{14} = -8 F^{13} + 9 F^{12} + 6 F^{11} - 5 F^{10} - 7 F^{9} - 8 F^{8} + 6 F^{7} + 2 F^{6} - F^{5} - 3 F^{4} - 3 F^{3} - F^{2} - 1 = 0,
\]
which, by adjunction of \( \sqrt{41} \), reduces to
\[
F^{3} = 4 F^{2} - \frac{7 + 3 i \sqrt{41}}{2} F^{1} - (11 + \sqrt{41}) F^{0} - \frac{13 + 3 i \sqrt{41}}{2} F^{1}.
\]
The equations satisfied by \( a, b, S, \) and \( D \) are
\[
\begin{align*}
\sigma^7 - 5 \sigma^6 - 6 \sigma^5 - 12 \sigma^4 - 12 \sigma^3 - 10 \sigma^2 - 4 \sigma - 1 &= 0, \\
\beta^7 - 5 \beta^6 - 18 \beta^5 + 20 \beta^4 + 116 \beta^3 + 50 \beta^2 - 238 \beta - 245 &= 0, \\
S_1^3 - 8 S_1^2 + 4 S_1 &= 3 S_1^3 - 23 S_1 + 9 S_2 - 6 S_3 - 7 = 0, \\
D_1^3 - 18 D_1 - 63 D_3^3 + 24 D_1 &= (9 D_2^4 + D_3^2 - 1)\sqrt{41}.
\end{align*}
\]

\( n = 295. \) \( (\text{G. II. 4).} \)

The equation satisfied by \( F_{295} \) is
\[
F^8 - 8 F^7 + 9 F^6 - 7 F^5 + 10 F^4 - 7 F^3 + 3 F - 1 = 0,
\]
which, by adjunction of \( \sqrt{5} \), is expressible in the form
\[
\left[ F^8 - 4 + \frac{1}{2} F - 7 + 3 \sqrt{5} F \right] = \frac{11 + 6 \sqrt{5}}{2} \left[ F^4 + 1 + \sqrt{5} \right].
\]

Here we have
\[
\sigma^7 - 7 \sigma^6 - 3 \sigma^5 - \sigma - 1 = 0, \quad \beta^7 - 6 \beta^6 - 2 \beta^5 + 3 \beta + 9 = 0,
\]
so that
\[
\sigma = \frac{7 + 3 \sqrt{5}}{4} + \sqrt{\frac{63 - 29 \sqrt{5}}{8}}, \quad \beta = \frac{3 + 2 \sqrt{5}}{2} + \sqrt{\frac{11 + 6 \sqrt{5}}{4}},
\]
\[
45 \sigma = 3 \beta^3 + 6 \beta^2 - 20 \beta - 12;
\]
also
\[
45 D_1 = 4 \beta^3 - 21 \beta^2 + 10 \beta + 42.
\]

\( n = 303. \) \( (\text{G. II. 5}). \)

The equation satisfied by \( F_{303} \) is
\[
F^{10} - 325 F^9 + 1302 F^8 - 756 F^7 + 720 F^6 - 447 F^5 - 173 F^4 - 46 F^3 - 36 F^2 - 2 F^1 - 1 = 0,
\]
which, by adjunction of \( \sqrt{101} \), reduces to
\[
F^{10} + 325 F^9 F^4 + 325 F^9 F - 1 = 0.
\]

The equations satisfied by \( a, \gamma, D, \) and \( S \) are
\[
a^7 - 10 a^6 + 5 a^5 + 2 a^3 - a - 1 = 0, \\
\gamma^7 - 95 \gamma^6 - 135 \gamma^5 + 2025 \gamma - 3375 = 0, \\
D_4^3 - 325 D_4^2 - 167 D_4^2 - 26 D_3 + 35 D_1 - 23 = 0, \\
S_3^3 - 255 S_3^2 - 70 S_3 = (33 S_4^2 + 38 S_5^2 - 9) \sqrt{101}.
\]

\( n = 311. \) \( (\text{G. I. 19}). \)

The equation satisfied by \( F_{311} \) is
\[
F^{10} - 4 F^9 + 16 F^8 - 37 F^7 + 42 F^6 - 38 F^5 + 10 F^4 + 10 F^3 + 25 F^2 + 18 F + 9 F + 10 F - 13 F + 14 F + 8 F - 5 F - 2 F = -1 = 0.
\]

I have not solved this equation.

\( n = 319. \) \( (\text{G. II. 5}). \)

The equation satisfied by \( F_{319} \) is
\[
F^{10} - 6 F^9 + 9 F^8 - 5 F^7 - F^6 - 2 F^5 - 10 F^4 - 14 F^3 - 11 F^2 - 5 F - 1 = 0
\]
which, by adjunction of \( \sqrt{29} \), reduces to
\[
F^8 - 3 + \sqrt{29} F + 11 + \frac{1}{2} F^3 - \frac{1 + \sqrt{29}}{2} F + F - \frac{5 + \sqrt{29}}{2} = 0.
\]

The equations satisfied by \( a \) is
\[
a^7 - 6 a^6 + 3 a^5 + a^3 - a = 1 = 0;
\]
also
\[
\beta = a + 2, \quad D_1 = a,
\]
and
\[
S_1^3 + 25 S_1^2 + 49 S_1 = (2 S_4^2 + 7 S_5^2 + 9) \sqrt{29},
\]
\( n = 327. \) \( (\text{G. II. 6}). \)

The equation satisfied by \( F_{327} \) is
\[ F^{14} = 423 F^{13} - 2573 F^{12} - 4844 F^{11} - 5524 F^{10} - 4006 F^{9} - 1436 F^{8} + 92 F^{7} - 174 F^{6} - 58 F^{5} - 63 F^{4} - 14 F^{3} - 1 = 0, \]

which, by adjunction of \( \sqrt{109} \), is expressible in the form

\[
\left[ F^9 - \frac{423}{4} \sqrt{109} F^8 + \frac{93}{2} \sqrt{109} F^7 - \frac{907}{4} \sqrt{109} F^6 + \frac{39}{2} \sqrt{109} F^5 \right]^2 = \frac{109}{2} \left[ \frac{199}{4} \sqrt{109} F^7 - \frac{21 + 2 \sqrt{109}}{2} F^6 + \frac{94}{2} \sqrt{109} F^5 \right].
\]

In terms of Berwick's cubic irrationality
\[ \theta^9 - 4 \theta^6 + 3 \theta - 3 = 0, \quad (\Delta = -327) \]

the equation reduces to
\[ F^9 = D_9 F^3^3 - \alpha^2 = 0, \]
where
\[ 2 \alpha^3 = 159 \theta^6 - 99 \theta^2 + 141 \sqrt{109} + 1661 \theta^8 - 1035 \theta + 1475 \sqrt{109}, \]
\[ 2 D_9 = 42 \theta^6 - 27 \theta^2 + 37 \sqrt{109} + 440 \theta^8 - 285 \theta + 403 \sqrt{109}. \]

It is also found that
\[ 2 \beta^3 = 6 \theta^6 - 2 \theta + 5 \theta^2 - 2 \theta - 14 \sqrt{109}. \]

It is to be remarked that
\[ 1661 \theta^8 - 1035 \theta + 1475 = (4 \theta^6 - 29 \theta + 41) (5 \theta^6 - 2 \theta - 14), \]
\[ 440 \theta^8 - 285 \theta + 403 = (12 \theta^6 - 7 \theta + 10) (5 \theta^6 - 2 \theta - 14), \]
\[ 5 \theta^2 - 2 \theta - 14 = \sqrt{109} (6 \theta^6 - 6 \theta - 4). \]

\[ n = 335. \quad \text{(G. II. 9).} \]

The equation satisfied by \( F_{331} \) is
\[ F^{14} - 4 F^{13} - 20 F^{12} - 55 F^{11} - 106 F^{10} - 144 F^{9} - 163 F^{8} - 174 F^{7} - 179 F^6 + 171 F^5 - 144 F^4 - 102 F^3 - 64 F^2 - 42 F^0 - 33 F^3 - 25 F^9 - 14 F^7 - 5 F^6 - 1 = 0, \]

which, by adjunction of \( \sqrt{5} \), reduces to
\[ F^9 - (2 + 2 \sqrt{5}) F^7 - (2 + 3 \sqrt{5}) F^5 - \frac{3 - 7 \sqrt{5}}{2} F^3 - \frac{1 + 5 \sqrt{5}}{2}, \]
\[ + \frac{3 + 3 \sqrt{5}}{2} F^3 - \frac{3 - 7 \sqrt{5}}{2} F^5 - \frac{1 + 5 \sqrt{5}}{2} = 0. \]

By means of Berwick's cubic irrationality
\[ \theta^9 - 2 \theta^7 - 5 \theta^5 - 5 \theta = 0 \quad (\Delta = -335) \]

the equation is further reducible to
\[ 6 \left( F^3 - (\theta^3 - 6 \theta + 4) F^2 - (\theta^2 - 3 \theta + 3) F + (\theta^2 + 3 \theta + 3) \right) \sqrt{5}; \]

it is to be noticed that
\[ \frac{\sqrt{5}}{\theta} = \sqrt{\theta^2 - 6 \theta + 3}. \]

Further, the equations satisfied by \( \alpha, \beta \) and \( D_i \) are
\[ \alpha^3 = 7 \alpha^2 - 14 \alpha + 21 \alpha^2 + 11 \alpha^2 - 2 \alpha^2 + \alpha^2 \alpha = 1 = 0, \]
\[ \beta^3 = -29 \beta^2 - 102 \beta + 173 \beta - 263 \beta + 376 \beta - 303 \beta - 133 \beta - 67 = 0, \]
\[ D_9 - 4 D_9^2 - 13 D_9^3 - 30 D_9^4 + D_9^5 + 32 D_9^6 + 95 D_9^7 + 3 D_9^2 - 45 D_9 - 155 = 0. \]

By adjunction of the cubic irrationality \( \theta \), these equations reduce to
\[ \alpha^3 = -2 (\theta^3 - 6 \theta + 7) \alpha^2 - (\theta^2 - 6 \theta + 4) \alpha^2 \alpha = 0, \]
\[ \beta^3 = (\theta^3 + 2 \theta^2 - 4 \theta - 6 + 16) \beta - (\theta^2 - 3 \theta + 18) \beta^2 = 0, \]
\[ D_9^3 - (\theta^3 - 6 \theta + 4) D_9^2 - (3 \theta^2 - 3 \theta + 12) D_9 - (3 \theta^2 - 2 \theta + 15) = 0. \]

Finally, \( \beta \) and \( D_i \) are expressible as rational functions of \( \alpha \) by the formulae
\[ 5 \beta = (\theta^3 - 4 \theta + 4) \alpha^2 + (2 \theta^2 - 6 \theta + 4) \alpha \alpha - (\theta^2 - 9 \theta + 8), \]
\[ 5 D_9 = (-6 \theta + 7) \alpha^2 + (3 \theta^2 - 7 \theta + 12) \alpha \alpha - (2 \theta^2 - 6 \theta + 4). \]

\[ n = 343. \quad \text{(G. I. 7).} \]

The equation satisfied by \( F_{341} \) is
\[ F^1 - 7 F^9 - 7 F^8 - 7 F^7 - \cdots - \cdots - 1 = 0, \]
its solution is given by Watson [12]. The equation is very easily obtainable from Schiller's modular equation of order 7 which connects \( F_{65} \) with \( F_7 \).

\[
n = 351. \quad \text{(G. I. 6)}.
\]

The equation satisfied by \( F_{65} \) is

\[
F^4 - 555 F^3 + 133 \sqrt{13} F^2 - 447 + 123 \sqrt{13} F - 177 + 49 \sqrt{13} = 0,
\]

which, by adjunction of \( \sqrt{13} \), is expressible in the form

\[
(7 + 2\sqrt{13}) \left[ \frac{153 + 39 \sqrt{13}}{4} F^3 + \frac{117 + 33 \sqrt{13}}{4} F^2 + \frac{47 + 13 \sqrt{13}}{2} F - 1 \right] = 0.
\]

In terms of Berwick's cubic irrationality

\[
\theta^4 + \theta + 3 \theta - 3 = 0 \quad (\lambda = -351)
\]

the equation reduces to

\[
F^3 - D_2 F^2 - a_2 = 0,
\]

where

\[
2 a_2 = 71 \theta^2 + 58 \theta + 261 + \frac{257 \theta + 210 \theta^2 + 943}{\sqrt{13}}.
\]

\[
2 D_2 = 110 \theta^2 + 91 \theta + 405 + \frac{400 \theta + 327 \theta^2 + 1463}{\sqrt{13}}.
\]

it is also found that

\[
2 \lambda^5 = 2 \theta^6 + 9 \theta + 7 + \frac{6 \theta^4 + 9 \theta + 25}{\sqrt{13}}.
\]

It is to be remarked that

\[
257 \theta^6 + 210 \theta^5 + 943 = (33 \theta^3 + 27 \theta^2 + 121) \sqrt{13}(\theta^2 + 4)
\]

\[
400 \theta^3 + 327 \theta^2 + 1463 = (51 \theta^2 + 42 \theta + 188) \sqrt{13}(\theta^2 + 4)
\]

\[
6 \theta^4 + 9 \theta + 25 = \sqrt{13}(\theta^2 + 4)^3
\]

\[
= (\theta^2 + 4)(2 \theta^2 + 3 \theta + 4).
\]

\[
n = 359. \quad \text{(G. I. 19)}.
\]

The equation satisfied by \( F_{65} \) is

\[
F^{15} - 14 F^{14} - 59 F^{13} + 91 F^{12} + 19 F^{11} - 90 F^{10} + 51 F^9 + 2 F^8 - 9 F^7 + 22 F^6 + 7 F^5 - 14 F^4 + 3 F^3 + 2 F^2 + 2 F - 1 = 0.
\]

I have not solved this equation.

\[
n = 367. \quad \text{(G. I. 9)}.
\]

The equation satisfied by \( F_{65} \) is

\[
F^5 - 9 F^4 + 3 F^3 - 2 F^2 + 1 + 2 F^4 - 6 F^3 + F^2 + 2 F - 1 = 0;
\]

it is solved by Watson [12].

\[
n = 375. \quad \text{(G. I. 5)}.
\]

The equation satisfied by \( F_{65} \) is

\[
F^{15} = 705 F^{14} - 1200 F^{13} + 1815 F^{12} - 344 F^{11} - 330 F^{10} - 35 F^9 + 35 F^8 + 10 F^7 - 1 = 0,
\]

which, by adjunction of \( \sqrt{5} \), reduces to

\[
F^{15} = \frac{705 - 319 \sqrt{5}}{2} F^{12} - (200 + 89 \sqrt{5}) F^9 + \frac{245 + 111 \sqrt{5}}{2} F^5
\]

\[
- (25 + 11 \sqrt{5}) F^3 - 29 + 13 \sqrt{5} = 0.
\]

The equations satisfied by \( a, \gamma, D_2 \) and \( S_4 \) are

\[
a^5 - 15 a^4 + 5 a^3 - 5 a^2 - 1 = 0,
\]

\[
\gamma^5 - 165 \gamma^4 + 955 \gamma^3 - 3540 \gamma^2 + 8395 \gamma - 7743 = 0,
\]

\[
D_2^8 = 705 D_2^4 - 180 D_2^3 - 175 D_2^2 + 15 D_2 - 9 = 0,
\]

\[
S_4^5 - 620 S_4^4 - 615 S_4^3 = (319 S_4^2 + 385 S_4 + 99) \sqrt{5}.
\]

\[
n = 383. \quad \text{(G. I. 17)}.
\]

The equation satisfied by \( F_{83} \) is

\[
F^{15} = 6 F^{14} - 24 F^{13} - 42 F^{12} - 31 F^{11} - 23 F^{10} - 7 F^9 + 2 F^8 - 5 F^7 - 3 F^6 + 2 F^5 - 2 F^4 + 2 F^3 + 2 F^2 - 2 F - 1 = 0.
\]
\[ F^4 - 11F^3 - 7F^2 - 13F - F^3 = 0. \]

I have not solved this equation.

\[ n = 391. \quad (G. \ II. 7). \]

The equation satisfied by \( F_{391} \) is

\[ F^{14} - 8F^{13} - 12F^{12} - 11F^{11} + \cdots + 4F^3 - 3F^2 - 8F^7 + 6F^6 + 15F^5 + 7F^4 - 5F^3 - 5F^2 + \cdots + 1 = 0, \]

which, by adjunction of \( \sqrt{17} \), reduces to

\[ F^2 - 4 - \frac{\sqrt{17}}{2} F^3 + 2 - \frac{\sqrt{17}}{2} F^4 - \frac{\sqrt{17}}{2} F^5 - \frac{\sqrt{17}}{2} F^6 - \frac{\sqrt{17}}{2} F^7 - \frac{\sqrt{17}}{2} F^8 = 0, \]

The equations satisfied by \( \alpha, \beta, S_1, \) and \( D_1 \) are

\[ \alpha^3 - 9\alpha^2 + 10\alpha^1 - 14\alpha^0 + 8\alpha^1 - 6\alpha^2 + 2\alpha - 1 = 0, \]

\[ \beta^3 - 9\beta^2 + 29\beta^1 - 17\beta^2 + 80\beta^1 - 180\beta^2 + 161\beta^3 - 324\beta - 129 = 0, \]

\[ S_1^3 - 8S_1^2 - 21S_1^1 - 20S_1^0 - 16S_1^1 - 25S_1^2 + 9S_1^3 - 1 = 0, \]

\[ D_1^3 - D_1^2 + 22D_1^1 + 11D_1 = (2D_1^1 + 4D_1^0 + 17D_1^2 + 1) \sqrt{17}. \]

\[ n = 399. \quad (G. \ IV. 4). \]

The equation satisfied by \( F_{399} \) is

\[ F^{48} - 896F^{47} - 5260F^{46} - 6165F^{45} - 7523F^{44} + 7376F^{43} + 13199F^{42} + 11486F^{41} + 360F^{40} + 3022F^{39} + 2490F^{38} + 641F^{37} + 361F^{36} + 330F^{35} + 125F^{34} + 19F^{33} + 1 = 0, \]

which, by adjunction of \( \sqrt{21}, \sqrt{57} \), and \( \sqrt{133} \), reduces to

\[ -4F^{15} - (896 + 198\sqrt{21} + 120\sqrt{57} + 78\sqrt{133})F^3 = 0. \]

In terms of the quartic irrationality \( \eta \), defined as

\[ \eta = -\frac{3}{2} + \sqrt{23 + 21\sqrt{133}} \]

so that

\[ 6\eta - 6\eta^2 - 2\eta^3 - 21\eta - 21 = 0, \]

we have

\[ F_{398}F_{393} + F_{393}F_{2119} = 6\eta^2 - 5, \]

\[ F_{2119}F_{393} + F_{2119}F_{398} = \eta^3 - 5\eta^2 + 49 - 15, \]

\[ F_{398}F_{3119} + F_{3119}F_{393} = 0, \]

\[ (F_{398}F_{3119}F_{393}F_{3119})^2 = 36\eta^3 - 40\eta^2 - 123\eta + 155. \]

\[ n = 407. \quad (G. \ II. 8). \]

The equation satisfied by \( F_{407} \) is

\[ F^{14} - 10F^{13} + 4F^{12} - 33F^{11} + 17F^{10} - 19F^9 - 4F^{10} - \cdots - 12F^4 - 15F^3 + 2F^2 - 9F^4 - F^2 - F - 1 = 0, \]

which, by the adjunction of \( \sqrt{37} \), is expressible in the form

\[ \left[ F^4 - \frac{5 + \sqrt{37}}{2} F^3 + \frac{4 + \sqrt{37}}{2} F^2 + \frac{3 + \sqrt{37}}{4} F - \frac{7 + \sqrt{37}}{4} \right]^2 = 91 + 15\sqrt{37} - \frac{8}{9} (\sqrt{37} - 5) F^3 - \frac{7 - \sqrt{37}}{2} F^2 + F^1. \]

We also have

\[ \left[ a^2 + \frac{7 + \sqrt{37}}{4} (a + 1) \right]^2 \quad 91 + 15 \frac{\sqrt{37}}{8} (a + 1)^2. \]
\[
\left[ \beta - \frac{11 + 3\sqrt{37}}{4}, \frac{7 - \sqrt{37}}{4} \right] = \frac{91 + 15\sqrt{37}}{8} \left[ \sqrt{37} - 5 \right] \frac{1}{2} 8 + (\sqrt{37} - 4)\right].
\]
\[
D_1^2 = \frac{5 + \sqrt{37}}{2}, D_1 + \frac{15 + 3\sqrt{37}}{4}
\]
\[
= \frac{91 + 15\sqrt{37}}{8} \left[ \sqrt{37} - 5 \right] D_1 - \frac{9 - \sqrt{37}}{2}\right]^2.
\]

\[n = 415. \text{ (G. II. 5).}\]

The equation satisfied by \( F_{111} \) is
\[
F^{11} - 10 F^3 - F^4 - 7 F^3 - 11 F^3, + 2 F^5 - 14 F^4 + 2 F^3
\]
\[
- 6 F^3 - \ast - 1 = 0,
\]
which, by adjunction of \( \sqrt{5} \), reduces to
\[
F^5 - (5 + 3 \sqrt{5}) F^4 + 19 + 7 \sqrt{5} F^3
\]
\[
- 17 + 9 \sqrt{5} F^3 - (5 + 2 \sqrt{5}) F - (2 + \sqrt{5}) = 0,
\]

The equations satisfied by \( \alpha, \beta, D_1 \) and \( S \) are
\[
\alpha^3 - 13 \alpha^2 + 9 \alpha^3 + \ast + \alpha - 1 = 0,
\]
\[
\beta^3 - 9 \beta^2 - 4 \beta^3 + 69 \beta^2 - 5 \beta - 147 = 0,
\]
\[
D_1^3 - 10 D_1 + 12 D_1 - 27 D_1 + 20 D_1 - 5 = 0.
\]
\[
S_1^3 + 26 S_1^3 + 14 S_1 = (6 S_1^3 + 15 S_1^3 + 1) \sqrt{5}.
\]

\[n = 423. \text{ (G. II. 5).}\]

The equation satisfied by \( F_{423} \) is
\[
F^{11} - 1140 F^{11} + 2434 F^{11} - 2432 F^{11} + 939 F^{11} - 140 F^{11} + 446 F^{11}
\]
\[
- 274 F^8 + 41 F^8 - F^3 + 1 = 0,
\]
which, by adjunction of \( \sqrt{141} \), reduces to

\[S_{10}^2 = \left( \frac{570 + 48 \sqrt{141}}{1} \right) F^{12} + \left( 1199 + 101 \sqrt{141} \right) F^3 - \left( 1354 + 114 \sqrt{141} \right) F^6
\]
\[
+ \frac{1223 + 103 \sqrt{141}}{2} F^3 - (95 + 8 \sqrt{141}) = 0.
\]

The equation satisfied by \( \gamma, \tau, S_9 \) and \( D_9 \) are
\[
\gamma^2 - 1291 \gamma^2 + 23318 \gamma^2 + 1 38743 \gamma^2 + 2 25823 \gamma - 1 68155 = 0
\]
\[
S_9^5 - 1140 S_9^5 + 1445 S_9^5 - 1382 S_9^5 + 370 S_9 - 325 = 0,
\]
\[
D_9^5 + 3351 D_9^5 + 3708 D_9 = (96 D_9^5 + 420 D_9^5 + 99) \sqrt{141}.
\]

\[n = 431. \text{ (G. I. 21).}\]

The equation satisfied by \( F_{431} \) is
\[
F^{11} - 12 F^{10} - 15 F^{19} + 16 F^{18} + 40 F^{17} - 21 F^{16} - 10 F^{15} - 44 F^{14}
\]
\[
- 83 F^{13} - 41 F^{12} - 66 F^{11} + 2 F^{10} + 14 F^9 + 30 F^8 + 36 F^7
\]
\[
- 10 F^6 + 4 F^5 - 9 F^4 - 9 F^3 - 6 F^2 - 3 F - 1 = 0.
\]

In terms of the cubic irrationality
\[
\phi^2 - \ast - 6 - 8 = 0, \quad (\Delta = 1724)
\]
we have
\[
F^3 - (2 \phi + 4) F^6 - \frac{5 \phi^2 + 3 \phi + 16}{2} F^3 - \frac{2 \phi^2 + 6 \phi + 2}{4} F^4
\]
\[
- \frac{6 \phi + 5 + 11}{2} F^3 = \left( (\phi^2 + 6 + 1) F^3 - (2 \phi + 1) F - 1 \right) = 0.
\]

This cubic irrationality is connected with Berwick's cubic irrationality
\[
2 \phi^2 - \phi^2 + 3 \phi - 2 = 0 \quad (16 \Delta = 431)
\]
by the relation
\[
\phi = \frac{2}{6 + 1}.
\]

I have not solved the septic equation.
n = 439. (G. I. 15).

The equation satisfied by \( F_{439} \) is
\[
F^{15} - 13 F^{14} + 23 F^{13} - 7 F^{12} - 24 F^{11} - 20 F^{10} - 13 F^9 - 28 F^8 - 29 F^7 + F^6 - 17 F^5 + 7 F^4 - 9 F^3 - 11 F^2 - 5 F - 1 = 0;
\]
its solution is given by Watson [12].

n = 447. (G. II. 7).

The equation satisfied by \( F_{447} \) is
\[
F^{43} - 141 F^{42} + 13302 F^{41} - 38101 F^{40} + 53826 F^{39} - 44566 F^{38} + 28940 F^{37} - 23117 F^{36} - 20245 F^{35} - 13246 F^{34} - 5664 F^{33} + 1516 F^{32} - 243 F^{31} - 22 F^4 - 1 = 0,
\]
which, by adjunction of \( \sqrt[149]{1} \), reduces to
\[
F^{31} = \frac{1419 + 117 \sqrt{149}}{2} F^{42} = \frac{5359 + 439 \sqrt{149}}{2} F^{31} = \frac{8409 + 689 \sqrt{149}}{2} F^{32} = \frac{6797 + 557 \sqrt{149}}{2} F^{33} = \frac{[1488 + 122 \sqrt{149}] F^3}{2} = \frac{61 + 5 \sqrt{149}}{2} = 0.
\]

The equations satisfied by \( a, \gamma, D_9 \) and \( S_9 \) are
\[
a^5 - 20 a^4 - 8 a^3 + 4 a^2 - 6 a - 2 = 0,
\]
\[
\gamma^7 - 248 \gamma^6 + 1368 \gamma^5 - 1886 \gamma^4 + 4105 \gamma^3 - 7665 \gamma^2 + 12321 \gamma - 51867 = 0,
\]
\[
D_9 = 1417 D_9 - 5275 D_9 - 6971 D_9 - 4608 D_9 - 2664 D_9 - 1458 D_9 - 405 = 0,
\]
\[
S_9 = 5443 S_9 - 10850 S_9 + 2310 S_9 = 117 S_9 + 829 S_9 + 566 S_9 + 25) \sqrt{149}.
\]

n = 455. (G. IV. 5).

The equation satisfied by \( F_{455} \) is
\[
F^{45} - 6 F^{44} + 50 F^{43} - 142 F^{42} - 200 F^{41} + 129 F^{40} + 38 F^{39} + 191 F^{38} + 246 F^{37} + 194 F^{36} + 76 F^{35} + 30 F^3 + 73 F^2 + 57 F + 1 = 0,
\]
which, by adjunction of \( \sqrt{5} \) and \( \sqrt{13} \), reduces to
\[
F^2 - 3 \sqrt{13} \left( \sqrt{5} \right) \left( \sqrt{5} \right) F^2 = 1.\]

Further, if
\[
F_{455} F_{455} F_{455} F_{455} F_{455} = a,
\]
\[
F_{455} F_{455} F_{455} F_{455} F_{455} = b,
\]
\[
F_{455} F_{455} F_{455} F_{455} F_{455} = c,
\]
then also
\[
F_{455} F_{455} F_{455} F_{455} F_{455} = b,
\]
and \( a, b, c \) are the quintic irrationalities given by the equations
\[
a^5 - 12 a^4 - 41 a^3 - 105 a^2 - 119 a - 49 = 0,
\]
\[
b^5 - 12 b^4 - 41 b^3 - 105 b^2 - 119 b - 49 = 0,
\]
\[
c^5 - 12 c^4 - 41 c^3 - 105 c^2 - 119 c - 49 = 0.
\]

n = 463. (G. I. 7).

The equation satisfied by \( F_{463} \) is
\[
F^{11} - 11 F^4 - 9 F^3 - 8 F^4 - 7 F^3 - 3 F - 1 = 0;
\]
its solution is given by Watson [12].

n = 471. (G. II. 8).

The equation satisfied by \( F_{471} \) is
\[
F^{45} - 1775 F^{44} - 4085 F^{43} + 29217 F^{42} - 60470 F^{41} + 44222 F^{40} - 36601 F^{39} + 22281 F^{38} - 18987 F^{37} + 9629 F^{36} - 4756 F^{35} - 942 F^{34} - 382 F^{33} + 106 F^9 - 62 F^8 - 6 F^3 - 1 = 0,
\]
which, by adjunction of \( \sqrt{157} \), is expressible in the form
\[
F^{45} = (1775 + 141 \sqrt{157}) F^3 - (12169 + 971 \sqrt{157}) F^3 - (9357 + 747 \sqrt{157}) F^3 - (5012 + 400 \sqrt{157}) F^3.
\]
which, by adjunction of $\sqrt{5}$ and $\sqrt{33}$, reduces to

\[4 F^{12} - (2200 + 984 \sqrt{5} + 384 \sqrt{33} + 172 \sqrt{165}) F^6 \]
\[+ (3402 + 1520 \sqrt{5} + 590 \sqrt{33} + 264 \sqrt{165}) F^5 \]
\[+ (1935 + 667 \sqrt{5} + 261 \sqrt{33} + 117 \sqrt{165}) F^4 \]
\[+ (234 + 104 \sqrt{5} + 40 \sqrt{33} + 18 \sqrt{165}) = 0.\]

We also have

\[F_{403} F_{318} + F_{318} F_{403} = 4 + 2 \sqrt{5} + \frac{3 + \sqrt{5}}{2} \sqrt{3 + 9 \sqrt{5}} \]
\[+ \frac{605 + 271 \sqrt{5} + \sqrt{3 + 9 \sqrt{5}}}{2},\]

\[F_{403} F_{318} F_{403} F_{318} = \frac{13 + 5 \sqrt{5}}{4} + 2 \sqrt{5} + \frac{3 + 9 \sqrt{5}}{2},\]

\[F_{403} F_{318} F_{318} F_{403} = \frac{15 + 7 \sqrt{5}}{4} + 2 \sqrt{5} + \frac{3 + 9 \sqrt{5}}{2}.\]

\[n = 503.\quad (G. I. 21).\]

The equation satisfied by $F_{381}$ is

\[F^{21} - 18 F^{20} + 69 F^{19} - 87 F^{18} + 44 F^{17} + 171 F^{16} - 106 F^{15} - 74 F^{14} + 92 F^{13} + 19 F^{12} - 27 F^{11} - 30 F^{10} + 23 F^9 + 10 F^8 - 12 F^7 + 3 F^6 - 7 F^5 + 6 F^4 + F^3 - F^2 - 1 = 0,\]

its solution is given by Watson [12].

\[n = 495.\quad (G. IV. 4).\]

The equation satisfied by $F_{401}$ is

\[F^{14} - 2200 F^{13} - 5184 F^{12} - 2553 F^{11} + 1023 F^{10} + 9360 F^9 + 5623 F^8 - 1895 F^7 + 5751 F^6 - 2515 F^5 + 2338 F^4 + 1170 F^3 - 282 F^2 - 235 F^1 + 81 F^0 - 5 F^1 + 1 = 0,\]
by the relation
\[ 2 \psi^3 - 5 \psi^2 + 5 \psi - 4 = 0 \]  
\[ \psi = \frac{20}{9 + 1}. \]

I have not solved the septimic equation.

\( n = 511. \) (G. II. 7.)

The equation satisfied by \( F_{511} \) is
\[ F^{14} - 16 F^{13} + 36 F^{12} - 56 F^{11} + 77 F^{10} - 84 F^9 + 70 F^8 - 37 F^7 + 16 F^6 - 21 F^5 + 35 F^4 - 35 F^3 + 21 F^2 - 7 F + 1 = 0, \]
which, by adjunction of \( \sqrt[7]{3} \), reduces to
\[ F^4 - (8 + \sqrt[7]{3}) F^3 + \frac{45 + 5 \sqrt[7]{3}}{2} F^2 - \frac{61 + 2 \sqrt[7]{3}}{2} F + 25 + 3 \sqrt[7]{3} \]  
\[ - \frac{27 + 3 \sqrt[7]{3}}{2} F^2 + \frac{7 + \sqrt[7]{3}}{2} F - 1 = 0. \]

The equation satisfied by \( \alpha \) is
\[ \alpha^7 - 16 \alpha^6 + 20 \alpha^5 - 16 \alpha^4 + 9 \alpha^3 - \alpha^2 - \alpha - 1 = 0; \]
also
\[ \beta + 2 = S_1 = \alpha, \]
\( D_1^7 + 70 D_1^6 + 331 D_1^5 + 376 D_1^4 + (2 D_1^3 + 22 D_1^2 + 50 D_1 + 21) \sqrt[7]{3}. \)

\( n = 519. \) (G. II. 9.)

The equation satisfied by \( F_{315} \) is
\[ F^{18} - 2709 F^{17} - 16545 F^{16} + 6267 F^{15} + 49945 F^{14} + 136972 F^{13} - 150834 F^{12} + 113720 F^{11} - 79908 F^{10} + 41992 F^9 - 30286 F^8 + 20926 F^7 - 12763 F^6 + 7806 F^5 - 3683 F^4 + 1178 F^3 + 109642 F^2 - 49954 F + 136972 = 0, \]
which, by adjunction of \( \sqrt{173} \), reduces to
\[ F^{18} - \frac{2079 + 207 \sqrt{173}}{2} F^{17} + \left(1002 + 76 \sqrt{173}\right) F^{16} + 951 + 75 \sqrt{173}\]
\[ F^{15} - \frac{1233 + 27 \sqrt{173}}{2} F^{14} - \frac{233 + 27 \sqrt{173}}{2} F^{13} + 333 + 27 \sqrt{173} F^{12} - \frac{233 + 19 \sqrt{173}}{2} F^{11} + (38 + 3 \sqrt{173}) F^{10} - \frac{13 + \sqrt{173}}{2} F^{9} = 0. \]

By adjunction of the cubic irrationality
\[ \sqrt[3]{3} - 2 \sqrt[3]{6} - 3 \sqrt[3]{5} - 3 = 0, \]
which is connected with Berwick's cubic irrationality
\[ 3 \psi^3 - 6 \psi^2 + 5 \psi - 3 = 0 \]
by the relation \( \psi = 1 + \theta \), the equation is further reducible to
\[ 2 F^3 - (177 \theta^6 + 219 \theta^5 + 167) F^2 + (87 \theta^4 + 108 \theta^3 + 81) F - (86 \theta^2 + 105 \theta + 80) \sqrt{173} \]
\[ = (2361 \theta^6 + 2841 \theta^5 + 2173) F^3 - (1153 \theta^2 + 1374 \theta + 1065) F^2 \]
\[ + (1136 \theta^5 + 1383 \theta + 1058). \]

Also \( \alpha, \gamma \) and \( D_\alpha \) are given in terms of \( \theta \) by the equations
\[ \alpha^7 - (2 \theta^3 + 6 \theta + 2) \alpha^6 - (6 \theta^2 + 2 \theta + 1) \alpha^5 - (6 \theta + 6 + 1) \alpha^4 - 9 \alpha^3 - \alpha^2 - \alpha - 1 = 0, \]
\[ \gamma^3 = (265 \theta^2 + 324 \theta + 246) \theta^2 - (77 \theta^2 + 126 \theta + 97) \theta - (66 \theta^2 + 99 \theta + 105) \theta + 1 = 0, \]
\[ D_\alpha^3 - (177 \theta^6 + 219 \theta + 167) D_\alpha^2 + (37 \theta^2 + 48 \theta + 34) D_\alpha \]
\[ - (265 \theta^2 + 324 \theta + 246) = 0, \]
\( n = 527. \) (G. II. 9.)

The equation satisfied by \( F_{315} \) is
\[ F^{18} - 12 F^{17} - 31 F^{16} - 24 F^{15} + 45 F^{14} + 15 F^{13} + 12 F^{12} + 20 F^{11} + 5 F^{10} \]
\[ - 32 F^9 - 10 F^8 - F^7 - 7 F^6 - 4 F^5 + 8 F^4 + 5 F^3 + 2 F^2 + \star + 1 = 0, \]
which, by adjunction of \( \sqrt{17} \), reduces to
\[ F^3 - \frac{7 + \sqrt{17}}{2} F^2 - \frac{5 + \sqrt{17}}{2} F^1 + (8 + 2 \sqrt{17}) F^0 + \frac{9 + \sqrt{17}}{2} F^1 \]
\[ - \frac{7 + \sqrt{17}}{2} F^2 - \frac{5 + \sqrt{17}}{2} F^1 - \star = 1 = 0. \]
By adjunction of Berwick's cubic irrationality
\[ \theta^3 + \alpha + 59 = 1 = 0, \quad (\Delta = -527) \]
the equation is further reducible to
\[ [2 F^3 - (3 \theta^2 + 9 + 14) F^2 + (4 \theta^2 + 2 \theta + 20) F - (2 \theta^2 + \theta + 10)] \sqrt{17} \]
\[ = (7 \theta^2 - 3 \theta + 46) F^3 + (18 \theta^2 + 2 \theta + 94) F + (8 \theta^2 - 6 + 38). \]

Also \( \alpha, \beta \) and \( S_i \) are given in terms of \( \theta \) by the equations
\[ \alpha^2 - (2 \theta^2 + 6 + 11) \theta - (2 \theta^2 + 11) \alpha - (\theta^2 + 5) = 0, \]
\[ \beta^2 - (3 \theta^2 - 6 + 16) \beta^2 - (2 \theta^2 - 2 \theta + 13) \beta - (\theta^2 + 3) = 0, \]
\[ S_i^2 - (3 \theta^2 + 9 + 14) S_i^2 - (2 \theta^2 + 6 + 11) S_i - (3 \theta^2 + 2 \theta + 15) = 0. \]

\( n = 535. \quad (G. II. 7). \)

The equation satisfied by \( F_{535} \) is
\[ F^{16} - 16 F^{15} + 20 F^{14} + 7 F^{13} - 14 F^{12} + 10 F^{11} - 37 F^{10} + 34 F^9 + 3 F^8 \]
\[ - F^7 - 16 F^6 + 16 F^5 + 3 F^4 - \theta - 1 = 0, \]
which, by adjunction of \( \sqrt[3]{5} \), reduces to
\[ F^7 - (8 + 4 \sqrt[3]{5}) F^6 + (18 + 8 \sqrt[3]{5}) F^5 - \frac{25 + 11 \sqrt[3]{5}}{2} F^4 + (1 + \sqrt[3]{5}) F^3 \]
\[ - (2 + \sqrt[3]{5}) F^2 + (3 + 2 \sqrt[3]{5}) F - (2 + \sqrt[3]{5}) = 0. \]

The equations satisfied by \( \alpha, \beta, D_i, \) and \( S_i \) are
\[ \alpha^2 - 19 \alpha^2 + \alpha + 11 \alpha - 11 \alpha^2 - \alpha^2 + 4 \alpha - 1 = 0, \]
\[ \beta^2 - 6 \beta^2 - 52 \beta^2 - 96 \beta^4 - 44 \beta^2 - 90 \beta - 315 \beta - 243 = 0, \]
\[ D_i^2 - 16 D_i^4 + 39 D_i^6 - 50 D_i^8 + 120 D_i^{10} - 189 D_i^{12} + 139 D_i^{14} - 71 = 0, \]
\[ S_i^2 + 33 S_i^4 + 16 S_i^6 + 3 S_i = (8 S_i^4 + 10 S_i^4 + 5 S_i^2 + 1) \sqrt[3]{5}. \]

\( n = 551. \quad (G. II. 6). \)

The equation satisfied by \( F_{551} \) is
\[ F^{16} - 3325 F^{15} + 20738 F^{14} - 26913 F^{13} + 26778 F^{12} + 24014 F^{11} \]
\[ - 14714 F^{10} + 6654 F^9 - 2895 F^8 + 929 F^7 - 206 F^6 + 21 F^5 - 1 = 0, \]
which, by adjunction of \( \sqrt[3]{181} \), is expressible in the form
\[ \left[ F^8 - \frac{3325 + 247 \sqrt[3]{181}}{2} F^7 - \frac{269 + 20 \sqrt[3]{181}}{2} F^6 + \frac{861 + 64 \sqrt[3]{181}}{2} F^5 - \right. \]
\[ = \frac{1987 + 1477 \sqrt[3]{181}}{2} \left[ F^4 - \frac{3 + 3 \sqrt[3]{181}}{2} F^3 - \frac{21 - \sqrt[3]{181}}{4} F^2 + \frac{11 + \sqrt[3]{181}}{4} \right]. \]

In terms of Berwick's cubic irrationality
\[ \theta^2 - \theta^2 + 2 \theta - 5 = 0 \quad (\Delta = -543) \]
the equation reduces to
\[ F^6 - D_i F^3 - \theta^2 = 0 \]
where
\[ 2 \alpha^2 = 3665 \theta^2 + 2345 \theta + 11775 + \frac{49309 \theta^2 + 31549 \theta + 1}{\sqrt[3]{181}} \]
\[ 2 D_i = 490 \theta^2 + 313 \theta + 1494 + \frac{6586 \theta^2 + 4213 \theta + 20084}{\sqrt[3]{181}}; \]
it is also found that
\[ 2 \beta = \theta^2 + \theta + 4 + \frac{21 \theta^2 + 15 \theta + 16}{\sqrt[3]{181}}. \]
It is to be remarked that
\[ 49309 \theta^2 + 31549 \theta + 1 = 50349 = \frac{(508 \theta^2 + 325 \theta + 1549) (21 \theta^2 + 15 \theta + 16)}, \]
\[ 7 (6586 \theta^2 + 4213 \theta + 20084) = \frac{(475 \theta^2 + 304 \theta + 1448) (21 \theta^2 + 15 \theta + 16)}, \]
\[ 21 \theta^2 + 15 \theta + 16 = \sqrt[3]{181} (6 \theta^2 + 3 \theta + 3). \]

\( n = 551. \quad (G. II. 13). \)

The equation satisfied by \( F_{551} \) is
\[ F^{16} - 10 F^{15} - 62 F^{14} - 237 F^{13} - 618 F^{12} - 1183 F^{11} - 1773 F^{10} \]
\[ - 2121 F^9 - 2049 F^8 - 1625 F^7 - 1011 F^6 - 416 F^5 - 25 F^4 \]
\[ + 148 F^3 + 111 F^2 + 95 F + 1193 F^5 + 469 F^4 + 504 F^3 \]
\[ - 437 F^2 - 316 F + 198 F^3 - 109 F^4 + 49 F^5 - 17 F^6 - 5 F - 1 = 0. \]
which, by adjunction of $\sqrt{29}$, reduces to

\[
F^{19} - (5 + \sqrt{29}) F^{13} - (29 + 6 \sqrt{29}) F^{11} - 179\sqrt{29} F^{10} + 33 = 0.
\]

\[
-\frac{1}{2} \left( 353 + 65 \sqrt{29} \right) F^9 - \left( 256 + 48 \sqrt{29} \right) F^8 - \frac{591}{4} + 109 \sqrt{29} F^7 + \frac{277}{2} + 51 \sqrt{29} F^6 - 155 + 29 \sqrt{29} F^5 - \left( 37 + 7 \sqrt{29} \right) F^4 - 27 + 5 \sqrt{29} F^3 - \frac{5}{2} + 1 \sqrt{29} F^2 = 0.
\]

The equations satisfied by $a$, $b$, $D$, $1, \text{and } S$ are

\[
a^{13} - 13 a^{12} + \cdots + 7 a^6 + 3 a^5 - 11 a^4 + 28 a^3 - 21 a^2 - 9 a + 9 a^4 - 14 a^3 - 3 a - 1 = 0,
\]

\[
b^{13} - 16 b^{12} - 57 b^{11} + 304 b^{10} + 1100 b^9 - 1692 b^8 - 9219 b^7 - 20671 b^6 + 28820 b^5 + 32129 b^4 - 20387 b^3 - 54511 b^2 - 31299 b - 6859 = 0,
\]

\[
D^{13} - 10 D^{12} + 49 D^{11} - 177 D^{10} + 483 D^9 - 463 D^8 - 1851 D^7 - 2259 D^6 - 2965 D^5 - 3499 D^4 - 3227 D^3 - 2417 D^2 - 1140 D - 247 = 0,
\]

\[
S_{10} - 67 S_{11} - 787 S_{12} + 3995 S_{13} - 3429 S_{14} + 667 S_{15} + 38 S = 0.
\]

The equations satisfied by $F_{19}$ is

\[
F^{19} - 20 F^{15} + 78 F^{14} - 161 F^{13} + 196 F^{12} - 139 F^{11} + 18 F^{10} - 83 F^9 - 123 F^8 + 98 F^7 + 51 F^6 + 14 F^5 - F^4 + 9 F^3 + 5 F^2 + 3 F + 1 = 0,
\]

which, by adjunction of $\sqrt{13}$, is expressible in the form

\[
\left[ F^4 - (5 + \sqrt{13}) F^3 + \frac{1 + 3 \sqrt{13}}{4} F^2 + \frac{9 + \sqrt{13}}{4} F - \frac{9 + 3 \sqrt{13}}{4} \right]^2.
\]

\[
= \frac{3 + 2 \sqrt{13}}{4} \left[ (\sqrt{13} + 1)^4 F^{13} - \sqrt{13} F^7 - 1 + \frac{1 + \sqrt{13}}{2} F^6 - \frac{3 + \sqrt{13}}{2} F^4 \right]^2.
\]

We also have

\[
\left[ a^2 - \frac{15 + 3 \sqrt{13}}{4} a - \frac{9 + 3 \sqrt{13}}{4} \right] = \frac{3 + 2 \sqrt{13}}{4} \left[ \frac{5 + \sqrt{13}}{2} a + \frac{3 + \sqrt{13}}{2} \right]^2.
\]

\[
\left[ b^2 - \frac{9 + 3 \sqrt{13}}{2} b - \frac{5 - \sqrt{13}}{4} \right] = \frac{3 + 2 \sqrt{13}}{4} \left[ \frac{1 + \sqrt{13}}{2} b + \frac{\sqrt{13} - 1}{2} \right]^2.
\]

\[
D_1^2 - (5 + \sqrt{13}) D_1 + \frac{8 + 3 \sqrt{13}}{2} = \frac{3 + 2 \sqrt{13}}{4} \left[ (1 + \sqrt{13}) D_1 - (2 + \sqrt{13}) \right]^2.
\]

\[
n = 567. \quad \text{(G. II. 6)}.
\]

The equation satisfied by $F_{19}$ is

\[
F^{19} - 4068 F^{15} - 3798 F^{13} - 2548 F^{11} + 4319 F^{10} + 7704 F^9 + 9324 F^8 + 1072 F^7 + 9324 F^6 + 7704 F^5 + 1547 F^4 + 297 F^3 + 27 F^2 + 1 = 0,
\]

which, by adjunction of $\sqrt{21}$, is expressible in the form

\[
\left[ 2 F^9 - (2034 + 444 \sqrt{21}) F^8 + (2034 + 444 \sqrt{21}) F^7 - (2030 + 443 \sqrt{21}) F^6 \right] = (27 + 6 \sqrt{21}) \left[ (276 + 60 \sqrt{21}) F^4 + (276 + 60 \sqrt{21}) F^3 + (275 + 60 \sqrt{21}) F^2 \right].
\]

In terms of Berwick's cubic irrationality

\[
0^3 - 3 = 0,
\]

the equation reduces to

\[
F^4 - S_a F^3 + a^2 = 0,
\]

where

\[
2 a^2 = 435 \delta^2 + 129 b - 298 + 1993 \delta + 555 b + 1824, \quad 2 \delta = 5 a^2 - 92 b + 295 + 1483 \delta + 414 b + 1353.
\]

it is also found that

\[
2 \delta = 5 a^2 - 92 b + 295 + 1483 \delta + 414 b + 1353.
\]

It is to be remarked that

\[
1993 \delta + 555 b + 1824 = (34 \delta^2 + 9 b + 31)(4 \delta^2 + 3 b + 6).
\]
1483 \theta^2 + 414 \theta + 1333 = (25 \theta^2 + 8 \theta + 22)(4 \theta^2 + 3 \theta + 6),
4 \theta^2 + 3 \theta + 6 = 21(13 \theta^2 + 4 \theta + 12).

n = 575. (G. II. 9).

The equation satisfied by \( F_{135} \) is
\[
F^{10} - 20 F^{11} - 64 F^{15} - 64 F^{19} + \cdots + 25 F^{12} - 60 F^{11} + \cdots - \cdots - F^{3} - 15 F^{7} + 6 F^{9} + 4 F^{13} + \cdots + F^{3} + 5 F^{3} + 4 F + 1 = 0,
\]
which, by adjunction of \( \sqrt{5} \), reduces to
\[
F^{3} - (10 + 2 \sqrt{5}) F^{3} - (8 + 9 \sqrt{5}) F^{7} - (224 + 4 \sqrt{5}) F^{11} - \frac{19 + 15 \sqrt{5}}{2} F^{15} - \frac{1}{2} F^{19} - \frac{1}{2} F^{23} = 0.
\]

It is convenient to use the cubic irrationality \( \sqrt{\frac{3}{2}} \), namely
\( \theta^3 - \theta^2 - 5 \theta - 1 = 0 \quad (\Delta = -23) \)
rather than Berwick's cubic irrationality
\( \tilde{\varphi} = \varphi^3 - 4 \varphi^2 - 5 \varphi = 0, \quad (\Delta = -575) \)
these irrationalities being connected by the relation
\[
\varphi = \frac{\theta^3 - 3}{2\theta + 1}.
\]
The equation is then reducible to
\[
2 F^{3} - 4(\theta^2 + 5 \theta + 1) F^{2} - (6 \theta^2 + 29 \theta + 2) F - (3 \theta^2 + 3 \theta + 1) = 0,
\]
\[
= (\theta^2 + \theta) (2 F^2 - (\theta^2 + 3 \theta + 1) F - 1) \sqrt{5}.
\]
The equations giving \( a, \beta \) and \( S_i \) in terms of \( \theta \) are
\[
a^3 - (2 \theta^2 + 5 \theta + 4) a^2 - (3 \theta + 2) a - (\theta^2 + 29 \theta + 1) = 0,
\]
\[
\beta^3 - (5 \theta^2 + 5 \theta + 1) \beta^2 - (10 \theta^2 + 29 \theta + 13) \beta - (10 \theta^2 + 15 \theta + 8) = 0,
\]
\[
S_i^3 - 4(\theta^2 + 5 \theta + 1) S_i^2 - (2 \theta^2 + 5 \theta + 5) S_i - (2 \theta^2 + 29 \theta + 1) = 0.
\]

\( n = 583. \) (G. II. 4).

The equation satisfied by \( F_{583} \) is
\[
F^{3} - 16 F^{5} - 12 F^{7} - 11 F^{9} + 12 F^{15} + 5 F^{19} - 3 F^{13} - 4 F - 1 = 0,
\]
which, by adjunction of \( \sqrt{53} \), is expressible in the form
\[
F^{3} - \frac{8 + \sqrt{53}}{2} (F + 1)^2 = \frac{3 + \sqrt{53}}{2} (F + 1)^2.
\]
Here we have
\[
\alpha = \frac{8 + \sqrt{53}}{2} + \sqrt{131 + 18 \sqrt{53}} / 4
\]
so that
\[
\alpha^3 - 16 \alpha^2 + 4 \alpha^2 - 3 \alpha - 1 = 0;
\]
also
\[
\beta = \alpha + 2, \quad D_i = \alpha.
\]

\( n = 591. \) (G. II. 11).

The equation satisfied by \( F_{11} \) is
\[
F^{4} - 4945 F^{6} + 20955 F^{8} + 1 39902 F^{10} = 3 26851 F^{12} + 1 95297 F^{14} - 98265 F^{16} - 1 17126 F^{18} - 3 08141 F^{20} - 4 7480 F^{22} - 2 49560 F^{24} - 6 6786 F^{26} - 1 60633 F^{28} - 2 7245 F^{30} - 4 4701 F^{32} - 8 4929 F^{34} - 1 2199 F^{36} - 2 000 F^{38} - 1 851 F^{40} - 6 61 F^{42} + 18 F^{44} = 0.
\]
which, by adjunction of \( \sqrt{197} \), reduces to
\[
F^{12} - 4945 + 335 \sqrt{197} / 2 \cdot F^{10} + (1391 + 101 \sqrt{197}) F^{12} - (2622 + 190 \sqrt{197}) F^{14} - 5 135 + 367 \sqrt{197} / 2 \cdot F^{16} - 6 673 + 475 \sqrt{197} / 2 \cdot F^{18} - 4 229 + 301 \sqrt{197} / 2 \cdot F^{20} - (1768 + 127 \sqrt{197}) F^{22} - 1 631 + 115 \sqrt{197} / 2 \cdot F^{24} - 705 + 51 \sqrt{197} / 2 \cdot F^{26} - (71 + 5 \sqrt{197}) F^{28} - (14 + \sqrt{197}) F^{30} = 0.
\]
22. Ata Arithmetica, 1, 2.
The equations satisfied by \( a, \gamma, D_k \) and \( S_n \) are

\[
\begin{align*}
\gamma^{11} & = 35 a^{10} + 7 a^9 - 11 a^8 - 19 a^7 - 26 a^6 - 35 a^5 - 37 a^4 - 38 a^3 - 25 a^2 \\
& + 8 a - 1 = 0,
\gamma^{11} & = 572 \gamma^{10} - 5098 \gamma^9 - 21002 \gamma^8 - 92050 \gamma^7 - 60689 \gamma^6 - 288165 \gamma^5 \\
& - 836165 \gamma^4 - 1516875 \gamma^3 - 17408952 \gamma^2 - 11273013 \gamma \\
& - 3727863 = 0,
D_k^{11} & = 4945 D_k^{10} + 21216 D_k^9 - 80787 D_k^8 + 7341 D_k^7 - 68745 D_k^6 - 19866 D_k^5 \\
& - 9896 D_k^4 - 20092 D_k^3 - 23460 D_k^2 - 22419 = 0,
S_n^{11} & = 15654 S_n^{10} + 601 S_n^9 - 9373 S_n^8 - 8196 S_n^7 - 4806 S_n^6 \\
& = (353 S_n^{10} + 949 S_n^8 + 771 S_n^5 + 292 S_n^4 + 648 S_n^3 + 81) \gamma^{197}.
\end{align*}
\]

I have not simplified these equations in any way.

\( n = 599. \) (G. I. 25).

The equation satisfied by \( F_{100} \) is

\[
\begin{align*}
F^{19} & = 16 F^{18} - 25 F^{17} + 3 F^{16} + 82 F^{15} - 92 F^{14} - 63 F^{13} \\
& - 28 F^{12} + 66 F^{11} + 24 F^{10} - 35 F^{9} + 7 F^{8} + 1 F^{7} \\
& + 6 F^{6} - 58 F^{5} - 22 F^{4} + 36 F^{3} + 13 F^{2} - 32 F^{1} \\
& + 5 F + 17 F^{3} - 6 F^{2} - 8 F^{1} + 3 F^{2} + 2 F - 1 = 0.
\end{align*}
\]

I have not solved this equation.

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