Universal Waring theorems with cubic summands.

By

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1. Introduction. We shall obtain systematically[1]) 116 cubic polynomials $f(x)$ with rational coefficients such that $f(x)$ has an integral value $\geqq 0$ for every integer $x \geqq 0$ and such that every positive integer is proved to be a sum of nine values of $f(x)$ for integers $x \geqq 0$. The proof avoids the use of other papers. For several of the f, we obtain facts which indicate that it is highly probable that (instead of 9) 5 or 4 values suffice.

The triangular and pyramidal numbers are

$$T(x) = \frac{1}{2}(x^2 - x), \qquad P(x) = \frac{1}{6}(x^3 - x).$$

THEOREM 1. *The following functions $F(y)$ are integers $\geqq 0$ for all integers $y \geqq k$, while every integer $\geqq 0$ is a sum of nine values of $F(y)$ for integers $y \geqq k$:*

$P(y)$, $P+1$, $P+y$ for $k=0$; $P+y+1$, $k=-1$;

$P-y+1$, $k=0, -1, -2, -3$;

$P-2y+3$, $k=-2, -3, -4$; $P-4y+8$, $k=-2, -3, -4, -5$;

$P-5y+11$, $-6 \leqq k \leqq 0$; $P-7y+18$, $-7 \leqq k \leqq 0$;

[1]) The general theory applies to many further $f(x)$, for which it is improbable that 4 or 5 summands suffice.

$P-9y+26$, $-8 \leqq k \leqq 2$; $P-11y+35$, $-9 \leqq k \leqq 1$;

$P-14y+50$, $-10 \leqq k \leqq 2$: $P-16y+61$, $-11 \leqq k \leqq 4$;

$P-20y+85$, $-12 \leqq k \leqq 3$; $P-27y+133$, $-14 \leqq k \leqq 4$.

Such a theorem concerning $F(y)$ for integers $y \geqq k$ is equivalent to the like theorem concerning $F(x+k)$ for integers $x \geqq 0$. For example, if $F=P-9y+26$, $k=-3$, then $F(x-3)$ is

$$G(x) = P(x) - 3T(x) - 6x + 49. \tag{1}$$

It is shown that every integer[2]) from 0 to 30,000 inclusive is a sum of four values of $G(x)$ for integers $x \geqq 0$. Then in Lemma 3 we have $m=247$ and conclude that every integer from 0 to 2,478,752 is a sum of five such values. Both facts[3]) evidently hold also for $G(x-t)$ when $t=1, 2, 3, 4$ or 5, since $G(-t)>0$, $G(-6)=-13$.

When $F=P-7y+18$, $k=-4$, $F(x-4)$ is

$$H(x) = P(x) - 4T(x) - x + 36. \tag{2}$$

It is shown that every integer from 0 to 20,000 inclusive is a sum o four values of $H(x)$ for integers $x \geqq 0$. In Lemma 3 we have $m=199$ and conclude that every integer $\leqq 1{,}351{,}900$ is a sum of five such values. Both facts hold also for $H(x-t)$ with $t=1, 2$ or 3, since $H(-t)>0$, $H(-4)=-10$,

When $F=P-11y+35$, $k=-3$, $F(x-3)$ is

$$J(x) = P(x) - 3T(x) - 8x + 64.$$

Every integer $\leqq 25{,}000$ is a sum of four values of $J(x)$ for integers $x \geqq 0$. Thus every integer $\leqq 1{,}895{,}771$ is a sum of five such values by Lemma 3 with $m=226$. Both facts hold also for $J(x-t)$ with $t=1, \dots, 6$, since $J(-t)>0$.

Four summands suffice to 6000 for $P-16y+61$, $y \geqq -6$ (§ 7).

2. Sums of nine values of $f(x) = P(x) + gx$.

[2]) Since we used 59 values of $G(x)$ our result is to be compared with a Waring problem on cubes to $59^3 = 205{,}379$.

[3]) Their extensions to a larger range are more likely to hold than the facts for $G(x)$ since we now have available new summands.

LEMMA 1. *Given the positive integers n and s, and any integer h, we can find an integer m such that*

$$s \equiv f(3m) \pmod{3^n}, \quad h \leqq m < 3^n + h.$$

By induction on n, we see that $f(x+3r) - f(x)$ is not divisible by 3^n if r is not. Let j and k be any two distinct ones of the integers

$$h,\ h+1,\ \ldots,\ h+3^n-1. \tag{4}$$

Then $r=j-k$ is not divisible by 3^n. Also take $x=3k$. Then

$$f(3j)-f(3k)=f(x+3r)-f(x) \not\equiv 0 \pmod{3^n}.$$

Hence when m ranges over the 3^n integers (4), the values of $f(3m)$ are incongruent modulo 3^n, whence s is congruent to one of those values. A simple computation yields

LEMMA 2. *If* $0 \leqq h \leqq 234$, $g < 15{,}773$, $m < 3^n + h$, $n \geqq 8$, *then* $f(3m) < 5 . 3^{3n}$.

If s and C are given positive numbers, we can evidently choose a positive integer n so that

$$C . 27^n \leqq s \leqq C . 27^{n+1}.$$

Then s is one of the integers s_i of the three sub-intervals

$$3^{i-1} C\, 3^{3n} \leqq s_i < 3^i C\, 3^{3n} \qquad (i=1,\,2,\,3). \tag{5}$$

By Lemma 1 we can choose an integer m_i so that

$$s_i = f(3m_i) + 3^n M_i, \quad h \leqq m_i < 3^n + h, \tag{6}$$

where M_i is an integer. Let $f(3m_i) \geqq 0$. Using also Lemma 2, we get

$$(3^{i-1} C - 5)\, 3^{2n} < M_i < 3^i C\, 3^{2n}.$$

Write $M_i = 3^{2n} + N_i$. Then

$$(3^{i-1} C - 6)\, 3^{2n} < N_i < (3^i C - 1)\, 3^{2n}. \tag{7}$$

Henceforth employ summands $f(x)$, $x \geqq t$ where $t = 3h$:

$$0 \leqq t \leqq 702, \quad -3^{13} \leqq g < 15773, \quad n \geqq 8. \tag{8}$$

Then $b_1=5$, $b_2=7$, $b_3=11$, $C=168$ satisfy the inequalities

$$\frac{9}{8} b_i^3 + \left(1 - \frac{t}{3^n}\right)^2 + 6 + \frac{S_i}{3^{2n}} \leqq 3^{i-1} C \leqq \frac{3}{8} b_i^3 + \frac{b_i}{2}\left(\frac{3}{2} b_i - \frac{t}{3^n}\right)^2 + \frac{1}{3} + \frac{S_i}{3^{2n+1}} \tag{9}$$

for $i=1,\,2,\,3$, where $S_i = \left(1 + \frac{1}{2} b_i\right)(6g-1)$. Then (7) imply

$$l_i \leqq N_i \leqq L_i, \quad l_i = \frac{9}{8} b_i^3\, 3^{2n} + (3^n - t)^2 + S_i, \tag{10}$$

$$L_i = \frac{9}{8} b_i^3\, 3^{2n} + \frac{3}{2} b_i \left(\frac{3}{2} b_i 3^n - t\right)^2 + S_i.$$

Write

$$A_i = 6\left[\frac{N_i + 1 - 6g}{3 b_i} - g\right] - \frac{9}{4} b_i^2\, 3^{2n} + 1, \quad G_i = A_i - \frac{2}{b_i}(3^n - t)^2. \tag{11}$$

These with (10) imply

$$G_i \geqq 0 \ (\text{whence } A_i \geqq 0), \quad \sqrt{\frac{1}{3} A_i} \leqq \frac{3}{2} b_i 3^n - t. \tag{12}$$

For any number v_i in the interval

$$\frac{3}{2} b_i 3^n + \sqrt{\frac{1}{3} G_i} \leqq v_i \leqq \frac{3}{2} b_i 3^n + \sqrt{\frac{1}{3} A_i}, \tag{13}$$

the final inequality (12) and the first one (13) give

$$t < v_i \leqq 3 b_i 3^n - t. \tag{14}$$

Employ the abbreviation

$$V_i = v_i - \frac{3}{2} b_i 3^n.$$

Thus (13) give

$$V_i \geqq \sqrt{\frac{1}{3} G_i}, \quad \sqrt{\frac{1}{3} A_i} \geqq V_i. \tag{15}$$

These imply

$$0 \leqq N_i + 1 - 6g - 3b_i\left[g + \frac{1}{6}\left\{3V_i^2 + \frac{9}{4}b_i^2\, 3^{2n} - 1\right\}\right] \leqq (3^n - t)^2.$$

Write

(16) $$B_i = 3b_i\left\{g + \frac{1}{6}\left[9b_i^2\, 3^{2n} - 1 - 3v_i(3b_i\, 3^n - v_i)\right]\right\}.$$

Then the last inequalities give

(17) $$0 \leqq N_i + 1 - 6g - B_i \leqq (3^n - t)^2.$$

Write

(18) $$w_i = 3b_i\, 3^n - v_i, \quad R_i = f(v_i) + f(w_i).$$

Hence $R_i = 3^n B_i$. The identity

$$\sum_{j=1}^{3}\left\{f(3^n - x_j) + f(3^n + x_j)\right\} = 3^{3n} + 3^n(Q_i - 1 + 6g), \quad Q_i = x_1^2 + x_2^2 + x_3^2,$$

and (6) show that $s_i = f(3m_i) + 3^n(3^{2n} + N_i)$ will be the sum of the values of $f(x)$ for the nine values

(19) $$3m_i,\ v_i,\ w_i,\ 3^n - x_j,\ 3^n + x_j \quad (j = 1, 2, 3)$$

of x provided only

(20) $$Q_i = N_i + 1 - 6g - B_i$$

is a sum of three squares x_j^2. In that case, (17) gives $3^n - x_j \geqq t$. By (14) and (18), both v_i and w_i are $\geqq t$. By Lemma 1, $3m_i \geqq t$ since $t = 3h$. Thus the nine arguments (19) are all $\geqq t$.

It remains only to prove that we can choose an integer v_i so that Q_i will be a sum of three integral squares.

Consider the difference D_i between the limits in (13):

(21) $$D_i = \sqrt{\frac{1}{3}A_i} - \sqrt{\frac{1}{3}G_i}, \quad p_i = \frac{2(3^n - t)^2}{b_i A_i}.$$

By (11_2) and $G_i \geqq 0$

$$\frac{G_i}{A_i} = 1 - p_i, \quad 0 < p_i \leqq 1.$$

Thus D_i is the product of $\sqrt{\frac{1}{3}A_i}$ by

$$1 - \sqrt{1 - p_i} = \frac{p_i}{1 + \sqrt{1 - p_i}} > \frac{p_i}{2},$$

whence

(22) $$D_i > \frac{(3^n - t)^2}{b_i\sqrt{3A_i}}.$$

By (7) for $C = 168$ and (11),

(23) $$3A_i < 18\left\{\frac{(168 \cdot 3^i - 1)\, 3^{2n} + 1 - 6g}{3b_i} - g\right\} - \frac{27}{4}b_i^2\, 3^{2n} + 3.$$

We readily find that each $D_i > 8$. Hence (13) holds for at least eight consecutive integers v_i. But

$$2B_i - 6b_i g = b_i F,$$

where F denotes the quantity in square brackets in (16). It involves the function $v(k - v)$, where $k = 3b_i\, 3^n$ is odd. Evidently $v(k - v)$ can be made congruent to any assigned even integer modulo 8 by choice of v. Hence in (20) we can choose v_i (mod 8) so that $2Q_i \equiv 2z$ (mod 8), where z is an arbitrary integer. Take $z = 1$. Then $Q_i \equiv 1$ (mod 4). But $Q_i \geqq 0$ by (17). Hence Q_i is a sum of three integral squares. This proves[4])

THEOREM 2. *Every integer* $\geqq 168 . 3^{24}$ *is a sum of nine values of* $f(x) = gx + \frac{1}{6}(x^3 - x)$ *for integral values* $\geqq t$ *of* x, *if* $0 \leqq t \leqq 702$, $-3^{13} \leqq g < 15773$, *and if* $f(x) \geqq 0$ *for every integer* $x \geqq t$.

3. LEMMA 3. *Let a polynomial* $f(x)$ *take an integral value* $\geqq 0$ *for every integer* $x \geqq t$, *where the given integer* t *may be negative. Make the hypothesis* (H) *that every integer* l *for which* $l < l \leqq g$ *is a sum of* $k - 1$ *values of* $f(x)$ *for integers* $x \geqq t$. *Let*

(24) $$f(j+1) - f(j) < g - l \qquad (j = t, \ldots, m),$$

[4]) When $t = 0$, I had proved that every integer $\geqq 171 . 3^{24}$ is a sum of nine values if $2g \leqq 3^{15}$; also a like theorem for $gx + AP(x)$. Trans. Amer. Math. Soc. vol. 36 (1934), p. 740; cf., pp. 1 — 12, 493 — 510.

where the integer m exceeds t. Then every integer which exceeds $l+f(t)$ and is $\leqq g+f(m+1)$ is a sum of k values of $f(x)$ for integers $x\geqq t$.

For a fixed j consider an integer I for which

$$g+f(j)<I\leqq g+f(j+1). \tag{25}$$

Write $i=I-f(j+1)$. By (24) and (25), $g\geqq i>g+f(j)-f(j+1)>l$. By (H), i is therefore a sum of $k-1$ values of $f(x)$, whence I is a sum of k values. Apply the latter result for $j=t, \ldots, j=m$ in turn, and note that each interval (25) ends just where the next begins. Hence every integer which exceeds $g+f(t)$ and is $\leqq g+f(m+1)$ is a sum of k values of $f(x)$. By (H), those from l to g are sums of $k-1$ values; employ the further value $f(t)$; hence all from $l+f(t)$ to $g+f(t)$ are sums of k values. The two conclusions together yield the lemma.

4. Proof of Theorem 1. For each function $F=P(y)-ry+s$ in Theorem 1, we have $-1\leqq r\leqq 27$, $0\leqq s\leqq 133$. We shall verify later that all integers from 0 to 2000 inclusive are sums of five values of $F(y)$ for integers $y\geqq t$, where $-2\leqq t\leqq 4$. Let a function F have the latter property when

$$-63\leqq t\leqq 21,\quad -15\leqq r\leqq 27,\quad 0\leqq s\leqq 133. \tag{26}$$

Apply Lemma 3 with $l=0$, $g=2000$, $k=6$. Since

$$F(j+1)-F(j)=\frac{1}{2}(j^2+j)-r,$$

condition (24) is equivalent to

$$(2j+1)^2<16001+8r$$

and holds if $-63\leqq j\leqq 62$. Hence for any t in (26), (24) holds if $m=62$ Then

$$g_1=g+F(63)=43664-63r+s,\quad F(t)<2000.$$

Hence Lemma 3 shows that every integer $\leqq g_1$ is a sum of 6 values of $F(y)$ for integers $y\geqq t$.

Apply Lemma 3 with $l=0$, $g=g_1$, $k=7$. Now (24) is

$$(2j+1)^2<349313-496r+8s.$$

For any r and s in (26), this holds if $(2j+1)^2\leqq(579)^2$. Thus for any t in (26), (24) holds if $m=289$. Then

$$g_2=g_1+F(290)=4108449-353r+2s,$$

and every integer $\leqq g_2$ is a sum of 7 values of $F(y)$ for integers $y\geqq t$. The next m is 2862, and

$$g_3=g_2+F(2863)=3915331000-3216r+3s.$$

All integers $\leqq g_3$ are sums of 8 values. Then $m=88488$, and all integers $\leqq 11{,}548{,}650\times 10^7$ are sums of 9 values. This number exceeds

$$168\times 3^{24}=4{,}744{,}816\times 10^7.$$

If N is a sum of 9 values of $f(y)$ then $N+9s$ is a sum of 9 values of $f(y)+s$. Theorem 2 implies a like result when t is negative. We have now proved

THEOREM 3. *Let all integers from 0 to 2000 inclusive be sums of five values of $F=\frac{1}{6}(y^3-y)-ry+s$ for integers $y\geqq t$, where r, s, t satisfy inequalities (26), and $F\geqq 0$ for every integer $y\geqq t$. Then every integer $\geqq 0$ is a sum of nine values of F for integers $y\geqq t$.*

This implies Theorem 1.

5. Conditions for a universal Waring theorem. Any cubic function with rational coefficients may evidently be written in the form

$$F(x)=AP(x)+BT(x)+Cx+D,\quad A\neq 0, \tag{27}$$

where $A, \ldots, D$ are rational numbers. We assume

$$F(x) \text{ is an integer } \geqq 0 \text{ for every integer } x\geqq 0. \tag{28}$$

The fact that $A, \ldots, D$ are integers follows from

$$F(0)=D,\quad F(1)=C+D,\quad F(2)=A+B+2C+D,$$
$$F(3)=4A+3B+3C+D.$$

Then (27) is an integer for every integer x. Also, $A>0$ by (28) with $x=\infty$. We desire that

$$\text{every integer } \geqq 0 \text{ shall be a sum of } v \text{ values of } F(x), \tag{29}$$

where $v\leqq 9$. The smaller A is, the more slowly will $F(x)$ increase with x, and the smaller v will be in general. Hence we shall take $A=1$.

By (28) and (29), $F(h)=0$ for some integer $h\geqq 0$. Let the transformation $y=x+h$ replace $F(y)$ by $f(x)$. Then $f(0)=F(h)=0$. Hence Waring's problem for $F(y)$ reduces to that for

(30) $$f(x)=P(x)+b\,T(x)+cx,\quad x\geqq -h.$$

The maximum h will be found tentatively in each case, as for (1)—(3). By (29), $f(z)=1$ for some integer z. Since all terms of $6f(z)$ are products of z by integers, z must divide 6, whence $z=\pm 1, \pm 2, \pm 3, \pm 6$. The cases $z=6$ and $z=-3$ are excluded since

$$f(6)=35+15b+6c=1,\quad f(-3)=-4+6b-3c=1$$

are impossible in integers, in fact, modulo 3.

6. **Case** $z=1$. Thus $c=1=f(1)$. If $b<0$, $f(3)=7+3b\geqq 0$ requires $b=-1$ or -2. Postponing to Section 12 less interesting special cases, let $b\geqq 2$. When $x=-3b-1$, $f(x)=x$. Also, $f(-3b)=$ $=\frac{1}{2}b(3b-5)>0$. Besides the root 0 und the root between $-3b-1$ and $-3b$ of $f(x)=0$, there is a root between 0 and 1 if $b\geqq 3$, but a root between $-\frac{1}{2}$ and 0 if $b=2$. Hence $f(x)\geqq 0$ for every integer $\geqq -3b$. If $b\geqq 3$, the least integral values of $f(x)$ are 0, 1, $b-1=$ $=f(-1)$. Thus $b-2$ summands 1 are required to produce the number $b-2$, and hence at least six summands are needed when $b\geqq 8$. We exclude this case.

To (30) apply the transformation $x=y-b$; we get

(31) $$F(y)=P(y)+\left\{1-\frac{1}{2}(b+b^2)\right\}y+f(-b).$$

Thus if $b\geqq 2$, $F(y)\geqq 0$ for every integer $y\geqq -2b$.

The most interesting case has $b=4$. Then

(32) $$F(y)=P(y)-9y+26.$$

Its values for $y=-9, -8, \ldots, 7$ are -13, 14, 33, 45, 51, 52, 49, 43, 35, 26, 17, 9, 3, 0, 1, 7, 19. Hence we have a universal Waring problem $F(x+h)$, for integers $x\geqq 0$, when $-8\leqq h\leqq 4$. We discard $h=4$, since 6 is not a sum of fewer than six values of $F(x+4)$. Also $h=3$, since 100 is not a sum of five values of $F(x+3)$, but all others $\leqq 506$ are sums of five.

When $h=2$, the only integers <506 which are not sums of four values of $F(x+2)$ for integers $x\geqq 0$ are

62, 89, 97, 99, 135, 181, 183, 190, 236, 263, 265, 328, 336, 391, 433, 437, 443, 445, 500.

We readily conclude that all integers $\leqq 2906$ are sums of five such values.

The least positive integer not a sum of four values of $F(x+h)$ for integers $x\geqq 0$ is 97 if $h=1$, 336 if $h=0$, 539 if $h=-1$, 7243 if $h=-2$.

By use of a new table of sums of three values of $F(x-3)$ for integers $x\geqq 0$ covering $0-3500$, $15000-18000$, it was verified that every positive integer $\leqq 30000$ is a sum of four such values. Note that $F(x-3)$ is the function (1) discussed in Section 1.

7. **Case** $z=-1$. Thus $b=c+1$ in (30). Also, $f(1)=c\geqq 0$. When $x=3c+2$, $f(-x)=x$; also

$$f(-3c-3)=\frac{1}{2}(c+1)(4-3c),\quad f(2)=2+3c,$$
$$f(-1)=1,\quad f(-2)=2+c.$$

Hence if $c\geqq 2$, $f(x)$ is $\geqq 0$ for every integer $x\geqq -3c-2$ and its least values are 0, 1, c. Thus $c-1$ is a sum of $c-1$, but not fewer values.

To (30) apply the transformation $x=y-c-1$; we get

(33) $$F(y)=P(y)-\frac{1}{2}(c^2+c+2)y+f(-c-1).$$

We saw that if $c\geqq 2$, $F(y)$ is $\geqq 0$ for every integer $y\geqq -2c-1$, but is negative if $y=-2c-2$.

First, let $c=3$. Then $F(y)=P(y)-7y+18$, $y\geqq -7$. The least positive integer L which is not a sum of four values of $F(v)$ is

$y\geqq$	3	2 or 1	0 or -1	-2	-3
L	19	43	203	2831	3437

while every integer $\leqq 20000$ is a sum of four values of $F(y)$ for integers $y\geqq -4$. Note that $F(x-4)$ is function (2). All integers $\leqq 15883$ are sums of five values of $F(y)$, $y\geqq 0$.

Second, let $c=4$. Then $F=P-11y+35$, $y\geqq -9$. Now the least integer not a sum of four values is 11 if $y\geqq 4$, 54 if $y\geqq 3$ or 2, and 363 if $y\geqq 1$, 0, -1 or -2. But every integer $\leqq 25000$ is a sum of four values of $F(y)$ for integers $y \geqq -3$. Since all < 363 are sums of four values of F for integers $y \geqq 1$, all $\leqq 3377$ are sums of five such values by Lemma 3.

Third, let $c=2$. Then $F=P-4y+8$, $y \geqq -5$. All integers $\leqq 200$ except 90,163, and 167 are sums of four values with $y \geqq -1$. All $\leqq 2000$ except only 562, 710, 881, 1869, and 1893 are sums of four values with $y \geqq -2$. All but 1869 of these five exceptions become sums of four values with $y \geqq -4$. Since $F(-5)=8=F(0)$, 1869 is not a sum of four values with $y \geqq -5$.

Fourth, let $c=1$. Then $F=P-2y+3$, $y \geqq -4$. For $y \geqq 2$ (or $y \geqq 1$), 22 is not a sum of five values. The only useful case is $y \geqq -2$. Then all $\leqq 543$ are sums of four values except 191, 331, 334. It follows readily that all $\leqq 4335$ are sums of five.

Fifth, let $c=0$. Then $F=P-y+1$, and

$$F(-4)=-5,\ F(-3)=0=F(1),\ F(-2)=F(-1)=2=F(3).$$

Hence we may take $y \geqq 0$. The integers $\leqq 609$, except twenty seven, are sums of four values. From them we find that $0-4718$ are all sums of five values.

Sixth, let $c=5$. Then $F=P-16y+61$, $y \geqq -11$. If $y \geqq 5$, 14 requires six summands. The least integer not a sum of four values is 33 if $y \geqq 4$ (or $y \geqq 3$), 63 if $y \geqq 2$, 175 if $y \geqq 1$ or $y \geqq 0$, 955 if $y \geqq -1$ or $y \geqq -2$ or $y \geqq -3$, 2221 if $y \geqq -4$ or $y \geqq -5$. But all $\leqq 6000$ are sums of four values of $F(y)$ for $y \geqq -6$. We have not yet used the available summands

$$F(-7)=117,\ F(-8)=105=F(11),\ F(-9)=85,$$

$$F(-10)=56,\ F(-11)=17=F(3).$$

All integers $\leqq 3515$ are sums of five values of F for $y \geqq +4$.

8. Case $z=2$. Thus $b+2c=0$, $f(1)=c \geqq 0$. If $c=0$, then $f(x)=P(x)$, $f(-2)=-1$, $f(-1)=0=f(0)$, and we may take $x \geqq 0$. While 17 is not a sum of four values of $P(x)$, every positive integer $N \leqq 7000$ is a sum of five pyramidal numbers [5].

Next, let $c \geqq 1$. Then $f(3)=4-3c \geqq 0$ only when $c=1$. Then $b=-2$. Take $x=y+2$. Then $f(x)$ becomes $1+P(y)$. By the result quoted, $N+5$ is a sum of five values of $1+P(y)$ for $y \geqq 0$ and hence of five values of $f(x)$ for $x \geqq 2$. Hence for $0 \leqq M \leqq 7005$, M is a sum of five values of $f(x)$ for $x \geqq 0$. But 56 is not a sum of four values of $f(x)$.

[5]) K. C. Yang, Chicago Dissertation, 1928.

9. Case $z=3$. Thus $b=-1-c$. By $f(4)=4-2c \geqq 0$, $c=0,1$, or 2. If $c=0$, $f(x)=\frac{1}{6}x(x-1)(x-2)$ is pyramidal. If $c=1$, then $b=-2$ (end of § 8). If $c=2$, $b=-3$; taking $x=y+3$, we get $P-y+1$ (case $c=0$ of § 7).

10. Case $z=-6$. Thus $1=21b-6c-35$, $b=2B$, $c=7B-6$, whence $B \geqq 1$ since $f(1)=c \geqq 0$. But $f(-5)=10-5B \geqq 0$, whence $B \leqq 2$. By $f(-4)=14-8B \geqq 0$, $B \neq 2$. Hence $B=1$, $b=2$, $c=1$ (duplicate of fourth case $c=1$ in § 7).

11. Case $z=-2$. Thus $1=3b-2c-1$, $b=2B$, $c=3B-1$, $B \geqq 1$. By $f(-1)=1-B \geqq 0$, $B=1$, $b=2$, $c=2$. For $x=y-2$, $f(x)$ becomes $P-y+1$ (case $c=0$ of § 7).

12. Case $z=1$ **concluded.** If $b=0$, $f=P(x)+x$. Since $f(-1)=-1$, $x \geqq 0$. Except only 37, 115, 122, 166, 334, 372, 541, every positive integer $\leqq 2030$ is a sum of four values of f. Then by Lemma 3 all integers between 541 and $A=28236$ are sums of five values. Employ

$$B=f(55)=27775,\quad C=f(54)=26289,\quad D=f(22)=1793.$$

Then $B+541=C+D+234$ is a sum of five, since 234 is a sum of three, values. Hence by adding B to $461-2030$, we conclude that all integers from A to 29805 are sums of five values. Similarly, by adding in turn $f(56), \ldots, f(64)$, we see that all $\leqq 45774$ are sums of five.

When $b=-1$, take $x=y+1$; we get $G=P+y+1$. Let t range over the former exceptions $37, \ldots, 541$. Thus all integers from 4 to 2034 except the seven $4+t$ are sums of four values of G for integers $y \geqq 0$. But

$$41=G(4)+G(5),\quad 119=G(5)+G(8),\quad 126=3G(6),$$

$$170=G(6)+2G(7),\quad 338=G(6)+G(7)+G(11).$$

Since $G(-1)=0$, all integers $\leqq 2034$ except only 376 and 545 are sums of four values of $G(y)$ for integers $y \geqq -1$. Evidently all $\leqq 45779$ are sums of five such values.

If $b=1$, $f(x)$ is the pyramidal number $P(x+1)$.

If $b=2$, we have the fourth case $c=1$ of Section 7.

If $b=-2$, we have the second case of Section 8.

Let $b=3$. By (31), $F=P-5y+11$, $y\geqq-6$. For $y\geqq 3$, 31 is not a sum of five values. The least positive integer not a sum of four is 27 if $y\geqq 2$ or $y\geqq 1$, 53 if $y\geqq 0$ or $y\geqq -1$, 696 if $y\geqq -2$, 1631 if $y\geqq -3$, 1652 if $y\geqq -4$ or $y\geqq -5$ or $y\geqq -6$. For $y\geqq 0$, 53, 85, 217, 351, 391, 472 are the only integers $\leqq 501$ which are not sums of four values of F. We readily conclude that all $\leqq 2700$ are sums of five values.

Let $b=5$. By (31), $F=P-14y+50$, $y\geqq -10$. The least integer not a sum of five values of F is 37 if $y\geqq 4$, and 63 if $y\geqq 3$. Also 19 is not a sum of four values with $y\geqq -10$. Using the twenty-four integers $\leqq 500$ which are not sums of four values of F for $y\geqq 2$, we find that all $\leqq 3000$ are sums of five.

Let $b=6$. Then $F=P-20y+85$, $y\geqq -12$. Then 13 is not a sum of four values. For $y\geqq 4$, 122 is not a sum of five. All integers $\leqq 3775$ are sums of five values of F for $y\geqq 3$.

Finally, let $b=7$. Then $F=P-27y+133$, $y\geqq -14$. Then 5 is not a sum of four values. For $y\geqq 5$, 43 is not a sum of five. Every integer $\leqq 10000$ is a sum of five values of F for $y\geqq 4$.

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On the arithmetical density of the sum of two sequences one of which forms a basis for the integers.

By

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Let $a_1, a_2, \ldots$ be any given sequence of positive steadily increasing integers and suppose there are $x=f(n)$ of them not exceeding a number n, so that

$$a_x \leqq n < a_{x+1}.$$

The density δ of the sequence is defined by Schnirelmann as the lower bound of the numbers $f(n)/n$, $n=1, 2, \ldots$. Thus if $a_1 \neq 1$, $\delta=0$.

Clearly $f(n)\geqq \delta n$.

Suppose also that the steadily increasing set

$$A_0=0,\ A_1,\ A_2, \ldots$$

forms a basis of order l of the positive integers. This means that every positive integer can be expressed as the sum of at most l of the A's. I prove the following

Theorem: *If δ' is the density of the sequence $a+A$, i. e. of the integers which can be expressed as the sum of an a and an A, then*

$$\delta' \geqq \delta + \frac{\delta(1-\delta)}{2l}.$$

Particular cases of this theorem have been proved by Khintchine and Buchstab in an entirely different and more complicated way.