Aus (101), (110) und (114) folgt für $n \geqq n_1 \geqq n_0$, $n \equiv m \pmod 3$ und $n \equiv 0 \pmod 2$ für $\varepsilon = 1$, $n \equiv 1 \pmod 2$ für $\varepsilon = -1$,

$$\frac{k_{n+1} \log k_{n+1}}{k_n^2} \leqq A \frac{k_{n+1} \log k_{n+1}}{k_n^3}.$$

Da dies unmöglich ist, kann keine der Ungleichungen (111) für alle hinreichend grossen r erfüllt sein. Wegen (104) sind damit (8) und (9) bewiesen.

(Eingegangen am 6. Juli 1934.)

Note on Dirichlet's L-functions.

By

S. Chowla (Waltair, India).

Let

$$L(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \qquad (s > 0)$$

where $\chi(n)$ is a real non principal character mod k;

$$S_1(x) = \sum_{n \leqq x} \chi(n), \quad S_m(x) = \sum_{n \leqq x} S_{m-1}(n) \qquad (m \geqq 2).$$

Let $m = m(\chi)$ be the least positive integer (if any) such that

$$S_m(x) \geqq 0 \qquad (x \geqq 1).$$

If m exists, then

(1) $$L(s) > 0 \qquad (s > 0).$$ [1]

For $S_m(1) = 1$, $S_m(n) \geqq 0$ $(n = 2, 3, \ldots)$, whence

$(s > 0)$ $$L(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \sum_{n=1}^{\infty} S_1(n) \{n^{-s} - (n+1)^{-s}\}$$

$$= \sum_{n=1}^{\infty} S_2(n) \{n^{-s} - 2(n+1)^{-s} + (n+2)^{-s}\} = \ldots$$

(2) $$= \sum_{n=1}^{\infty} S_m(n) \sum_{t=0}^{m} (-1)^t \frac{m!}{t!(m-t)!} (n+t)^{-s}$$

[1]) By a theorem of Hecke, a proof of (1) yields important consequences on the magnitude of the class-number of binary quadratic forms of a given negative discriminant. See E. Landau „Über die Klassenzahl imaginär-quadratischer Zahlkörper" [Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1918, p. 285—295], p. 287.

$$= s(s+1)\ldots(s+m-1)\sum_{n=1}^{\infty} S_m(n)\int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 (n+u_1+ + u_2 + \ldots + u_m)^{-s-m}\, du_m > 0.$$

At the writer's suggestion K. Subba Rao calculated m for the primitive real characters corresponding to

$$k = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47, 53, 59, 61, 71, 73, 79, 83, 89, 97$$ [2]).

He showed that $m=3$ for $k=53$ and in the other cases $m \leqq 2$.

I. Chowla proved that m is finite for the real primitive characters corresponding to

$$k = 15, 21, 33, 35, 39, 51, 55, 57, 77, 87, 91, 95, 101, 103, 105, 107, 127, 131, 191, 203, 421$$ [3]),

m being $=3$ for $k=91$, $=7$ for $k=77$ and $\leqq 2$ otherwise.

I have been unable to find a real non principal character χ for which $m(\chi)$ does not exist, *i. e.* for which $m(\chi)=\infty$.

If m is finite, we obtain from (2)

$$(3) \quad L(1) \geqq \sum_{t=0}^{m} (-1)^t \frac{m!}{t!\,(m-t)!}(1+t)^{-1} = \int_0^1 (1-u)^m\, du = \frac{1}{m+1}.$$

But if the extended Riemann hypothesis is true there exist real primitive characters $\chi(n)$ mod k for some arbitrarily large k such that

$$(4) \quad L(1) < \frac{c}{\log\log k},$$

c being a certain absolute positive constant [4]).

By (3) and (4)

$$m(\chi) = \Omega(\log\log k) \qquad (\chi \text{ primitive}),$$

on the extended Riemann hypothesis.

[2]) These are all but two odd primes < 100.

[3]) These values were chosen at random.

[4]) J. E. Littlewood „On the class-number of the corpus $P(\sqrt{-k})$ „[Proceedings of the London Mathematical Society, ser. 2, vol. 27, 1927, p. 358—372], Theorem 2.

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The representation of a number as a sum of four squares and a prime.

By

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Let

$$(1) \qquad N_{r,s}(n) = \sum_{n_1^2+\ldots+n_r^2+p_1+\ldots+p_s=n} 1$$

the number of representations of n as a sum of r squares and s primes. In this paper I show that

$$(I) \qquad N_{4,1}(n) \sim \frac{\pi^2 n^2}{2\log n} \prod_{\substack{p\mid n\\ p>2}} \frac{(p-1)^2(p+1)}{p^3-p^2+1} \prod_{p>2}\left(1+\frac{1}{p^2(p-1)}\right)$$

where p denotes a typical prime. This is the second half of Conjecture J of Hardy and Littlewood's „Partitio Numerorum, III" [1]).

From (I) I easily derive the formula

$$(II) \qquad \sum_{n_1^2+\ldots+n_4^2+p=n} \log p \sim \frac{\pi^2 n^2}{2} \prod_{\substack{p\mid n\\ p>2}} \frac{(p-1)^2(p+1)}{p^3-p^2+1} \prod_{p>2}\left(1+\frac{1}{p^2(p-1)}\right).$$

The above results and more general ones on $N_{r,s}(n)$ $[r \geqq 3, s \geqq 1]$ were proved by G. K. Stanley [2]) on the assumption of unproved hypotheses concerning the zeros of Dirichlet's L-functions.

[1]) „On the expression of a number as a sum of primes" [Acta mathematica 44 (1922), 1—70], (5.452) and (5.4521).

[2]) „On the representation of a number as a sum of squares and primes„ [Proceedings of the London Mathematical Society, ser. 2, 29 (1929), 122—144].