hence
$$N \geqq n\left(\delta + \frac{\delta(1-\delta)}{2l}\right),$$
and this is the theorem.

I can prove in the same way that if a sequence $a_1, a_2, \ldots$ is given and there are $f(n)$ of the a's not exceeding n, then in the set $|a \pm A|$, there are at least
$$f(n) + \frac{f(n)\,(n - f(n))}{2l}$$
numbers not exceeding n.

Before closing my paper I would express my sincere gratitude to Prof. L. J. Mordell for having so kindly helped me with my ms.

(Received 11 March, 1935.)

A note on the distribution of primes.

By

A. E. Ingham (Cambridge).

1. If $\pi(x)$ denotes as usual the number of primes not exceeding x, then, by the prime number theorem, $\pi(x) \sim \operatorname{li} x$ (as $x \to \infty$), and, by a well known theorem of Littlewood [1]),

$$(1) \qquad \pi(x) - \operatorname{li} x = \Omega_{\pm}\left(\frac{x^{\frac{1}{2}}}{\log x} \log\log\log x\right).$$

The first aim of this paper is to give a proof of (1) without the use of the Phragmén-Lindelöf theorem which was an essential feature of Littlewood's original proof; and the second is to adapt the method to the proof of the following result, in which Θ denotes the upper bound of the real parts of the zeros of the Riemann zeta-function $\zeta(s) = \zeta(\sigma + it)$.

Theorem A. *If Θ is attained, i. e. if $\zeta(s)$ has a zero on the line $\sigma = \Theta$, then there exists an absolute constant $A > 1$ such that, for all $x > 1$, the interval $(x, A x)$ contains an integer n and an integer n' satisfying*
$$\pi(n) < \operatorname{li} n, \quad \pi(n') > \operatorname{li} n'.$$

The possibility of dispensing with the Phragmén-Lindelöf theorem and the resulting advantages for the detailed study of the difference

[1]) See E. Landau, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), II, 123—150; or A. E. Ingham, *The distribution of prime numbers* (Cambridge, 1932), Chapter V. [These books will be quoted as „*Vorlesungen*" and „*Prime numbers*" respectively]. The notation is that of *Prime numbers*.

$\pi(x) - \mathrm{li}\, x$, have already been demonstrated by Skewes [2]) in his account of the method by which he obtains (on the Riemann hypothesis) a numerical upper bound for the position of the first change of sign of this difference. The argument which replaces the Phragmén-Lindelöf theorem in the present paper is not very different in principle from that indicated by Skewes, the main difference of detail consisting in the use of the „Fejér kernel" in place of the „Poisson kernel". The method is suggested in part by the systematic use of the Fejér kernel by N. Wiener in his work on Tauberian theorems.

If we denote by $V(X)$ the number of changes of sign of the sequence $\pi(n) - \mathrm{li}\, n$ $(n = 2, 3, \ldots)$ or of the function $\pi(x) - \mathrm{li}\, x$ in the interval $(2, X)$, we infer at once from Theorem A that, if Θ is attained (in particular if $\Theta = \frac{1}{2}$, i. e. if the Riemann hypothesis is true), then

$$\text{(2)} \qquad \varliminf_{X \to \infty} \frac{V(X)}{\log X} \geqq \frac{1}{\log A} > 0.$$

The corresponding problem of the frequency of changes of sign of the „error term" in the asymptotic formula $\psi(x) \sim x$, where

$$\psi(x) = \sum_{p^m \leqq x} \log p = \sum_{n \leqq x} \Lambda(n)$$

(the p's being primes and the m's and n's positive integers) has been studied by Pólya. He has proved, *without hypothesis*, that, if $W(X)$ is the number of changes of sign of $\psi(n) - n$, or of $\psi(x) - x$, in $(2, X)$, then

$$\text{(3)} \qquad \varlimsup_{X \to \infty} \frac{W(X)}{\log X} > 0 \text{ [3]).}$$

His proof is based on a general theorem, which shows that similar results hold for various „averages" of $\psi(x) - x$. This is so, in particular, for the function

$$\sum_{n=1}^{\infty} (\Lambda(n) - 1)\, e^{-n/x}$$

---

[2]) S. Skewes, „On the difference $\pi(x) - \mathrm{li}(x)$ (I)", Journal London Math. Soc., 8 (1933), 277 — 283.

[3]) G. Pólya „Über das Vorzeichen des Restgliedes im Primzahlsatz", Göttinger Nachrichten (1930), 19 — 27. Prof. Pólya informs me, and asks me to take the opportunity of mentioning, that the statement „ . . . und nicht für $n > m$, wegen (24)" on p. 26 of this paper (fourth line after formula (24)) is false, but that in his opinion the error can easily be corrected.

(substantially the „Abel mean" of $\psi(x) - x$), and for this function Pólya has indeed proved, *on the Riemann hypothesis*, the stronger result with $\varliminf$ in place of $\varlimsup$ [4]). For $V(X)$ it is natural to expect that more special arguments will be required. For the proof that $\pi(x) - \mathrm{li}\, x$ changes sign at all turns essentially on showing that the „oscillating part" ultimately overpowers the „negative part"; and the situation is completely changed by averaging, at any rate if the Riemann hypothesis is true [5]). For the same reasons we cannot hope to obtain for the $\varliminf$ in (2) any such simple estimate in terms of the zeros of $\zeta(s)$ as in Pólya's theorems.

If Θ is not attained (in particular if $\Theta = 1$), I am unable to prove anything about $V(X)$ by the methods of this paper. But, if the general theorem of Pólya referred to above has a natural analogue for functions with logarithmic singularities (instead of poles), we could deduce from it that, if Θ is unattained $\left(\text{so that } \Theta > \frac{1}{2}\right)$, then

$$\varlimsup_{X \to \infty} \frac{V(X)}{\log X} = \infty,$$

and this combined with Theorem A would give at any rate the analogue of (3) without hypothesis.

2. Write

$$f(x) = \frac{\pi(x) - \mathrm{li}\, x}{x^{\Theta} (\log x)^{-1}}, \quad g(x) = \frac{\psi(x) - x}{x^{\Theta}},$$

$$F(u) = f(e^u), \qquad G(u) = g(e^u).$$

Then by trivial or classical arguments

$$\text{(4)} \qquad |f(x) - f(n)| < A_1 n^{-\Theta} \log n \quad (n \leqq x \leqq n+1;\ n = 2, 3, \ldots),$$

$$\text{(5)} \qquad |F(u) - G(u)| < A_2 e^{\left(\frac{1}{2} - \Theta\right)u} + A_3 (u+1)^{-1} \quad (u > 0),$$

where $A_1, A_2, \ldots$ are positive absolute constants [6]).

---

[4]) G. Pólya, „On polar singularities of power series and of Dirichlet series", Proc. London Math. Soc. (2), **33** (1932), 85 — 101 (§ 7). The ideas of this paper are further developed and illustrated in the paper which follows it: A. Bloch and G. Pólya, „On the roots of certain algebraic equations", Proc. London Math Soc. (2) **33** (1932), 102 — 114.

[5]) Cf. *Prime numbers*, 92 — 93 and 105 — 106.

[6]) (4) is trivial. For (5) cf. *Vorlesungen* II, 129—130; or *Prime numbers* 103—104. It is assumed (in both places) that $\Theta = \frac{1}{2}$, but the arguments give (5) without this hypothesis (the first term on the right arising from the contribution of prime powers higher than the first).

We start from the formula

$$\int_a^b \chi(x)\,(\psi(x)-x)\,dx = -\sum_\rho \frac{1}{\rho}\int_a^b \chi(x)\,x^\rho\,dx + \tag{6}$$

$$+\int_a^b \chi(x)\left(\frac{1}{2}\log\frac{1}{1-x^{-2}}-\frac{\zeta'}{\zeta}(0)\right)dx,$$

where $1<a<b<\infty$, $\chi(x)$ is any function whose behaviour is sufficiently regular, and the summation extends to all the non-trivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ arranged in order of increasing $|\gamma|$ (with repetitions for multiplicity). Formally this equation is obtained by multiplying the explicit formula for

$$\psi_0(x)=\frac{\psi(x+0)+\psi(x-0)}{2}$$

by $\chi(x)$ and integrating term by term. The process is legitimate if $\chi(x)$ is merely integrable in the sense of Lebesgue, since the series occurring in the formula for $\psi_0(x)$ is boundedly convergent in (a, b) [7]). In the proposed application, however, $\chi(x)$ will be a function possessing a continuous derivative, and in this case we need only know the explicit formula for

$$\psi_1(x)=\int_1^x \psi(v)\,dv,$$

which is much easier to prove than the formula for $\psi_0(x)$ and involves only absolutely and uniformly convergent series [8]). We multiply the formula for $\psi_1(x)$ by $\chi'(x)$, integrate term by term from a to b (by uniform convergence), perform an integration by parts in each term of the resulting equation, and eliminate the „terms at the limits" by means of the explicit formulae for $\psi_1(a)$ and $\psi_1(b)$; the infinite series in (6) will be absolutely convergent.

Now let

$$\mathfrak{k}(x)=\begin{cases}1-|x| & (|x|\leqq 1)\\ 0 & (|x|>1)\end{cases},\qquad \mathfrak{K}(y)=\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2,$$

[7]) *Vorlesungen* II, 116—120; *Prime numbers*, 77—81.

[8]) *Prime numbers*, 73—74.

so that

$$\mathfrak{k}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathfrak{K}(y)\,e^{ixy}\,dy. \tag{7}$$

Let T, η, ω be any numbers satisfying

$$T>e,\quad \frac{1}{T}<2\eta\leqq\omega, \tag{8}$$

and write for brevity $\mathfrak{K}_T(v)=T\,\mathfrak{K}(Tv)$. Then, by an application of (6) and the substitution $x=e^u$, we have

$$\int_{\omega-\eta}^{\omega+\eta}\mathfrak{K}_T(u-\omega)\,G(u)\,du=-\sum_\rho\frac{1}{\rho}\int_{\omega-\eta}^{\omega+\eta}\mathfrak{K}_T(u-\omega)\,e^{(\rho-\Theta)u}\,du \tag{9}$$

$$+\int_{\omega-\eta}^{\omega+\eta}\mathfrak{K}_T(u-\omega)\,r(u)\,du,$$

where

$$r(u)=e^{-\Theta u}\left(\frac{1}{2}\log\frac{1}{1-e^{-2u}}-\frac{\zeta'}{\zeta}(0)\right).$$

In the last integral in (9) we have, since $\frac{1}{2}\omega\leqq\omega\pm\eta\leqq\frac{3}{2}\omega$,

$$\left|r(u)-\frac{1}{2}\overset{+}{\log}\frac{1}{\omega}\right|<A_4\,e^{-\frac{1}{2}\Theta\omega},$$

where $\overset{+}{\log}\,v$ is $\log v$ if $v>1$, and 0 otherwise. Also $0\leqq\mathfrak{K}(y)\leqq 4y^{-2}$, and $\mathfrak{k}(0)=1$, whence by the substitution $u=\omega+T^{-1}y$ and the equation (7)

$$\int_{\omega-\eta}^{\omega+\eta}\mathfrak{K}_T(u-\omega)\,du=\int_{-T\eta}^{T\eta}\mathfrak{K}(y)\,dy=2\pi-\frac{8\vartheta}{T\eta},$$

where $0\leqq\vartheta\leqq 1$. The integral in question is therefore

$$\frac{1}{2}\overset{+}{\log}\frac{1}{\omega}\left(2\pi+O\left(\frac{1}{T\eta}\right)\right)+O\left(e^{-\frac{1}{2}\Theta\omega}\right)$$

$$= \pi \overset{+}{\log} \frac{1}{\omega} + O\left(\frac{\log T}{T\eta}\right) + O\left(e^{-\frac{1}{2}\Theta\omega}\right),$$

since $1/\omega < T$ by (8). Here and in what follows O implies an absolute constant and an inequality valid for all T, η, ω satisfying (8). Thus (9) gives

$$\text{(10)} \qquad \int_{-T\eta}^{T\eta} \mathfrak{K}(y)\, G\left(\omega + \frac{y}{T}\right) dy = -\sum_\rho \frac{e^{(\rho-\Theta)\omega}}{\rho} \int_{-T\eta}^{T\eta} \mathfrak{K}(y)\, e^{\frac{\rho-\Theta}{T} y}\, dy$$

$$+ \pi \overset{+}{\log} \frac{1}{\omega} + O\left(\frac{\log T}{T\eta}\right) + O\left(e^{-\frac{1}{2}\Theta\omega}\right).$$

3. Suppose first $\Theta = \frac{1}{2}$. Then $\rho - \Theta = i\gamma$, and we have

$$\left| \int_{T\eta}^{\infty} \mathfrak{K}(y)\, e^{\frac{\rho-\Theta}{T} y}\, dy \right| \begin{cases} \leqq \displaystyle\int_{T\eta}^{\infty} \mathfrak{K}(y)\, dy < \frac{4}{T\eta}, \\ = \left| \displaystyle\int_{T\eta}^{\infty} \mathfrak{K}'(y) \frac{T}{i\gamma} \left(e^{i\gamma\eta} - e^{\frac{i\gamma y}{T}}\right) dy \right| < \dfrac{A_5}{T\eta} \dfrac{T}{|\gamma|}, \end{cases}$$

with similar inequalities for the range $(-\infty, -T\eta)$. Hence the ranges of integration on the right of (10) may be extended to $(-\infty, \infty)$ with an error

$$O\left(\frac{1}{T\eta} \sum_{|\gamma| \leqq T} \frac{1}{|\gamma|}\right) + O\left(\frac{1}{\eta} \sum_{|\gamma| > T} \frac{1}{\gamma^2}\right) = O\left(\frac{\log^2 T}{T\eta}\right); \text{ [9]}$$

and if the denominators ρ are then replaced by $i\gamma$ the error thereby involved will be

$$O\left(\sum_\rho \frac{1}{\gamma^2} \int_{-\infty}^{\infty} \mathfrak{K}(y)\, dy\right) = O(1).$$

Thus we infer from (10), in virtue of (7), that

[9] See, e. g., *Prime numbers*, 70 (Theorem 25 *b*).

$$\text{(11)} \quad \frac{1}{2\pi} \int_{-T\eta}^{T\eta} \mathfrak{K}(y)\, G\left(\omega + \frac{y}{T}\right) dy = -S_T(\omega) + \frac{1}{2} \overset{+}{\log} \frac{1}{\omega} + O\left(\frac{\log^2 T}{T\eta}\right) + O(1),$$

where

$$S_T(u) = \sum_\rho \frac{e^{i\gamma u}}{i\gamma}\, \mathfrak{k}\left(\frac{\gamma}{T}\right) = 2 \sum_{0 < \gamma \leqq T} \mathfrak{k}\left(\frac{\gamma}{T}\right) \frac{\sin \gamma u}{\gamma}.$$

Now we satisfy the conditions (8) by taking (for any $T > e$) $2\eta = \omega = \tau$, where

$$\tau = \tau_T = \frac{\log^2 T}{T}.$$

Thus (11) yields in particular, since $0 < \tau < A_6$,

$$O(1) = -S_T(\tau) + \frac{1}{2} \log \frac{1}{\tau} + O(1) + O(1) + O(1),$$

or, since $S_T(u)$ is an odd function of u,

$$\text{(12)} \qquad S_T(\pm\tau) = \pm \frac{1}{2} \log \frac{1}{\tau} + O(1). \text{ [10]}$$

Now let $\mathfrak{U}(T, q)$, for any positive integer q, be the class of all real numbers $U \geqq 2$ with the property that

$$\text{(13)} \qquad \left| \frac{\gamma_n U}{2\pi} \right| < \frac{1}{q} \pmod 1 \qquad (n = 1, 2, \ldots, N),$$

where $\gamma_1, \gamma_2, \ldots \gamma_N$ are the γ in the range $0 < \gamma \leqq T$ (enumerated with repetitions for multiplicity), so that $N = N(T)$ in the usual notation. Let U be any number of $\mathfrak{U}(T, q)$. Then for all real u, we have by (13)

$$|S_T(U+u) - S_T(u)| = \left| 2 \sum_{n=1}^{N} \mathfrak{k}\left(\frac{\gamma_n}{T}\right) \frac{\sin(\varphi_n + \gamma_n u) - \sin \gamma_n u}{\gamma_n} \right| \left(|\varphi_n| < \frac{2\pi}{q} \right)$$

$$\text{(14)} \qquad \leqq 2 \sum_{1}^{N} \frac{|\varphi_n|}{\gamma_n} < \frac{4\pi}{q} \sum_{0 < \gamma \leqq T} \frac{1}{\gamma} < A_7 \frac{\log^2 T}{q}.$$

[10] The behaviour of $S_T(u)$ for small $|u|$ could also be inferred directly from the definition and the known facts about the distribution of the γ' s. (Cf. *Vorlesungen* II 139-140; *Prime numbers*, 98-99).

Now combine (11), (14), (12), taking

$$\eta = \tau,\ \omega = U \pm \tau,\ u = \pm\tau,\ q = q_T = [\log^2 T] + 1,$$

and noting that, for all sufficiently large T, $T^{-1} < 2\tau \leqq U \pm \tau$, and $U \pm \tau > 1$, so that the conditions (8) are fulfilled and the $\overset{+}{\log}$ term in (11) is 0; we deduce that, for all $T > A_8$ and all U belonging to $\mathfrak{U}(T, q_T)$

$$\frac{1}{2\pi}\int_{-T\tau}^{T\tau} \mathfrak{K}(y)\left\{\mp G\left(U \pm \tau + \frac{y}{T}\right)\right\} dy > \frac{1}{2}\log\frac{1}{\tau} - A_9 .$$

Since $\mathfrak{K}(y) \geqq 0$ and $\mathfrak{k}(0) = 1$, we may, by (5) and (7), replace G in this formula by F if we change A_9 to $A_9 + A_2 + A_3 = A_{10}$. We thus deduce (using again (7), $\mathfrak{k}(0) = 1$, $\mathfrak{K}(y) \geqq 0$) that

$$\int_{-T\tau}^{T\tau} \mathfrak{K}(y)\left\{\mp F\left(U \pm \tau + \frac{y}{T}\right)\right\} dy > \left(\frac{1}{2}\log\frac{1}{\tau} - A_{10}\right)\int_{-T\tau}^{T\tau} \mathfrak{K}(y)\, dy ,$$

provided that T is large enough (say $T > A_{11} \geqq A_8$) to make the first factor on the right positive. From this it follows (again because $\mathfrak{K}(y) \geqq 0$) that there must exist a u and a u' such that

(15) $$U < u < U + 2\tau , \quad -F(u) > \frac{1}{2}\log\frac{1}{\tau} - A_{10} ,$$

(16) $$U - 2\tau < u' < U , \quad F(u') > \frac{1}{2}\log\frac{1}{\tau} - A_{10} .$$

To obtain (1) we appeal to Dirichlet's theorem on Diophantine approximation. From this we infer that there exists a U_T of $\mathfrak{U}(T, q_T)$ satisfying

(17) $$q_T^{N(T)} \leqq U_T \leqq q_T^{2N(T)} .$$

When $T \to \infty$, we have, using the relation $2\pi N(T) \sim T \log T$ and (17),

$$\log\frac{1}{\tau} \sim \log T \sim \log N(T) \sim \log\log U_T ,$$

and it follows at once from (15) and (16) that

$$\overline{\lim_{u\to\infty}}\ \frac{-F(u)}{\log\log u} \geqq \frac{1}{2} , \quad \overline{\lim_{u\to\infty}}\ \frac{F(u)}{\log\log u} \geqq \frac{1}{2} .$$

These relations clearly imply (1).

To obtain Theorem A we appeal to a theorem of Bohl on Diophantine approximation [11]). From this we infer the existence of a number $L = L(\gamma_1, \ldots, \gamma_{N(T)}; q_T) = L_T > 0$ such that every interval $(u_0, u_0 + L)$ with $u_0 \geqq 2$ contains a number U belonging to $\mathfrak{U}(T, q_T)$. It follows, by (15) and (16), that, if $T > A_{11}$ and $u_0 \geqq 2$, the interval $(u_0 - 2\tau, u_0 + L + 2\tau)$ contains a u and a u' satisfying

$$-F(u) > \frac{1}{2}\log\frac{1}{\tau} - A_{10},\ F(u') > \frac{1}{2}\log\frac{1}{\tau} - A_{10} ;$$

from which we deduce, taking account of (4), that if u_0 is sufficiently large, the interval $(e^{u_0 - 2\tau}, e^{u_0 + L + 2\tau})$ contains an integer n $(= [e^u]$ or $[e^u] + 1)$ and an integer n' satisfying

(18) $$-f(n) > \frac{1}{2}\log\frac{1}{\tau} - A_{12} ,\ f(n') > \frac{1}{2}\log\frac{1}{\tau} - A_{12} ,$$

where A_{12} may be taken to be $A_{10} + 1$. Now take T to be an absolute constant so large that (in addition to the conditions already imposed) the expression on the right of the inequalities (18) is positive. Then $e^{L+4\tau}$ is an absolute constant $A_{13} > 1$, and the interval $(x, A_{13}x)$, contains for every $x > A_{14}$ integers n and n' for which $f(n) < 0 < f(n')$. This clearly implies Theorem A in the form stated.

4. Next suppose that $\Theta > \frac{1}{2}$. In this case the relation (1), and indeed more, follows by a familiar argument from a well known theorem of Landau on Dirichlet's integrals [12]), and I have nothing to add to this.

To obtain Theorem A in this case we return to (10). Since, by partial integration,

---

[11]) The theorem in question is substantially equivalent to the theorem that a finite sum of purely periodic continuous functions of a real variable is almost periodic, and we might, alternatively, apply this theorem directly to $S_T(u)$. See H. Bohr, „Zur Theorie der fast periodischen Funktionen, I", Acta Mathematica 45 (1925), 29-127 (119-121).

[12]) *Vorlesungen* II, 130—132; *Prime numbers*, 90—91.

$$\left|\frac{e^{(\rho-\Theta)\omega}}{\rho}\int_{-T\eta}^{T\eta}\mathfrak{K}(y)\,e^{\frac{\rho-\Theta}{T}y}\,dy\right|<\frac{e^{(\beta-\Theta)\omega}}{|\rho|}A_{15}\frac{T}{|\rho-\Theta|}e^{-(\beta-\Theta)\eta}<A_{15}\frac{T\,e^{(\beta-\Theta)(\omega-\eta)}}{\gamma^2},$$

the terms of $\sum_\rho$ for which $\beta<\Theta$ form, for any fixed admissible values of T and η, an absolutely and uniformly convergent series for $\omega\geqq 2\eta$, in which each term tends to 0 when $\omega\to\infty$. Hence the sum of these terms tends to 0 when $\omega\to\infty$. In the terms for which $\beta=\Theta$ we extend the ranges of integration to $(-\infty,\infty)$ as in § 3, and we obtain

$$\frac{1}{2\pi}\int_{-T\eta}^{T\eta}\mathfrak{K}(y)\,G\left(\omega+\frac{y}{T}\right)dy=-\sum_{\substack{\beta=\Theta\\|\gamma|<T}}\frac{e^{i\gamma\omega}}{\rho}\,\mathfrak{k}\left(\frac{\gamma}{T}\right)+O\left(\frac{\log^2 T}{T\eta}\right)+o(1),$$

where $o(1)$ denotes a function which, for any fixed admissible T and η, tends to 0 when $\omega\to\infty$; and in this we may replace G by F in virtue of (5) since $\Theta>\frac{1}{2}$.

Now, assuming that Θ is attained, let $\Theta\pm i\gamma_0\;(\gamma_0>0)$ be the zeros nearest to the real axis on the line $\sigma=\Theta$, and let them be of order r_0. Take T to be an absolute constant such that the open segment $\sigma=\Theta$, $-T<t<T$ contains the zeros $\Theta\pm i\gamma_0$ but no others. Then taking η to be a sufficiently large absolute constant, and writing $\Theta+i\gamma_0=\rho_0=|\rho_0|e^{i\delta_0}$, we have

$$\frac{1}{2\pi}\int_{-T\eta}^{T\eta}\mathfrak{K}(y)\,F\left(\omega+\frac{y}{T}\right)dy=-\frac{r_0}{|\rho_0|}\,\mathfrak{k}\left(\frac{\gamma_0}{T}\right)(2\cos(\gamma_0\,\omega-\delta_0)+\vartheta),$$

where $|\vartheta|<1$ for all sufficiently large ω. Taking $\omega=\omega_m$ and $\omega=\omega_{m+1}$ where

$$\omega_m=\frac{m\pi+\delta_0}{\gamma_0},$$

we deduce (substantially as in § 3) first that, for every sufficiently large positive integer m, the interval $(\omega_m-\eta,\,\omega_{m+1}+\eta)$ contains a u and a u' satisfying

$$-F(u)>\frac{r_0}{|\rho_0|}\,\mathfrak{k}\left(\frac{\gamma_0}{T}\right),\qquad F(u')>\frac{r_0}{|\rho_0|}\,\mathfrak{k}\left(\frac{\gamma_0}{T}\right),$$

and thence, with the help of (4), that, if $A_{16}=e^{\frac{2\pi}{\gamma_0}+2\eta}$, every interval $(x,A_{16}\,x)$ with sufficiently large x contains an integer n and an integer n' satisfying

$$-f(n)>0,\qquad f(n')>0.$$

This completes the proof.

King's College, Cambridge, 9 July, 1935.

(Received 12 July, 1935).