Two-sided estimates for the approximation numbers of Hardy-type operators in $L^\infty$ and $L^1$

by

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Abstract. In [2] and [3] upper and lower estimates and asymptotic results were obtained for the approximation numbers of the operator $T: L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by $(Tf)(x) = \nu(x) \int_0^x u(t)f(t) \, dt$ when $1 < p < \infty$. Analogous results are given in this paper for the cases $p = 1, \infty$ not included in [2] and [3].

1. Introduction. In [2] and [3] the operator $T: L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$Tf(x) = \nu(x) \int_0^x u(t)f(t) \, dt \tag{1.1}$$

was studied in the case $1 < p < \infty$, with $u, \nu$ real-valued functions and $u \in L^p_{\text{loc}}(\mathbb{R}^+)$, $\nu \in L^{p'}(\mathbb{R}^+)$, $p' = p/(p-1)$. Estimates for the approximation numbers $\alpha_n(T)$ of $T$ were obtained in [2], but the procedure for extracting the upper and lower bounds from the results is rather cumbersome to apply. This deficiency was overcome in [3] where asymptotic bounds for the approximation numbers which are easy to check in practice were determined. Specifically, it was proved that

$$\lim_{n \to \infty} n\alpha_n(T) = \frac{1}{\pi} \int_0^\infty |u(t)v(t)| \, dt \tag{1.2}$$

when $p = 2$; and when $p \neq 2$,

$$\frac{1}{4} \alpha_p \int_0^\infty |u(t)v(t)| \, dt \leq \liminf_{n \to \infty} n\alpha_n(T) \leq \limsup_{n \to \infty} n\alpha_n(T) \leq \alpha_p \int_0^\infty |u(t)v(t)| \, dt \tag{1.3}$$

for some constant $\alpha_p$ depending on $p$. Further in [3], two-sided estimates

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are given for the \( L^p \) and weak \( L^p \) norms of \( \{ a_n(T) \} \) when \( \alpha > 1 \); in the case \( p = 2 \), these results recover those in [5].

The analysis in [3] is no longer valid when \( p = \infty \) or 1, and, indeed, the result itself has to be modified in the following way: when \( p = \infty \), the function \( v \) in the integrals in (1.2), (1.3) is replaced by

\[
v_\varepsilon(t) = \lim_{\varepsilon \to 0} \|v\|_{L^\infty(t-\varepsilon, t+\varepsilon)} \int_{t-\varepsilon}^{t+\varepsilon} u(t) dt,
\]

while if \( p = 1 \), then \( u \) is replaced by \( u_\varepsilon \). Three critical ingredients of the proof in [2] and [3] are no longer available in these cases. The first is that the operator \( P \) defined by the integral mean over an interval \( I \subset \mathbb{R}^+ \), namely

\[
Pf := \frac{1}{|I|} \int_I f(x) dx,
\]

where \( |I| \) denotes the length of \( I \), is such that the distance from \( T \) to the one-dimensional operators on \( L^p(I) \) is comparable to \( \|T - P\| \|L^p(I) \| \). The second concerns the basic strategy which relies on a partition of \( \mathbb{R}^+ \) into intervals \( I_k \) which are defined by means of a compact set function \( L(I) \) with \( I = (c, d) \), decreasing as \( c \) increases and increasing as \( d \) increases. In the \( L^\infty \) and \( L^1 \) cases the analogue of \( L \) is no longer continuous and an alternative function, and technique, have to be found. Finally, the fact that the step functions are not dense in \( L^\infty \) causes difficulties, and indeed, it is this which dictates the form of the result noted above.

It is just as easy to consider a general interval \( (a, b) \) instead of \( \mathbb{R}^+ \), so that in this paper

\[
Tf(x) := \int_a^x u(t) f(t) dt, \quad a < x < b;
\]

this simple extension will have a useful consequence when \( T \) is considered as an operator on \( L^1 \), as we can then simply translate the dual of the \( L^\infty \) result. Also, as was observed in [3], the condition on \( v \) assumed there, namely \( v \in L^p(\mathbb{R}^+) \), can be weakened to \( v \in L^p(a, \infty) \) for all \( a > 0 \), and we incorporate this fact in the present paper.

Finally, to give some insight into the significance of the function \( v_\varepsilon \) in the \( L^\infty(a, b) \) case, we show that, with the operator in (1.4) denoted by \( T_{u,v} \), the following is possible:

\[
\begin{align*}
\|T_{u,v}\| &= \|T_{u,v}\| = \|T_{u,v} - T_{u,v - v}\|, \\
\int_a^b |u(t)v(t)| dt &\neq \int_a^b |(u(t)v_\varepsilon(t)| dt, \\
\limsup_{a} n\alpha_n(T_{u,v}) &< \limsup_{a} n\alpha_n(T_{u,v - v}), \\
\liminf_{a} n\alpha_n(T_{u,v}) &\geq \liminf_{a} n\alpha_n(T_{u,v - v}),
\end{align*}
\]

where the symbol \( \asymp \) indicates that the quotient of the two sides is bounded above and below by positive constants. Analogous possibilities exist in the \( L^1(a, b) \) case.

2. Preliminaries. In most of the paper we shall be concerned with the operator \( T \) defined in (1.4) as a map from \( L^\infty(a, b) \) into itself. The assumptions made on \( u, v \) in this case are that, for all \( (a, b) \),

\[
(2.1) \quad u \in L^1(a, b),
(2.2) \quad v \in L^\infty(a, b).
\]

The results for \( T \) acting between \( L^1(a, b) \) will follow on taking duals, and for this part of the paper alternative conditions to (2.1) and (2.2) will be required.

For \( I = (c, d) \subset (a, b) \), define

\[
J(I) := \sup_{x \in I} \left\{ \int_c^x |u(t)| |v(x) - v_\varepsilon(t)| dt \right\},
\]

where \( \chi_{S} \) denotes the characteristic function of the set \( S \), and \( \| \cdot \|_p \) denotes the norm on \( L^p(a, b) \); we shall write \( \| \cdot \|_{p, I} \) for the usual norm on \( L^p(I) \), \( 1 \leq p \leq \infty \), but use \( \| \cdot \|_p \) when \( I = (a, b) \). It is easy to see that

\[
J(I) = \text{ess sup}_{x \in I} \left\{ \int_c^x |u(t)| |v(x)| dt \right\}.
\]

We also have

\text{Lemma 2.1. Suppose that (2.1) and (2.2) are satisfied. Then the function}

\( J(\cdot, d) \) \text{ is continuous and non-increasing on } (a, d), \text{ for any } a < b.

\text{Proof. Given } x \in (a, b) \text{ and } \varepsilon > 0, \text{ there exists}

\( h = h(x, \varepsilon) \in (0, \min \{ \frac{1}{2}(x + a), b - x \}) \)

\text{such that}

\[
\int_{x-h}^{x-h+\varepsilon} \|u(t)\| dt < \min \left( \frac{\varepsilon}{\|v\|_{\infty, \infty, (x-a)/2, d}}, \frac{\varepsilon}{\|v\|_{\infty, \infty, (x-a)/2, d}} \right).
\]

Then

\[
J(x, d) \leq J(x-h, d) - J(x-h, d)
\]

\text{max} \left\{ \sup_{x-h < \tau < x} \int_{x-h}^{x-h+\varepsilon} |u(t)| \|v(x) - v_\varepsilon(t)| dt \right\},

\sup_{x-h < \tau < x} \left( \int_{x-h}^{x-h+\varepsilon} |u(t)| \|v(x) - v_\varepsilon(t)| dt \right) 
\leq \max \{ \varepsilon, \varepsilon + J(x, d) \} = \varepsilon + J(x, d) \]
and so $0 < J(x-h,f, a)-J(x,d) < \varepsilon$. Similarly, $0 < J(x) - J(x+h) < \varepsilon$ and
the continuity is established. It is obvious that $J(\cdot, d)$ is non-increasing and
hence the lemma is proved. ■

The following result is known (see [4] and [6]):

**Proposition 2.2.** The operator $T$ in (1.4), with $u, v$ satisfying (2.1) and
(2.2), is bounded as a map from $L^\infty(a, b)$ into $L^\infty(a, b)$ if and only if
$J(a, b) < \infty$. It is compact if and only if $\lim_{xt\to a, b} J(a, c) = \lim_{xt\to a, b} J(d, b) = 0$.

In [2], the analogue of the function $J$ in (2.3) could have been used to
construct the partition of $(a, b)$ into the intervals $I_i$, which feature so
prominently in the analysis; see the Remark at the end of §4 in [2]. However,
in the $L^\infty$ case, for the reason given in the introduction, we need to use
directly the function

$$A(I) := \sup_{f \in L^\infty(I), f \neq 0} \inf_{\alpha \in \mathbb{R}} \|Tf - \alpha v\|_{\infty, I} / \|f\|_{\infty, I}$$

if $v(I) > 0$,

$$0$$

if $v(I) = 0$,

where $v(I) := \int_I v(t) \, dt$. If $v$ is continuous, it can be shown that $A(\cdot, b)$ is
continuous, but in general, this is not so. For, consider the example

$$v(x) = \begin{cases} 1 & \text{for } x \in (0, 1) \cup (2, \infty), \\ 0 & \text{otherwise,} \end{cases}$$

with $(a, b) = (0, \infty)$. Then $A(\cdot, \infty) = 0$ for $x > 1$, but for $x < 1$,

$$A(x, \infty) \geq \inf_{\alpha \in \mathbb{R}} \left( \int_0^x u(t) - \alpha \, dt \right) \|v(y)\|_{\infty, (x, \infty)} = \inf_{\alpha \in \mathbb{R}} \max\{|\alpha|, 1 - |\alpha|\} = \frac{1}{2}.$$ 

It is of interest to note that if (2.1) and (2.2) are satisfied and $v \notin L^\infty(a, b)$,
then, since $\int_0^x u(t) f(t) \to \infty$ as $x \to a_+$ for every $f \in L^\infty(a, b)$, we must have
if $\alpha \neq 0$, $\|Tf - \alpha v\|_{\infty, (a, \infty)} = \infty$ for $c \in (a, b)$. Hence, with $I = (a, c)$,

$$A(a, c) = \sup_{\|f\|_{\infty, I} = 1} \|Tf\|_{\infty, I} = \sup_{\|f\|_{\infty, I} = 1} \sup_{\alpha \in \mathbb{R}} |v(\alpha)| \int_0^a |u(t)| \, dt = J(a, c)$$

by (2.4).

We now define, for any interval $I \subseteq (a, b)$ and $\varepsilon > 0$,

$$M(I, \varepsilon) := \inf \left\{ n: I = \bigcup_{i=1}^n I_i, A(I_i) \leq \varepsilon \right\}.$$

Observe that if $I \subseteq (a, b)$, then we have $M(I, \varepsilon) < \infty$. For, since $J(c, d) \leq ||u||_{L^\infty(I, c, d)} ||v||_{\infty, I}$ for any $(c, d) \subseteq I$ and $\|\cdot\|_I$ is absolutely continuous, it
follows that the number

$$N(I, \varepsilon) := \inf \left\{ n: I = \bigcup_{i=1}^n I_i, A(I_i) \leq \varepsilon \right\}$$

is finite, and

$$A(I) \leq \sup_{f \in L^\infty(I), f \neq 0} \frac{||Tf||_{\infty, I}}{||f||_{\infty, I}} \leq \sup_{f \in L^\infty(I), f \neq 0} \frac{\text{ess sup} \{ |v(x)| \int_0^a |u(t)| \, dt \}}{||f||_{\infty, I}} \leq J(I)$$

by (2.4); thus $M(I, \varepsilon) \leq N(I, \varepsilon) < \infty$. If $I = (a, b)$, we still have
$M(I, \varepsilon) < \infty$ if

$$\lim_{x \to a_+} J(a, x) = \lim_{x \to a_+} J(x, b) = 0$$

since $N(I, \varepsilon) < \infty$ and (2.9) remains valid.

**Lemma 2.3.** Suppose that (2.1) and (2.2) are satisfied and let $M(I, \varepsilon) = m \leq \infty$ for $I \subseteq (a, b)$ and $\varepsilon > 0$. Then we have:

(i) if $m = 2n$, there exist intervals $I_i$, $i = 1, \ldots, n$, such that $I = \bigcup_{i=1}^n I_i$ and $A(I_i) > \varepsilon$;

(ii) if $m = 2n + 1$, there exist intervals $I_i$, $i = 1, \ldots, n + 1$, such that $I = \bigcup_{i=1}^{n+1} I_i$, $A(I_i) > \varepsilon$, $i = 1, \ldots, n$, and $A(I_{n+1}) \leq \varepsilon$.

**Proof.** From the definition of $M(I, \varepsilon)$ in (2.7) there exist $I_i, i = 1, \ldots, m,
$ such that $A(I_i) \leq \varepsilon$ and $A(I_i \cup I_{i+1}) > \varepsilon$. Now set $J_1 = I_1 \cup I_2$, $J_2 = I_3 \cup I_4, \ldots$, with $J_{m+1} = I_m$ in case (ii).

The final preliminary result is the following critical lemma which will
yield a one-dimensional approximation to $T$ on $I$.

**Lemma 2.4.** There exists $\omega \in L^\infty(I)$ such that $\omega_I(1) = 1$, $||\omega_I||_{L^\infty(I)} = 1$ and, for all $f \in L^\infty(I)$,

$$\inf_{\alpha \in \mathbb{R}} \| v - \alpha v\|_{\infty, I} \leq ||(f - \omega_I(f))v||_{\infty, I} \leq 2 \inf_{\alpha \in \mathbb{R}} \| (f - \alpha)v\|_{\infty, I}.$$ 

**Proof.** For $0 < \gamma < ||v||_{\infty, I}$ and $A_\gamma := \{ x : v(x) > \gamma \}$, define $\omega_\gamma \in L^\infty(I)^*$ by

$$\omega_\gamma(f) := \frac{1}{|A_\gamma|} \int_{A_\gamma} f(x) \, dx, \quad f \in L^\infty(I).$$

Then $\omega_\gamma(1) = 1$, $||\omega_\gamma||_{L^\infty(I)^*} = 1$ and

$$||\omega_\gamma(f)|| \leq \frac{1}{\gamma} ||f||_{\infty, I}.$$ 

The set $W := \{ W_\beta : 0 < \beta < ||v||_{\infty, I} \}$, where $W_\beta := \{ \omega_\gamma : \gamma > \beta \}$, is a filter base whose members $W_\beta$ are subsets of the unit ball in $L^\infty(I)^*$. 
Hence, by the weak* compactness of this unit ball, \( W \) has an adherent point, \( \omega_f \) say. It follows that \( \omega_f(1) = 1, \| \omega_f \|_{(L^\infty'(I))^*} = 1 \) and, from (2.11), for all \( \beta \in (0, \| v \|_\infty, I), \)

\[
|\omega_f(f)| \leq \frac{1}{\beta} \| fv \|_\infty, I, \quad f \in L^\infty(I).
\]

Consequently, for any \( \delta \in \mathbb{R}, \)

\[
\inf_{\alpha \in \mathbb{R}} \| (f - \alpha)v \|_\infty, I \leq \| (f - \omega_f(f))v \|_\infty, I \\
\leq \| (f - \delta)v \|_\infty, I + |\omega_f(f - \delta)v|_\infty, I \\
\leq \| (f - \delta)v \|_\infty, I \left\{ 1 + \frac{\| v \|_\infty, I}{\beta} \right\}.
\]

Since \( \delta \in \mathbb{R} \) and \( \beta \in (0, \| v \|_\infty, I) \) are arbitrary, the lemma follows. ■

3. Bounds for the approximation numbers. We recall that, given any \( m \in \mathbb{N}, \) the \( m \)th approximation number of a bounded operator \( T, a_m(T), \)

is defined by

\[
a_m(T) := \inf \| T - F \|,
\]

where the infimum is taken over all bounded linear maps \( F : L^\infty(a, b) \to L^\infty(a, b) \) with rank less than \( m \). General information on approximation numbers may be found in [3]. Since \( L^\infty(a, b) \) has the approximation property, \( T \) is compact if and only if \( a_m(T) = 0 \) as \( m \to \infty \).

The first two lemmas of this section give estimates for \( a_m(T) \) which are the analogues of those obtained in [2]. Hereafter, until §7, we shall always assume (2.1) and (2.2).

**Lemma 3.1.** Suppose that \( T : L^\infty(a, b) \to L^\infty(a, b) \) is bounded. Let \( \varepsilon > 0 \) and suppose that there exist \( N \in \mathbb{N} \) and numbers \( a_k, k = 0, 1, \ldots, K, \)

with \( a = a_0 < a_1 < \ldots < a_N = b \), such that \( A(I_k) \leq \varepsilon \) for \( k = 0, 1, \ldots, N - 1, \)

where \( I_k = (a_k, a_{k+1}). \) Then \( a_{N+1}(T) \leq 2\varepsilon. \)

**Proof.** Let \( f \in L^\infty(a, b) \) be such that \( \| f \|_\infty = 1, \) and write

\[
Pf := \sum_{i=0}^{N-1} P_{I_k} f
\]

where the \( P_{I_k} \) are the one-dimensional operators

\[
P_{I_k} f(x) := \chi_{I_k}(x)v(x)\omega_{I_k}\left(\int_a^x uf dt\right), \quad k = 0, 1, \ldots, N - 1,
\]

and

\[
\omega_{I_k}\left(\int_a^x uf dt\right) = \int_a^b u f dt + \omega_k\left(\int_a^x uf dt\right),
\]

with \( \omega_k \in (L^\infty(I_k))^* \) the functionals in Lemma 2.4.

It is obvious that \( P_{I_k}, k = 1, \ldots, N - 2, \) are bounded. With \( k = 0 \) or \( N - 1 \) we have on \( I = (a, c_1) \) or \( (c_N, b), \)

\[
\left| \int_a^x \omega_k\left(\int_a^u uf dt\right) \right| \leq \| \omega_k \|_{(L^\infty'(I))^*} \| v(x) \| \| \int_a^x uf dt \|_\infty, I
\]

and hence \( P \) is bounded in view of Proposition 2.2 and (2.4). We have

\[
\| T - Pf \|_\infty = \sup_{k \in \{0, 1, \ldots, N-1\}} \| T - P_{I_k} f \|_\infty, I_k
\]

\[
= \sup_{k \in \{0, 1, \ldots, N-1\}} \left\| \left.v(x)\omega_k\left(\int_a^x uf dt\right)\right|_{I_k}\right\|_\infty, I_k
\]

\[
\leq 2 \sup_{k \in \{0, 1, \ldots, N-1\}} A(I_k) \| f \|_\infty, I_k \leq 2\varepsilon \| f \|_\infty, I
\]

by Lemma 2.4. Since \( \text{rank} P \leq N, \) the lemma follows. ■

**Lemma 3.2.** Suppose that \( T : L^\infty(a, b) \to L^\infty(a, b) \) is bounded. Let \( \varepsilon > 0 \) and suppose that there exist \( N \in \mathbb{N} \) and numbers \( a_k, k = 0, 1, \ldots, K, \)

with \( a = a_0 < a_1 < \ldots < a_K \), such that \( A(I_k) \leq \varepsilon \) for \( k = 0, 1, \ldots, K - 1, \)

where \( I_k = (a_k, a_{k+1}). \) Then \( a_K(T) \geq \varepsilon. \)

**Proof.** Let \( \lambda \in (0, 1). \) From the definition of \( A(I_k) \) we see that there exists \( \phi_k \in L^\infty(I_k) \) with \( \| \phi_k \|_{\infty, I_k} = 1 \) and such that

\[
\inf_{\alpha \in \mathbb{R}} \| T\phi_k - \alpha v \|_{\infty, I_k} > \lambda A(I_k) \geq \lambda \varepsilon.
\]

Set \( \phi_k(x) = 0 \) for \( x \notin I_k. \) Let \( P : L^\infty(a, b) \to L^\infty(a, b) \) be bounded and rank \( P \leq K - 1. \) Then there are constants \( \phi_0, \ldots, \phi_{K-1}, \) not all zero, such that

\[
P\left(\sum_{k=0}^{K-1} \lambda_k \phi_k \right) = 0.
\]

Put \( \phi = \sum_{k=0}^{K-1} \lambda_k \phi_k. \) Then

\[
\| T\phi - Pf \|_{\infty} = \| T\phi \|_{\infty}
\]

\[
\geq \sup_{k \in \{0, 1, \ldots, K-1\}} \| T\phi_k + \alpha_k v \|_{\infty, I_k}
\]

\[
\geq \sup_{k \in \{0, 1, \ldots, K-1\}} \inf_{\alpha \in \mathbb{R}} |\lambda_k| \| T\phi_k - \alpha v \|_{\infty, I_k}
\]

\[
\geq \sup_{k \in \{0, 1, \ldots, K-1\}} |\lambda_k| \| \phi_k \|_{\infty, I_k}
\]

\[
\geq \sup_{k \in \{0, 1, \ldots, K-1\}} |\lambda_k| \varepsilon = \varepsilon \lambda \| \phi \|_{\infty}
\]

\[
\geq 0.
\]
by (3.1). This implies that $a_K(T) \geq \lambda \varepsilon$, whence the result since $\lambda \in (0,1)$ is arbitrary.

**Corollary 3.3.** Suppose that $T$ is compact (see Proposition 2.2). Then, for $\varepsilon \in (0, A(a,b))$,

$$
\phi_{M(\varepsilon)+1}(T) \leq 2\varepsilon, \quad a_{M(\varepsilon/2)+1}(T) > \varepsilon,
$$

where $M_\varepsilon \equiv M((a,b),\varepsilon)$ is defined in (2.7) and $[\cdot]$ denotes integer part.

**Proof.** This is an immediate consequence of Lemmas 3.1 and 3.2. 

**4. Local asymptotic results.** We need some preliminary results and the functions $v_\varepsilon$ mentioned in §1, namely

$$
v_\varepsilon(x) := \lim_{\varepsilon \to 0^+} \|v\|_{\infty,(x-\varepsilon,x+\varepsilon)}
$$

for $x \in (a,b)$.

**Lemma 4.1.** For any $I \subseteq (a,b)$, we have $J(I;u,v) = J(I;u,v_\varepsilon)$ and $A(I;u,v) = A(I;u,v_\varepsilon)$, where $J(I;u,v)$ and $A(I;u,v)$ are the functions defined in (2.3) and (2.6) respectively.

**Proof.** For any continuous function $\phi$, it is readily shown that $\|v_\varepsilon \phi\|_{\infty,I} = \|v \phi\|_{\infty,I}$, and this fact yields the lemma.

**Lemma 4.2.** Let $I \subseteq (a,b)$, and let $\theta_n = \{I_n^{(n)}\}_{n=1}^{(n)}$ be a partition of $I$ by intervals $I_n^{(n)}$ such that each $I_n^{(n)}$ is a subinterval of some $x_n \in \theta_n$, and $|I_n^{(n)}| \to 0$ as $n \to \infty$. Define

$$
v_\varepsilon^n(t) := \sum_{i=1}^{i=n} \chi_{I_i(t)}(t)c_\varepsilon^n, \quad c_\varepsilon^n := \|v_\varepsilon\|_{\infty,I^n}.
$$

Then for a.e. $t \in I$,

(i) $\|v_\varepsilon\|_{\infty,I} \geq v_\varepsilon^n(t) \geq v_\varepsilon(t)$,

(ii) $v_\varepsilon^n(t) \searrow v_\varepsilon(t)$ as $n \to \infty$,

(iii) $\lim_{n \to \infty} \int_I u(t)\|v_\varepsilon^n(t) - v_\varepsilon(t)\|dt = 0$.

**Proof.** Since $v_\varepsilon$ is upper semi-continuous and bounded, it is known that it can be approximated from above by a decreasing sequence of step functions. However, we shall give a proof of the lemma for completeness and subsequent reference.

If $t \in \text{int } I^n$, the interior of $I^n$, then $v_\varepsilon^n(t) = \|v_\varepsilon\|_{\infty,I^n}$ satisfies

$$
v_\varepsilon(t) \leq v_\varepsilon^n(t) \leq \|v_\varepsilon\|_{\infty,I^n}.
$$

This establishes (i), the exceptional set being $S = \bigcup_{n \in \mathbb{N}} S_n$, where $S_n$ is the set of end-points of the intervals $I_n^{(n+1)} \subseteq \theta_n$. If $t \in \text{int } I_n^{(n+1)} \subseteq \theta_n$, say, we have $c_\varepsilon^{n+1} \leq c_\varepsilon^n$ and so $v_\varepsilon^{n+1}(t) \leq v_\varepsilon^n(t)$ for $t \in I \setminus S$. Also, if $t \in \text{int } I_n^{(n)}$, we have $v_\varepsilon^n(t) = \|v_\varepsilon\|_{\infty,I_n^{(n)}} = \|v\|_{\infty,I_n^{(n)}} \geq v(t)$ as observed in the proof of Lemma 4.1. Moreover, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$
v_\varepsilon(t) > \|v\|_{\infty,I \setminus [t-\varepsilon_0,t+\varepsilon_0]} - \delta.
$$

Now choose $N$ such that for all $n \geq N$,

$$
t \in \text{int } I_n^{(n)} \subseteq (t-\varepsilon_0,t+\varepsilon_0).
$$

Then we have, for all $n \geq N$,

$$
0 < v_\varepsilon^n(t) - v_\varepsilon(t) < \delta
$$

and hence $v_\varepsilon^n(t) \to v_\varepsilon(t)$ for all $t \in I \setminus S$.

Finally, (iii) follows by the dominated convergence theorem since $u \in L^1(I)$ and $\|v_\varepsilon^n\|_{\infty,I} = \|v_\varepsilon\|_{\infty,I} = \|v\|_{\infty,I} < \infty$.

**Lemma 4.3.** Let $u,v$ be constant on $I \subseteq (a,b)$. Then

$$
A(I) = \frac{1}{2}\|u\|_{I} + \|v\|_{I}.
$$

**Proof.** We have, if $I = (c,d)$,

$$
A(I) = \|u\|_{I} \inf_{\alpha} \|x - c - \alpha\|_{\infty,I} = \|v\|_{I} \|x - c - \frac{1}{2}(d-c)\|_{\infty,I} = \frac{1}{2}\|u\|_{I} + \|v\|_{I}.
$$

Let $f \in L^\infty(I)$ and set $F(x) = \int_a^x f dt$. Then there exist $x_0,x_1 \in [c,d]$ such that

$$
F(x_0) \leq F(x) \leq F(x_1), \quad x \in [a,b],
$$

and hence

$$
\inf_{\alpha} \|F - \alpha\|_{\infty,I} \leq \|F - \frac{1}{2}(F(x_0) + F(x_1))\|_{\infty,I} = \frac{1}{2}\|F(x_1) - F(x_0)\|_{\infty,I} = \frac{1}{2}\int_{x_0}^{x_1} f dt.
$$

This yields

$$
A(I) \leq \sup_{\|f\|_{\infty,I} = 1} \left\{ \frac{1}{2}\int_{x_0}^{x_1} f dt \right\} \leq \frac{1}{2}\|I\|
$$

and the lemma is proved.

**Lemma 4.4.** Let $I \subseteq (a,b)$ and $u_1,u_2 \in L^1(I)$. Then

$$
|A(I;u_1,v) - A(I;u_2,v)| \leq \|u_1 - u_2\|_{1,I} \|v\|_{\infty,I}.
$$

**Proof.** We have

$$
|A(I;u_1,v) - A(I;u_2,v)| \leq \sup_{\|f\|_{\infty,I} = 1} \left| \inf_\alpha \inf_\beta \|v(x)f dt - \alpha\|_{\infty,I} - \inf_\alpha \inf_\beta \|v(x)f dt - \alpha\|_{\infty,I} \right|.
$$

Two-sided estimates
Suppose \( f \) is such that
\[
\inf_{\alpha} \left\| v(x) \left( \frac{\partial}{\partial a} u_1 f dt - \alpha \right) \right\|_{\infty,L} \geq \inf_{\alpha} \left\| v(x) \left( \frac{\partial}{\partial a} u_2 f dt - \alpha \right) \right\|_{\infty,L}.
\]

Given \( \varepsilon > 0 \) there exists \( \alpha_0 \in \mathbb{R} \) such that
\[
\inf_{e} \left\| v(x) \left( \frac{\partial}{\partial a} u_1 f dt - \alpha \right) \right\|_{\infty,L} > \left\| v(x) \left( \frac{\partial}{\partial a} u_2 f dt - \alpha_0 \right) \right\|_{\infty,L} - \varepsilon.
\]

Hence
\[
0 \leq \inf_{\alpha} \left\| v(x) \left( \frac{\partial}{\partial a} u_1 f dt - \alpha \right) \right\|_{\infty,L} - \inf_{\alpha} \left\| v(x) \left( \frac{\partial}{\partial a} u_2 f dt - \alpha_0 \right) \right\|_{\infty,L} + \varepsilon
\]
\[
\leq \left\| v(x) \left( \frac{\partial}{\partial a} u_1 f dt - \alpha_0 \right) \right\|_{\infty,L} - \left\| v(x) \left( \frac{\partial}{\partial a} u_2 f dt - \alpha_0 \right) \right\|_{\infty,L} + \varepsilon
\]
\[
\leq \left\| v(x) \left( u_1 - u_2 \right) f dt \right\|_{\infty,L} + \varepsilon
\]
\[
\leq \left\| v(x) \right\|_{\infty,L} \left\| u_1 - u_2 \right\|_{1,L} \left\| f \right\|_{\infty,L} + \varepsilon.
\]

This remains valid if the inequality (4.3) is reversed, and so
\[
|A(I; u_1, v) - A(I; u_2, v)| \leq \left\| v(x) \right\|_{\infty,L} \left\| u_1 - u_2 \right\|_{1,L} + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, the lemma is proved. \( \blacksquare \)

In the next lemma \( \psi \) denotes the non-increasing rearrangement of a function \( g \) on an interval \( I \): \( \psi \) is the generalised inverse of the non-increasing distribution function \( g_* \) of \( g \), namely
\[
(4.4) \quad \psi(x) := \inf \{ t \in I : g_*(t) \geq x \}
\]
where
\[
(4.5) \quad g_*(t) := \{ x \in I : g(x) \geq t \}.
\]

Note that since we have \( \geq \) in the definitions above, \( g_* \) and \( \psi \) are left-continuous functions.

**Lemma 4.5.** Let \( I \subset (a, b) \) and \( \gamma, \delta \in \mathbb{R} \) with \( \delta \geq v_0(t) \geq 0 \) on \( I \). Then
\[
(4.6) \quad A(I; \gamma, \delta) \geq A(I; \gamma, v_0) \geq \frac{1}{2} \gamma \| (\psi \psi) \|_{\infty,(0,|I|)}.
\]

**Proof.** The first inequality in (4.6) is obvious. The set
\[
M_\delta := \{ y \in I : v_0(y) \geq \delta \}
\]
is relatively closed in \( I \). For if \( \{ y_n \} \subset M_\delta \) and \( y_n \rightarrow y \in I \) as \( n \rightarrow \infty \), then given \( \varepsilon > 0 \) there exists \( N \) such that \( (y - \varepsilon, y + \varepsilon) \supset (y_n - \frac{1}{2} \varepsilon, y_n + \frac{1}{2} \varepsilon) \) for \( n > N \). Hence
\[
\| v \|_{\infty,(y - \varepsilon, y + \varepsilon)} \geq \| v \|_{\infty,(y_n - \frac{1}{2} \varepsilon, y_n + \frac{1}{2} \varepsilon)} \geq \| v \|_{y_0} \geq \beta
\]
whence \( v_0(y) \geq \beta \) and \( y \in M_\beta \). From the observed left continuity of (4.4) and (4.5), we have
\[
\| (\psi \psi) \|_{\infty,(0,|I|)} = \max_{t \in (0,|I|)} \| (\psi \psi)(t) \| = \| (\psi \psi)(t_0) \| t_0
\]
for some \( t_0 \in (0,|I|) \), and there exists \( \beta > 0 \) such that \( |M_\beta| = t_0 \). Choose the optimal \( \alpha_0, \alpha_0 \) such that \( M_\beta \subset [\alpha_0, \alpha_0] \subset I \). Then, with \( I = (c, d) \),
\[
A(I; \gamma, \delta) \geq \| \gamma \|_{\infty,L} \| (\psi \psi)(t) \|_{\infty,L} + \varepsilon
\]
\[
\geq \| \gamma \|_{\infty,L} \| \psi \psi \|_{1,L} \| f \|_{\infty,L} + \varepsilon
\]
\[
\geq \frac{1}{2} \beta \| \gamma \|_{\infty,L} \| \psi \psi \|_{1,L} \| f \|_{\infty,L} + \varepsilon.
\]

The lemma is therefore proved. \( \blacksquare \)

**Lemma 4.6.** Let \( I \subset (a, b) \) and \( \gamma, \delta \in \mathbb{R} \) with \( \delta \geq v_0(t) \geq 0 \) on \( I \). Then, for any \( \alpha > 1 \),
\[
(4.7) \quad A(I; \gamma, \delta) - A(I; \gamma, v_0) \leq \frac{\alpha}{2} \int_I \| (\psi \psi)(t) \| dt + \frac{\delta |I|}{2 \alpha}.
\]

**Proof.** We first observe that
\[
(4.8) \quad (\psi \psi)(t) \geq v_0(t) := \left( \frac{\delta - V \alpha}{\gamma |I|} \right) \chi_{(0,|I|)}.
\]
where \( V = |\gamma| \int_I (\psi \psi)(t) dt \). For, with \( S := \{ x : v_0(x) < \delta - V \alpha / (\gamma |I|) \} \),
\[
V \gamma I > \int_S (\delta - \delta + V \alpha / |I|) dt = \frac{V \alpha}{|I|} |S|,
\]
which implies that
\[
\left\{ x : v_0(x) > \delta - V \alpha / |I| \right\} \geq |I| - \frac{|I|}{\alpha}
\]
and hence (4.8). Note that (4.8) is trivially true if \( \delta - V \alpha / (\gamma |I|) < 0 \). On
using (4.1) and (4.6),

\[
0 \leq A(I; \gamma, \delta) - A(I; \gamma, v_\eta) \leq \frac{1}{2} |\gamma| |\delta| I - \frac{1}{2} |\gamma| \|v_\eta \chi_I^*(t)\|_{\infty,(0,|I|)} I
\]

\[
\leq \frac{1}{2} |\gamma| |\delta| I - \frac{1}{2} \max_{(0,|I|)} (tv_0(t))
\]

\[
= \frac{1}{2} |\gamma| |\delta| I - \frac{1}{2} |\gamma| \left( \delta - \frac{V\alpha}{|\gamma| I} \right) \left( |I| - \frac{|I|}{\alpha} \right)
\]

\[
= \frac{\alpha V}{2} + \frac{|\gamma| |\delta| I}{2\alpha} - \frac{V}{2}
\]

\[
\leq \frac{\alpha}{2} \int |\gamma| (\delta - v_\eta(t)) dt + \frac{|I|}{2\alpha} |\gamma| |\delta|,
\]

which is (4.7). \( \square \)

**Theorem 4.7.** For any \( I \subset (a, b) \),

\[
\frac{1}{2} \int_I |u(t)| v_\eta(t) dt \leq \liminf_{\varepsilon \to 0^+} \varepsilon M(I, \varepsilon)
\]

\[
\leq \limsup_{\varepsilon \to 0^+} \varepsilon M(I, \varepsilon) \leq \int_I |u(t)| v_\eta(t) dt.
\]

**Proof.** On using Lemma 4.2, we infer that for each \( \eta > 0 \) there exist step functions \( u_\eta, v_\eta \) on \( I \) such that

\[
\|u - u_\eta\|_{1, I} < \eta, \quad \int_I |u(t)| (v_\eta(t) - v_\eta(t)) dt < \eta
\]

and

\[
\|v_\eta\|_{\infty, I} \geq v_\eta(t) \geq v_\eta(t)
\]

on \( I \). We may assume that

\[
u_\eta = \sum_{j=1}^{m} \xi_j \chi_{W(j)}, \quad v_\eta = \sum_{j=1}^{m} \eta_j \chi_{W(j)},
\]

where the \( W(j) \) are disjoint subintervals of \( I \), and \( \eta_j \geq 0 \).

Let \( \varepsilon > 0, M \equiv M(I, \varepsilon) \), and let \( c_k \equiv c_k(\varepsilon), k = 1, \ldots, M+1 \), be the endpoints of the intervals in (2.7): with \( I = [c, d] \) and \( I_k \equiv I_k(\varepsilon) = [c_k, c_{k+1}] \), we have \( c = c_1 < < c_{M+1} = d \) and

\[
A(I_k) \equiv A(I_k; u, v) \leq \varepsilon, \quad k = 1, \ldots, M,
\]

\[
A(I_k \cup I_{k+1}) > \varepsilon, \quad k = 1, \ldots, M - 1.
\]

Then

\[
(4.10) \quad \left| \int_I |u(t)| v_\eta(t) dt - \int_I |u(t)| v_\eta(t) dt \right|
\]

\[
\leq \int_I |u(t)| (v_\eta(t) - v_\eta(t)) dt + \int_I |u(t)| (v_\eta(t) - v_\eta(t)) dt
\]

\[
< \eta(1 + \|v_\eta\|_{\infty, I}) \leq \eta(1 + \|v_\eta\|_{\infty, I}).
\]

Next, let \( K := \{ k : \text{there exist } j \text{ such that } I_{2k} \cup I_{2k+1} \subset W(j) \} \). Then

\[
\#K \geq \frac{M}{2} - 2m \geq \frac{M}{2} - 1 - m, \text{ and, by Lemmas 4.4 and 4.6,}
\]

\[
\left( \frac{M}{2} - 1 - 2m \right) \varepsilon \leq \sum_{k \in K} A(I_{2k} \cup I_{2k+1}; u, v)
\]

\[
\leq \sum_{k \in K} \{ A(I_{2k} \cup I_{2k+1}; u_\eta, v_\eta)
\]

\[
+ A(I_{2k} \cup I_{2k+1}; u, v) - A(I_{2k} \cup I_{2k+1}; u_\eta, v_\eta)
\]

\[
+ (A(I_{2k} \cup I_{2k+1}; u, v) - A(I_{2k} \cup I_{2k+1}; u_\eta, v_\eta))
\]

\[
\leq \frac{1}{2} \sum_j |\xi_j| |v_j| W(j)
\]

\[
+ \sum_j \left\{ |u - u_\eta|_{1, W(j)} \|v_\eta\|_{\infty, W(j)} + \frac{\alpha}{2} \int_I |v_j| (v_\eta - v_\eta) dt + \frac{|\xi_j| |v_j|}{2\alpha} |W(j)| \right\}
\]

\[
\leq \frac{1}{2} \int_I |u_\eta| v_\eta dt + |u - u_\eta|_{1, I} \|v_\eta\|_{\infty, I}
\]

\[
+ \frac{\alpha}{2} \int_I |u_\eta| (v_\eta - v_\eta) dt + \frac{1}{2|\xi_j|} |u_\eta| v_\eta dt
\]

\[
\leq \frac{1}{2} \int_I |u_\eta| v_\eta dt + K \left( \alpha \eta + \frac{1}{\alpha} \right)
\]

\[
\leq \frac{1}{2} \int_I |u(t)| v_\eta(t) dt + K \left( \alpha \eta + \frac{1}{\alpha} \right)
\]

by (4.10), for some constant \( K \) independent of \( \varepsilon \). We therefore conclude that

\[
\limsup_{\varepsilon \to 0^+} \varepsilon M(I, \varepsilon) \leq \int_I |u(t)| v_\eta(t) dt + K \left( \alpha \eta + \frac{1}{\alpha} \right)
\]

and the right-hand inequality in (4.9) follows since \( \eta > 0 \) and \( \alpha > 1 \) are arbitrary.
For the left-hand inequality in (4.9), we add the endpoints of the intervals \( W(j), j = 1, \ldots, m \), to the \( c_k, k = 1, \ldots, M - 1 \), to form the partition \( c = e_1 < \cdots < e_n = d \), say, where \( n \leq M + 1 + m \). Note that each interval \( J_i := [e_i, e_{i+1}] \) is a subinterval of some \( W(j) \) and hence \( u_{e_i}, v_{e_i} \) have constant values on each \( J_i \). We again use Lemmas 4.3, 4.4 and 4.6 to get

\[
\frac{1}{2} \int \left| u_{e_i} v_{e_i} \right| \, dt = \sum_{j=1}^{m} \sum_{J_i \subseteq W(j)} A(J_i; u_{e_i}, v_{e_i}) \\
\leq \sum_{i=1}^{n} \left\{ A(J_i; u, v) + \| u - u_{e_i} \|_{L_1} \| v \|_{L_\infty, J_i} \right\} \\
+ \frac{\alpha}{2} \int_{J_i} \left| u_{e_i} v_{e_i} \right| \, dt + \frac{\alpha}{2} \int_{J_i} \left| u_{e_i} v_{e_i} \right| \, dt \\
\leq (M + 1 + m) \varepsilon + K \left( \alpha \eta + \frac{1}{\alpha} \right).
\]

Hence, from (4.10),

\[
\frac{1}{2} \int \left| u(t) v(t) \right| \, dt \leq (M + 1 + m) \varepsilon + K \left( \alpha \eta + \frac{1}{\alpha} \right)
\]

and the left-hand inequality in (4.9) follows.

5. The main result. With \( U(x) := \int_{\xi}^{x} |u(t)| \, dt \), we define \( \xi_k \in \mathbb{R}^+ \) by

\[
U(\xi_k) = 2^k;
\]

if \( u \notin L^1(a, b) \), then \( k \) may be any integer, but if \( u \in L^1(a, b) \), then \( 2^k \leq \| u \|_1 \).

For each admissible \( k \) we set

\[
\sigma_k := \| u \|_{L_\infty, Z_k}, \quad Z_k = (\xi_k, \xi_{k+1}),
\]

so that

\[
2^k \| u \|_{L_\infty, Z_k} \leq \sigma_k \leq 2^{k+1} \| u \|_{L_\infty, Z_k}.
\]

For non-admissible \( k \) we set \( \sigma_k = 0 \). The sequence \( \{ \sigma_k \} \) is the analogue of that defined in [3, 33], which in turn was motivated by a similar sequence introduced in [5].

The following technical lemma has a central role in this section.

**Lemma 5.1.** Let \( k_0, k_1, k_2 \in \mathbb{Z} \) with \( k_0 < k_1 < k_2 \), and let \( I_j = (a_j, b_j) \) \((j = 0, 1, \ldots, l)\) be intervals in \( (a, b) \) which are non-overlapping and such that \( I_j \subset Z_{k_2} \) \((j = 1, \ldots, l)\), \( a_0 \in Z_{k_0}, b_0 \in Z_{k_2} \). Let \( x_j \in I_j \) \((j = 0, 1, \ldots, l)\) and \( x_0 \in Z_{k_1} \). Then, if \( \alpha \geq 1 \),

\[
S := \sum_{j=0}^{l} \left( \int_{a_j}^{x_j} |u(t)| \, dt \right)^\alpha \| v \|_{L_\infty, (x_j, x_j + 1)}^\alpha \leq (2^\alpha + 1) \max_{k_3 \leq n \leq k_2} \sigma_n^\alpha.
\]

**Proof:** On using Jensen's inequality, we have

\[
S \leq \left\{ \int_{\xi_{k+1}}^{\xi_{k+2}} |u(t)| \, dt \right\}^{\alpha} \| v \|_{L_\infty, (\xi_k, \xi_{k+2})}^{\alpha} + \sum_{j=1}^{l} \left( \int_{I_j} |u(t)| \, dt \right)^\alpha \| v \|_{L_\infty, I_j}^{\alpha}
\]

\[
\leq \left( 2^{k+1 - 2k} \max_{k_1 \leq n \leq k_2} \sigma_n^2 \right)^\alpha + \left( \int_{\xi_{k+2}}^{\xi_{k+3}} |u(t)| \, dt \right)^\alpha \| v \|_{L_\infty, Z_{k_2}}^{\alpha} \text{ by (5.3),}
\]

\[
\leq \left( 2^{k_2 + 2k_3} \sigma_{k_2} \right)^\alpha + \left( \sigma_{k_3} \right)^\alpha,
\]

whence (5.4).

**Lemma 5.2.** The quantity \( J(a, b) \) defined in (2.3) satisfies

\[
\frac{1}{2} J(a, b) \leq \sup_k \sigma_k \leq 2 J(a, b).
\]

**Proof:** From (2.4) and Lemma 5.1,

\[
J(a, b) \leq 3 \sup_k \sigma_k.
\]

Also,

\[
\sigma_k \leq 2^{k+1} \| u \|_{L_\infty, Z_k} \leq 2 \int |u(t)| \, dt \| v \|_{L_\infty, (\xi_k, b)} \leq 2J(a, b).
\]

**Corollary 5.3.** The operator \( T : L^\infty(a, b) \to L^\infty(a, b) \) is bounded if and only if the sequence \( \{ \sigma_k \} \) is bounded, in which case their norms are equivalent:

\[
||T|| \asymp ||\{ \sigma_k \}||.
\]

Also, \( T \) is compact if and only if \( \lim_{k \to \infty} \sigma_k = 0 \).

**Proof:** The first part is an immediate consequence of Proposition 2.2 and Lemma 5.2. We also have from Lemma 5.2, as in its proof,

\[
\frac{1}{2} J(a, \xi_{k_3}) \leq \max_{n \leq k_2} \sigma_n \leq 2J(a, \xi_{k_2+1})
\]

and

\[
\frac{1}{2} J(\xi_{k_2}, b) \leq \max_{n \geq k_0} \sigma_n \leq 2J(\xi_{k_2-1}, b).
\]

Since \( \xi_{k_2} \to a \) if and only if \( k_2 \to -\infty \) and \( \xi_{k_0} \) tends to \( b \) if and only if \( k_0 \) tends to \( \infty \) in the case \( u \notin L^1(a, b) \) and otherwise to the largest admissible value of \( k \) in the definition of \( \sigma_k \), the corollary follows.
The main result is

**Theorem 5.4.** Suppose that (2.1) and (2.2) are satisfied, T is compact, and that \( \sum_{n \in \mathbb{Z}} \sigma_n \) is convergent. Then

\[
\frac{1}{4} \int_{a}^{b} |u(t)| v_s(t) \, dt \leq \liminf_{n \to \infty} n \alpha_n(T) \leq \limsup_{n \to \infty} n \alpha_n(T) \leq 2 \int_{a}^{b} |u(t)| v_s(t) \, dt.
\]

**Proof.** Let \( I = [c, d] \subset (a, b) \) and suppose that \( c \in [\xi_0, \xi_{k_0+1}] \) and \( d \in [\xi_{k_1}, \xi_{k_1+1}] \). With \( I_j^c, j = 1, \ldots, M(e) \), the covering of \((a, b)\) in (2.7), where \( M(e) \equiv M((a, b), e) \), let

\[
m_0(e) = \#\{j : I_j^c \subset [a, c]\}, \quad m_1(e) = \#\{j : I_j^c \subset [a, d]\}.
\]

Then

\[
m_1(e) - m_0(e) \leq M(I, e) + 1
\]

and

\[
\frac{e}{2} (M(e) - M(I, e) - 9) \leq \varepsilon (m_0(e)/2 + [M(e)/2 - m_1(e)/2] - 2)
\]

\[
\leq \sum_{j=1}^{m_0(e)/2} A(I_{2j-1} \cup I_{2j}; u, v) + \sum_{j=m_1(e)/2 + 2}^{[M(e)/2]} A(I_{2j-1} \cup I_{2j}; u, v)
\]

\[
\leq \sum_{j=1}^{m_0(e)/2} J(I_{2j-1} \cup I_{2j}; u, v) + \sum_{j=m_1(e)/2 + 2}^{[M(e)/2]} J(I_{2j-1} \cup I_{2j}; u, v)
\]

\[
\leq 3 \sum_{n \leq k_0} \nu_n + 3 \sum_{n \geq k_1} \nu_n
\]

on using (2.9) and (5.5).

It follows from Theorem 4.7 that

\[
\limsup_{\varepsilon \to 0^+} M(e) \leq \int_{\xi_{k_0}}^{\xi_{k_1+1}} |u(t)| v_s(t) \, dt + 3 \left( \sum_{n \leq k_0} \sigma_n + \sum_{n \geq k_1} \sigma_n \right),
\]

which yields

\[
\limsup_{\varepsilon \to 0^+} M(e) \leq \int_{a}^{b} |u(t)| v_s(t) \, dt.
\]

On setting \( n = M(e) + 1 \) in Corollary 3.3, we get \( \varepsilon \geq \frac{1}{2} \nu_n(T) \) and hence

\[
\limsup_{n \to \infty} n \alpha_n(T) \leq \frac{b}{a} \int_{a}^{b} |u(t)| v_s(t) \, dt.
\]

Similarly, from Theorem 4.7,

\[
\liminf_{\varepsilon \to 0^+} M(e) \geq \frac{1}{2} \int_{a}^{b} |u(t)| v_s(T) \, dt
\]

and from Corollary 3.3,

\[
\liminf_{n \to \infty} n \alpha_n(T) \geq \frac{1}{4} \int_{a}^{b} |u(t)| v_s(T) \, dt.
\]

**6.15 and weak-1^q estimates.** In this section we show that the sequences \( \{a_n(T)\}_{n \in \mathbb{N}} \) and \( \{\sigma_n\}_{n \in \mathbb{N}} \) belong to \( l_1^q \) and weak-1^q sequence spaces with the same exponent \( q \), and have equivalent norms. We first need some preparatory results.

**Lemma 6.1.** Let \( I = [c, d] \subset (a, b) \) and, for \( \varepsilon > 0 \), suppose that

\[
\sigma(e) := \{k \in \mathbb{Z} : Z_k \subset I, \, \sigma_k > \varepsilon\}
\]

has at least 4 distinct elements. Then \( A(I) > \varepsilon/8 \).

**Proof.** Let \( Z_{k_1}, Z_{k_2}, Z_{k_3}, Z_{k_4} \), with \( k_1 < k_2 < k_3 < k_4 \), be distinct members of \( \sigma(e) \), and set \( I_1 = (\xi_{k_1}, \xi_{k_2}), I_2 = (\xi_{k_3}, \xi_{k_4}) \). Then, with \( f_0 = \chi_{I_1} + \chi_{I_2} \),

\[
A(I) \geq \inf_{\alpha} \left\| v(x) \left( \int_{c}^{x} |u(t)| f_0(t) \, dt - \alpha \right) \right\|_{\infty, I}
\]

\[
\geq \sup_{\alpha} \max \left\{ \left\| v \right\|_{\infty, Z_{k_1}} \left\| \int_{I_1} |u(t)| \, dt - \alpha \right\|_{L^1(I_1) \cap L^2(I_1, I_2)}, \left\| v \right\|_{\infty, Z_{k_2}} \left\| \int_{I_1 \cup I_2} |u(t)| \, dt - \alpha \right\|_{L^1(I_1 \cup I_2)} \right\}
\]

\[
\geq \frac{e}{2 k_4 + 1}\left(\frac{2^{k_2} - 2^{k_1} - \alpha}{2^{k_4 + 1}}\right) \geq \frac{e}{8}.
\]

**Lemma 6.2.** Let \( \varepsilon > 0 \) and \( M(e) = M((a, b), e) \). Then

\[
\#\{k \in \mathbb{Z} : \sigma_k > 8\varepsilon\} \leq 5M(e) + 3.
\]
Proof. Clearly, with \( I_i = (c_i, c_{i+1}) \) the intervals in (2.7) when \( I = (a, b) \),
\[
\# \{ k \in \mathbb{Z} : c_i \in \bar{Z}_k \text{ for some } i \in \{1, \ldots, M(\varepsilon)\} \} \leq 2M(\varepsilon).
\]
Also, for every \( k \in \mathbb{Z} \) not included in the above set, we have \( \bar{Z}_k \subset I_i \) for some \( i \in \{1, \ldots, M(\varepsilon)\} \). Hence, by Lemma 6.1,
\[
\# \{ k \in \mathbb{Z} : \sigma_k > 8\varepsilon \} \leq 2M(\varepsilon) + 3M(\varepsilon) + 1 = 5M(\varepsilon) + 3.
\]

**Lemma 6.3.** For all \( t > 0 \),
\[
\# \{ k \in \mathbb{Z} : \sigma_k > t \} \leq 10\# \{ k \in \mathbb{N} : a_k(T) > t/8 \} + 23.
\]

*Proof.* By Corollary 3.3,
\[
\# \{ k \in \mathbb{N} : a_k(T) > \varepsilon \} \geq \frac{M(\varepsilon)}{2} - 2.
\]
Hence, by Lemma 6.2,
\[
\# \{ k \in \mathbb{Z} : \sigma_k > t \} \leq 5M(t/8) + 3 \leq \# \{ k \in \mathbb{N} : a_k(T) > t/8 \} + 23.
\]

**Lemma 6.4.** For all \( q > 0 \),
\[
\| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} \leq 10 \cdot 8^q \| \{ a_k(T) \} \|_{l^q(\mathbb{N})} + 23 \| \{ \sigma_k \} \|_{l^{\infty}(\mathbb{Z})}.
\]

*Proof.* Let \( \lambda = \| \{ \sigma_k \} \|_{l^{\infty}(\mathbb{Z})} \). Then, by Lemma 6.3,
\[
\| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} = \int_0^\lambda q \int_0^{t^{-1} \# \{ k \in \mathbb{Z} : \sigma_k > t \} } dt
\]
\[
\leq 10 \lambda \int_0^{q^{-1} \# \{ k \in \mathbb{N} : a_k(T) > t/8 \} } dt + 23 \lambda^q
\]
\[
\leq 10 \cdot 8^q \| \{ a_k(T) \} \|_{l^q(\mathbb{N})} + 23 \lambda^q.
\]

**Corollary 6.5.** For any \( q > 0 \) there exists a constant \( C > 0 \) such that
\[
\| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} \leq C \| \{ a_k(T) \} \|_{l^q(\mathbb{N})}.
\]

*Proof.* By (6.6),
\[
\| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} \leq C \| T \| = CA_1(TE) \leq C \| \{ a_k(T) \} \|_{l^q(\mathbb{N})}.
\]
The result then follows from Lemma 6.4.

**Theorem 6.6.** For \( q \in (1, \infty) \), we have \( \{ a_k(T) \} \in l^q(\mathbb{N}) \) if and only if \( \{ \sigma_k \} \in l^q(\mathbb{Z}) \), and
\[
\| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} \approx \| \{ a_k(T) \} \|_{l^q(\mathbb{N})}.
\]

*Proof.* Let \( I_i, i = 1, \ldots, N(\varepsilon) \), be the intervals in (2.8) with \( I = (a, b) \) and \( N(\varepsilon) = N((a, b), \varepsilon) \); note that in view of Lemma 2.1, we have \( J(I_i) = \varepsilon \). We group the intervals \( I_i \) into families \( F_j, j = 1, 2, \ldots \), such that each \( F_j \) consists of the maximal number of those intervals satisfying the hypothesis of Lemma 5.1; they lie within \( (\xi_k, \xi_{k+1}) \) for some \( k_0, k_2 \), and the next interval \( I_k \) intersects \( Z_{k_2+1} \). Hence, by Lemma 5.1, there is a positive constant \( c \) such that
\[
\varepsilon \# F_j \leq c \max_{k_0 \leq k \leq k_2} \sigma_n = c\sigma_{k_j},
\]
say. It follows that, with \( n_j = \lfloor c\sigma_{k_j}/\varepsilon \rfloor \),
\[
N(\varepsilon) = \sum_j \# F_j \leq \sum_j n_j \sum_{n_j + j n_j \geq n_j} 1 = \sum_{n_j = 1}^\infty \# \{ j : c\sigma_{k_j}/\varepsilon \geq n_j \} \leq \sum_{n_j = 1}^\infty \# \{ k : \sigma_k \geq n_j/\varepsilon \}.
\]
Thus, if \( \{ \sigma_k \} \in l^q(\mathbb{Z}) \) for some \( q \in (1, \infty) \), then
\[
q \int_0^{t^{-1} N(\varepsilon)} dt \leq q \sum_{n_j = 1}^\infty \int_0^{t^{-1} \# \{ k : \sigma_k > n_j/\varepsilon \} } dt
\]
\[
= q \int_0^{t^{-1} \# \{ k : \sigma_k > \varepsilon \} } dt
\]
\[
\leq \| \{ \sigma_k \} \|_{l^q(\mathbb{Z})}.
\]
where \( \preceq \) stands for less than or equal to a constant multiple of what follows. From Corollary 3.3, \( a_M(\varepsilon) + \| T \| \leq 2\varepsilon \) and so
\[
\# \{ k \in \mathbb{N} : a_k(T) > t \} \leq M(t/2) + 1 \leq N(t/2) + 1.
\]
This yields
\[
\| \{ a_k(T) \} \|_{l^q(\mathbb{N})} = \int_0^\infty q \int_0^{t^{-1} \# \{ k \in \mathbb{N} : a_k(T) > t \} } dt
\]
\[
\leq q \int_0^{t^{-1} \# \{ k \in \mathbb{N} : a_k(T) > t \} } dt
\]
\[
\leq \| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} + \| T \| ^q \leq \| \{ \sigma_k \} \|_{l^q(\mathbb{Z})}.
\]
by (6.6) and since \( \| T \| \leq \| \{ \sigma_k(T) \} \|_{l^{\infty}(\mathbb{Z})} \leq \| \{ \sigma_k \} \|_{l^q(\mathbb{Z})} \), by (6.6). The theorem follows from (6.4).

The final result in this section concerns the weak \( l^q \) spaces, which we denote by \( l_w^q \) \((l^q_{\infty}, \infty) \) in the Lorentz scale. Recall that \( l_w^q(\mathbb{Z}) \) is the space of sequences \( x = (x_k) \) such that
\[
\| x \|_{l_w^q(\mathbb{Z})} := \sup_{t > 0} \{ t \# \{ k \in \mathbb{Z} : \| x_k \| > t \} \}^{1/q} < \infty.
\]
The space \( l_w^q(\mathbb{N}) \) is defined analogously.
Theorem 6.7. For $q \in (1, \infty)$, we have \( \{a_k(T)\} \in l^q(\mathbb{N}) \) if and only if \( \{\sigma_k\} \in l^q(\mathbb{Z}) \), and
\[
\|\{\sigma_k\}\|_{l^q(\mathbb{Z})} \gg \|\{a_k(T)\}\|_{l^q(\mathbb{N})}.
\]

Proof. Suppose \( \{\sigma_k\} \in l^q(\mathbb{Z}) \). From Corollary 3.3 and (6.5),
\[
\|\{a_k(T)\}\|_{l^q(\mathbb{N})} \leq \sup_{t>0} \{t^q M(t)\} \leq \sup_{t>0} \{t^q N(t)\} \leq \sum_{n=1}^{\infty} t^q \# \{k : \sigma_k \geq nt/c\} \leq \sum_{n=1}^{\infty} \|\{\sigma_k\}\|_{l^q(\mathbb{Z})}^q (c/n)^q \leq \|\{\sigma_k\}\|_{l^q(\mathbb{Z})}^q.
\]

Now suppose that \( \{a_k(T)\} \in l^q(\mathbb{N}) \). From Lemma 6.3,
\[
\sup_{t>0} \{t^q \# \{k \in \mathbb{Z} : \sigma_k > t\}\} \leq \sup_{t>0} \{t^q \# \{k \in \mathbb{N} : a_k(T) > t/8\} + 1\}.
\]

Since
\[
\# \{k \in \mathbb{N} : a_k(T) > t/8\} \geq \frac{M(t/8)}{2} - 2 \geq 1
\]
for sufficiently small \( t \), we conclude that
\[
\sup_{t>0} \{t^q \# \{k \in \mathbb{Z} : \sigma_k > t\}\} \leq \sup_{t>0} \{t^q \# \{k \in \mathbb{N} : a_k(T) > t/8\}\}.
\]
This implies that \( \{\sigma_k\} \in l^q(\mathbb{Z}) \) and \( \|\{\sigma_k\}\|_{l^q(\mathbb{Z})} \leq \|\{a_k(T)\}\|_{l^q(\mathbb{N})} \). The theorem is therefore proved. \( \blacksquare \)

7. The operator \( T \) on \( L^1 \). In this case the assumptions (2.1) and (2.2) on \( u \) and \( v \) are replaced by
\[
(7.1) \quad u \in L^{\infty}(a, b), \quad v \in L^1(a, b),
\]
for all \( x \in (a, b) \). On setting \( a = -B, b = -A, \hat{f}(x) = f(-x) \), and similarly for \( u, v \) in (1.4), we see that
\[
T \hat{f}(x) = \hat{u}(x) \int_{-B}^{B} \hat{v}(t) \hat{f}(t) \, dt, \quad A \leq x \leq B.
\]
But this is the adjoint of the map \( S : L^{\infty}(A, B) \to L^{\infty}(A, B) \) defined by
\[
Sg(x) = \hat{u}(x) \int_{-A}^{A} \hat{v}(t) g(t) \, dt, \quad A \leq x \leq B.
\]
Hence, \( T \) and \( S \) have the same norms and their approximation numbers are equal if one, and hence both, are compact (see [1; Proposition II.2.5]). The results for \( T : L^1(a, b) \to L^1(a, b) \) therefore follow from those proved for the \( L^{\infty}(a, b) \) case on interchanging \( u \) and \( v \). Before stating the results, we need some new terminology.

Let \( \eta_k \in \mathbb{R}^+ \) be defined by
\[
V(x) := \int_{-\infty}^{x} |v(t)| \, dt, \quad V(\eta_k) = 2^k,
\]
where \( k \in \mathbb{Z} \) if \( v \in L^1(a, b) \), but otherwise \( 2^k \leq \|v\|_1 \). Set
\[
\zeta_k := \|uv\|_{\infty, W_k}, \quad W_k = (\eta_k, \eta_{k+1}),
\]
with \( \zeta_k = 0 \) if \( v \notin L^1(a, b) \) and \( 2^k > \|v\|_1 \).

Theorem 7.1. Suppose that (7.1) and (7.2) are satisfied. Then
(i) \( T \) in (1.4), as a map from \( L^1(a, b) \) into \( L^1(a, b) \), is bounded if and only if \( \{\zeta_k\} \in l^\infty(\mathbb{Z}) \), in which case
\[
\|T\| = \|\{\zeta_k\}\|_{l^\infty(\mathbb{Z})}.
\]
(ii) \( T \) is compact if and only if \( \lim_{k \to \infty} \zeta_k = 0 \);
(iii) if \( \{\zeta_k\} \in l^1(\mathbb{Z}) \) then
\[
\frac{1}{a} \int_{a}^{b} u_s(t) |v(t)| \, dt \leq \liminf_{n \to \infty} n a_n(T) \leq \limsup_{n \to \infty} n a_n(T) \leq \frac{1}{a} \int_{a}^{b} u_s(t) |v(t)| \, dt;
\]
(iv) for \( q \in (1, \infty) \), we have \( \{a_k(T)\} \in l^q(\mathbb{N}) \) if and only if \( \{\zeta_k\} \in l^q(\mathbb{Z}) \) and
\[
\|\{\zeta_k\}\|_{l^q(\mathbb{Z})} \propto \|\{a_k(T)\}\|_{l^q(\mathbb{N})}.
\]
(v) for \( q \in (1, \infty) \), we have \( \{a_k(T)\} \in l^q(\mathbb{N}) \) if and only if \( \{\zeta_k\} \in l^q(\mathbb{Z}) \) and
\[
\|\{\zeta_k\}\|_{l^q(\mathbb{Z})} \propto \|\{a_k(T)\}\|_{l^q(\mathbb{N})}.
\]

Remark 7.2. Let \( M \) be a dense subset of \((0, 1)\) with measure \( |M| = \alpha < 1 \) and let \( u = 1, v = \chi_M \). Then \( u_s = 1, v_s = 1 = 1 \) on \((0, 1)\) and so
\[
\|v\|_{\infty, (a, 1)} = \|v_s\|_{\infty, (a, 1)} = \|v - u_s\|_{\infty, (a, 1)}
\]
for any \( x \in (0, 1) \). Since
\[
\|T_{u,v} L^\infty(0,1) \to L^\infty(0,1)\| = \sup_{0 < \epsilon < 1} \left\{ \epsilon \int_{0}^{1} dt \|v\|_{\infty, (a, 1)} \right\}
\]
[see (6)], where \( T_{u,v} \) denotes the operator in (1.4), it follows that
\[
\|T_{u,v}\| = \|T_{u,0}\| = \|T_{0,v} - T_{u,v} - u\|,
\]
for the operator norms from \( L^\infty(0,1) \) to \( L^\infty(0,1) \). Also,
\[
\frac{1}{a} \int_{a}^{b} |u(t)v(t)| \, dt = |M| < 1 = \frac{1}{a} \int_{a}^{b} |u(t)v(t)| \, dt.
\]
The choice \( u = \chi_M, v = 1 \) gives an analogous example in the \( L^1(0, 1) \) case.
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References


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Corrigendum and addendum:
"On the axiomatic theory of spectrum II"

by
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Abstract. The main purpose of this paper is to correct the proof of Theorem 15 of [4], concerned with the stability of the class of quasi-Fredholm operators under finite rank perturbations, and to answer some open questions raised there.

Recall some notations and terminology from [4].

For closed subspaces $M, L$ of a Banach space $X$ we write $M \subseteq L$ ($M$ is essentially contained in $L$) if there is a finite-dimensional subspace $F \subset X$ such that $M \subseteq L + F$. Equivalently, $\dim M/(M \cap L) = \dim(M + L)/L < \infty$.

Similarly we write $M \subseteq M$ if $M \subseteq L$ and $L \subseteq M$.

For a (bounded linear) operator $T \in \mathcal{L}(X)$ write $R^\infty(T) = \cap_{n=0}^\infty \mathcal{R}(T^n)$ and $N^\infty(T) = \cup_{n=0}^\infty \mathcal{N}(T^n)$.

An operator $T \in \mathcal{L}(X)$ is called semiregular (essentially semiregular) if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$ ($N(T) \subseteq R^\infty(T)$, respectively). Further, $T$ is called quasi-Fredholm if there exists $d \geq 0$ such that $R(T^{d+1})$ is closed and $R(T) + N(T^d) = R(T) + N^\infty(T)$ (equivalently, $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$).

The proof of Theorem 15 of [4] relies on the following statement (where $d$ is the integer whose existence is postulated in the definition of quasi-Fredholm operators):

If $T$ is quasi-Fredholm and $F$ of rank 1 then $N(T) \cap R(T^d) \subseteq R^\infty(T+D)$.

This, however, need not be satisfied.

Counterexample. Let $H$ be the Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Define $T, F \in \mathcal{L}(H)$ by

\[ T e_1 = 0, \quad T e_n = e_{n-1} \quad (n \geq 2), \quad F e_2 = -e_1, \quad F e_n = 0 \quad (n \neq 2). \]

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