FINITE ELEMENT DISCRETIZATION
OF THE KURAMOTO–SIVASHINSKY EQUATION

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Abstract. We analyze semidiscrete and second-order in time fully discrete finite element methods for the Kuramoto–Sivashinsky equation.

1. Introduction. In this paper we study finite element approximations for the solution of the following periodic initial-value problem for the Kuramoto–Sivashinsky (KS) equation: For $T, \nu > 0$, we seek a real-valued function $u$ defined on $\mathbb{R} \times [0,T]$, 1-periodic in the first variable and satisfying

\begin{equation}
 u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0 \quad \text{in } \mathbb{R} \times [0,T]
\end{equation}

and

\begin{equation}
 u(\cdot,0) = u^0 \quad \text{in } \mathbb{R},
\end{equation}

where $u^0$ is a given 1-periodic function. We assume that (1.1)–(1.2) has a unique, sufficiently smooth solution (cf. [8], [17]).

The KS equation was derived independently by Kuramoto and Sivashinsky in the late 70’s and is related to turbulence phenomena in chemistry and combustion. It also arises in a variety of other physical problems such as plasma physics and two-phase flows in cylindrical geometries. For the mathematical theory and the physical significance of the KS equation as well as for related computational work we refer the reader to [7], [16], [3], [4], [17], [5], [6], [8], [9], [13], [14], [1] and the references therein; see also Temam [18] for an overview. In [1] the discretization of (1.1)–(1.2) by a Crank–Nicolson finite difference method and a linearization

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thereof by Newton’s method is studied. In the present paper we analyze a semidiscrete method and a second-order in time fully discrete finite element method. The discretization in space is based on the standard Galerkin method; for the time discretization the Crank–Nicolson scheme is used.

For \( m \in \mathbb{N} \) let \( H^m_{\text{per}} \) be the periodic Sobolev space of order \( m \), consisting of the 1-periodic elements of \( H^m_{\text{loc}}(\mathbb{R}) \). We denote by \( \| \cdot \|_m \) the norm over a period in \( H^m_{\text{per}} \), by \( \| \cdot \|_{L^2(0,1)} \) the norm in \( L^2(0,1) \), and by \( (\cdot, \cdot) \) the inner product in \( L^2(0,1) \).

A variational form of (1.1) is

\[
(1.3) \quad (u_t, v) + (uu_x, v) - (u_x, v') + \nu(u_{xx}, v'') = 0 \quad \forall v \in H^2_{\text{per}}.
\]

Taking \( v := u(\cdot, t) \) in (1.3) we obtain by periodicity

\[
(1.4) \quad \frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 = \|u_x(\cdot, t)\|^2 - \nu \|u_{xx}(\cdot, t)\|^2.
\]

Now, for \( v \in H^2_{\text{per}} \), \( \|v'\|^2 = -(v, v'') \), i.e.,

\[
(1.5) \quad \|v'\|^2 \leq \|v\|\|v''\|, \quad v \in H^2_{\text{per}}.
\]

Therefore,

\[
(1.6) \quad \|v'\|^2 \leq \nu \|v''\|^2 + \frac{1}{4\nu} \|v\|^2, \quad v \in H^2_{\text{per}},
\]

and (1.4) leads to

\[
\frac{d}{dt} \|u(\cdot, t)\|^2 \leq \frac{1}{2\nu} \|u(\cdot, t)\|^2,
\]

i.e.,

\[
(1.7) \quad \|u(\cdot, t)\| \leq \|u_0\| e^{t/(4\nu)}, \quad 0 \leq t \leq T.
\]

Moreover, using the well-known Wirtinger inequality

\[
(1.8) \quad \|v'\| \leq \frac{1}{2\pi} \|v''\|, \quad v \in H^2_{\text{per}},
\]

(cf. [12]), (1.4) yields

\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 \leq \left( \frac{1}{4\pi^2} - \nu \right) \|u_{xx}(\cdot, t)\|^2,
\]

and, consequently,

\[
(1.9) \quad \|u(\cdot, t)\| \leq \|u(\cdot, s)\|, \quad 0 \leq s \leq t \leq T, \quad \text{for } \nu \geq \frac{1}{4\pi^2}.
\]

We shall discretize (1.1)–(1.2) in space by the standard Galerkin method. To this end, let \( 0 = x_0 < x_1 < \ldots < x_J = 1 \) be a partition of \([0,1]\), \( h := \max_j (x_{j+1} - x_j) \), and \( h := \min_j (x_{j+1} - x_j) \). Setting \( x_{J+s} := x_s, \ j \in \mathbb{Z}, \ s = 0, \ldots, J-1 \), this partition is extended periodically to a partition of \( \mathbb{R} \). For integer \( r \geq 4 \), let \( S^r_h \) denote a space of continuously differentiable, 1-periodic splines of degree \( r - 1 \) in which approximations to the solution \( u(\cdot, t) \) of (1.1)–(1.2) will be sought for
$0 \leq t \leq T$. The following approximation property for the family $(S^r_h)_{0<h<1}$ is well known:

$$\inf_{\chi \in S^r_h} \sum_{j=0}^{2} h^j \|v - \chi\|_j \leq ch^s \|v\|_s, \quad v \in H^s_{per}, \quad 2 \leq s \leq r,$$

(cf., e.g., Schumaker [15], §8.1). Motivated by (1.3) we define the semidiscrete approximation $u_h(\cdot, t) \in S^r_h$, $0 \leq t \leq T$, to $u$ by

$$u_{ht}(\chi) + (u_{hx}, \chi') + \nu(u_{hxx}, \chi'') = 0 \quad \forall \chi \in S^r_h,$$

where $u_h(\cdot, 0) := u^0_h \in S^r_h$, and $u^0_h$ is such that

$$\|u^0 - u^0_h\| \leq ch^r.$$

In Section 2 we show existence and uniqueness of the semidiscrete approximation, and derive the optimal-order error estimate

$$\max_{0 \leq \xi \leq T} \|u(\cdot, t) - u_h(\cdot, t)\| \leq ch^r.$$

In analogy to the exact solution, for the semidiscrete approximation the following inequalities hold:

$$\|u_h(\cdot, t)\| \leq \|u^0_h\| e^{t/(4\nu)}, \quad 0 \leq t \leq T,$$

and

$$\|u_h(\cdot, t)\| \leq \|u_h(\cdot, s)\|, \quad 0 \leq s \leq t \leq T, \quad \text{for } \nu \geq \frac{1}{4\pi^2}.$$

Section 3 is devoted to a second-order in time fully discrete finite element method for (1.1)–(1.2). Let $N \in \mathbb{N}$, $k := T/N$, and $t^n := nk, n = 0, \ldots, N$. For $v(\cdot, t) \in L^2(0, 1), 0 \leq t \leq T$, let

$$v^n := v(\cdot, t^n), \quad \partial v^n := \frac{1}{k}(v^{n+1} - v^n), \quad \text{and } v^{n+1/2} := \frac{1}{2}(v^n + v^{n+1}).$$

The Crank–Nicolson approximations $U^n \in S^r_h$ to $u^n$ are then given by $U^0 := u^0_h$, and for $n = 0, \ldots, N - 1$

$$(\partial U^n, \chi) + (U^{n+1/2} U_x^{n+1/2}, \chi) - (U_x^{n+1/2}, \chi') + \nu(U_{xx}^{n+1/2}, \chi'') = 0 \quad \forall \chi \in S^r_h.$$

The following discrete analogs to (1.7) and (1.8), respectively, can be easily proved:

$$\|U^n\| \leq \|U^0\| e^{t/(4\nu)\alpha}, \quad \alpha > 1, \quad k \leq 8\nu \frac{\alpha - 1}{\alpha}, \quad n = 1, \ldots, N,$$

and

$$\|U^{n+1}\| \leq \|U^n\|, \quad n = 0, \ldots, N - 1, \quad \text{for } \nu \geq \frac{1}{4\pi^2}.$$
Further, we show existence of the Crank–Nicolson approximations for \( k < 8\nu \), and derive the optimal-order error estimate

\[
\max_{0 \leq n \leq N} \| u^n - U^n \| \leq c(k^2 + h^r).
\]

We also prove uniqueness of the fully discrete approximations under a mild mesh condition.

It is well known and easily seen that \( u(\cdot, t) \) is odd for \( 0 \leq t \leq T \) if the initial value \( u^0 \) is an odd function. This property carries over to the semidiscrete and the fully discrete approximations provided \( \chi \in S^r_h \) implies \( \chi(-\cdot) \in S^r_h \).

2. Semidiscretization. In this section we briefly study the semidiscrete approximation \( u_h \). The inequality (1.14) can be proved in the same way as (1.7).

Now, it is evident from (1.14) and the fact that \( S^r_h \) is finite-dimensional that an estimate of the form

\[
\max_{0 \leq t \leq T} \| u_h(\cdot, t) \|_{L^\infty} \leq c(h)
\]

is valid. Combining this with the fact that the “right-hand side” of the system of O.D.E.’s (1.11) is locally Lipschitz continuous we deduce existence and uniqueness of the semidiscrete approximation \( u_h \).

In the error estimation that follows we will compare the semidiscrete approximation with the elliptic projection of the exact solution. This projection \( P_E : H^2_{\text{per}} \to S^r_h \) is defined by

\[
\nu(v'' - (P_E v)''', \chi''') - (v' - (P_E v)', \chi') + \lambda(v - (P_E v), \chi) = 0 \quad \forall \chi \in S^r_h,
\]

where \( \lambda > 1/(2\nu) \). For the elliptic projection we have the following estimate:

\[
\sum_{j=0}^{2} h^j \| v - P_E v \|_j \leq c h^s \| v \|_s, \quad v \in H^s_{\text{per}}, \ 2 \leq s \leq r
\]

(cf. [11]). This estimate can be proved in the usual manner. First, using the fact that the bilinear form \( a \),

\[
a(v, w) := \nu(v'', w'') - (v', w') + \lambda(v, w),
\]

is continuous and coercive in \( H^2_{\text{per}} \) (cf. (1.5)), the Lax–Milgram lemma yields, in view of the approximation property (1.10),

\[
\| v - P_E v \|_2 \leq c h^{s-2} \| v \|_s, \quad v \in H^s_{\text{per}}, \ 2 \leq s \leq r.
\]

Next, to estimate \( \| v - P_E v \| \) consider the auxiliary problem

\[
a(\psi, w) = (v - P_E v, w) \quad \forall w \in H^2_{\text{per}}.
\]

Then, for \( \chi \in S^r_h \) we have

\[
\| v - P_E v \|^2 = a(\psi \chi, v - P_E v) \leq c \| \psi - \chi \|_2 \| v - P_E v \|_2.
\]
Therefore, the well-known regularity estimate
\[ \| \psi \|_4 \leq c \| v - P_E v \|, \]
easily established in our one-dimensional case, and (1.10), (2.3) yield
\[ \| v - P_E v \| \leq c h^s \| v \|_s, \quad v \in H^s_{\text{per}}, \ 2 \leq s \leq r. \]
The estimate (2.2) now follows from (2.3), (2.4) and (1.5).

**Theorem 2.1.** Let the solution \( u \) of (1.1)–(1.2) be sufficiently smooth, and let (1.12) hold. Then
\[ \max_{0 \leq t \leq T} \| u(\cdot, t) - u_h(\cdot, t) \| \leq c h^r. \]

**Proof.** Let \( W(\cdot, t) := P_E u(\cdot, t), \ \varrho(\cdot, t) := u(\cdot, t) - W(\cdot, t), \) and \( \vartheta(\cdot, t) := W(\cdot, t) - u_h(\cdot, t). \) Then \( u - u_h = \varrho + \vartheta \) and by (2.2)
\[ \max_{0 \leq t \leq T} \| \varrho(\cdot, t) \| \leq c h^r. \]
Thus, it remains to estimate \( \| \vartheta(\cdot, t) \|. \) Using (1.11), (2.1) and (1.3) we have, for \( \chi \in S^r \)
\[ \langle \vartheta_t, \chi \rangle + \nu \| \vartheta_{xx} \|_2 - \| \vartheta_x \|_2 \leq c h^2 r + c \| \vartheta \|_2 + \| \vartheta_x \|_2. \]
A straightforward consequence of the commutativity of \( P_E \) with time differenti-
3. Crank–Nicolson discretization. In this section we show existence of the
Crank–Nicolson approximations $U^1, \ldots, U^N$ for $k < 8\nu$, derive the optimal-order
error estimate (1.19), and under a mild mesh condition prove uniqueness of the
Crank–Nicolson approximations. We also briefly discuss the case of an odd initial
value.

Taking $\chi := U^{n+1/2}_n$ in (1.16) we obtain by periodicity
\begin{equation}
\|U^{n+1}\|_2^2 - \|U^n\|_2^2 = 2k\{\|U^{n+1/2}_x\|^2 - \nu\|U^{n+1/2}_{xx}\|^2\},
\end{equation}
and (1.18) follows using (1.8). Further, using (1.6) we obtain from (3.1),
\begin{equation}
\|U^{n+1}\|_2^2 - \|U^n\|_2^2 \leq \frac{k}{8\nu} \|U^{n+1/2}\|^2,
\end{equation}
i.e.,
\begin{equation}
\left(1 - \frac{k}{8\nu}\right)\|U^{n+1}\| \leq \left(1 + \frac{k}{8\nu}\right)\|U^n\|,
\end{equation}
for $\alpha > 1$ obviously
\begin{equation}
\frac{8\nu + k}{8\nu - k} \leq 1 + \frac{\alpha - 1}{4\nu k}
\end{equation}
and (1.17) follows easily from (3.2).

Existence. We shall use the following well-known variant of the Brouwer fixed-
point theorem (see, e.g., Browder [2]).

**Lemma 3.1.** Let $(H, (\cdot, \cdot)_H)$ be a finite-dimensional inner product space and
denote by $\|\cdot\|_H$ the induced norm. Suppose that $g : H \to H$ is continuous and
there exists an $\alpha > 0$ such that $(g(x), x)_H > 0$ for all $x \in H$ with $\|x\|_H = \alpha$.
Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \leq \alpha$.

The proof of existence of $U^0, \ldots, U^N$ for $k < 8\nu$ is by induction. Assume that
$U^0, \ldots, U^n$, $n < N$, exist and let $g : S^*_{h} \to S^*_{h}$ be defined by
\begin{equation}
(g(V), \chi) = 2(V - U^n, \chi) + k(VV', \chi) - k(V', \chi') + \nu k(V''', \chi'')
\end{equation}
for $V, \chi \in S^*_{h}$. This mapping is obviously continuous. Furthermore, by periodicity we have
\begin{equation}
(g(V), V) = 2(V - U^n, V) - k\{\|V'\|^2 - \nu\|V''\|^2\},
\end{equation}
and via (1.6),
\begin{equation}
(g(V), V) \geq 2\|V\|\left(1 - \frac{k}{8\nu}\right)\|V\| - \|U^n\| \quad \forall V \in S^*_{h}.
\end{equation}
Therefore, assuming $k < 8\nu$, for $\|V\| = \frac{8\nu}{8\nu - k} \|U^n\| + 1$ obviously $(g(V), V) > 0$
and the existence of a $V^* \in S^*_{h}$ such that $g(V^*) = 0$ follows from Lemma 3.1.
Then $U^{n+1} := 2V^* - U^n$ satisfies (1.16).

Convergence. The main result in this paper is given in the following theorem.
THEOREM 3.1. Let the solution $u$ of (1.1)–(1.2) be sufficiently smooth, $U^0, \ldots, U^N$ satisfy (1.16), and (1.12) hold. Then, for $k$ sufficiently small,

$$\max_{0 \leq n \leq N} \|u^n - U^n\| \leq c(u)(k^2 + h^\gamma).$$

**Proof.** Let $W^n := W(\cdot, t^n)$, $\varrho^n := u^n - W^n$, and $\zeta^n := W^n - U^n$. Then $u^n - U^n = \varrho^n + \zeta^n$ and by (2.6),

$$\max_{0 \leq n \leq N} \|\varrho^n\| \leq c h^\gamma.$$

Thus it remains to estimate $\|\zeta^n\|$. Using (1.16), (2.1) and (1.3) we have, for $\chi \in S^+_{a}$,

$$(\partial \zeta^n, \chi) + a(\zeta^{n+1/2}, \chi) = (\partial W^n, \chi) + a(W^{n+1/2}, \chi) - (\partial U^n, \chi) - a(U^{n+1/2}, \chi) = (\partial W^n - u^{n+1/2} - \frac{1}{2}(u^n u^n_x + u^n u^n_x), \chi)$$

$$+ \lambda \varrho^{n+1/2} + \lambda \varrho^{n+1/2} + U^{n+1/2} U^{n+1/2}, \chi),$$

i.e.,

$$\|\partial \zeta^n, \chi\| = \|\zeta^{n+1/2}, \chi\| - (\zeta^{n+1/2}, \chi') = (\zeta^{n+1/2}, \chi') + (u^{n+1/2} \varrho^{n+1/2} + W^{n+1/2} \varrho^{n+1/2}, \chi'),$$

where $\omega^n = \omega^n_0 + \omega^n_1 + \omega^n_2 + \lambda \varrho^{n+1/2}$, and

$$\omega^n_0 := \partial W^n - \partial u^n,$$

$$\omega^n_1 := \partial u^n - u^n u^n_x,$$

$$\omega^n_2 := u^n u^n_x - \frac{1}{2}(u^n u^n_x + u^n u^n_x).$$

It is easily seen that

$$\max_{0 \leq n \leq N} \|\omega^n\| \leq c(k^2 + h^\gamma).$$

Taking $\chi := \zeta^{n+1/2}$ in (3.5) and using (3.4), (3.6) and (2.9) we obtain by periodicity

$$\frac{1}{2\kappa} \left( \|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 \right) + \nu \|\zeta^{n+1/2}\|^2 - \|\zeta^{n+1/2}\|^2 \leq c(k^2 + h^\gamma)^2 + c\|\zeta^{n+1/2}\|^2.$$ 

Therefore by (1.5) we see that

$$\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 \leq c k \left( (k^2 + h^\gamma)^2 + \|\zeta^{n+1}\|^2 + \|\zeta^n\|^2 \right)$$

and the discrete Gronwall lemma yields in view of (1.12) for $k$ sufficiently small

$$\max_{0 \leq n \leq N} \|\zeta^n\| \leq c(k^2 + h^\gamma),$$

which concludes the proof. □
Uniqueness. In addition to our assumptions on $S_h^k$ we suppose here for the corresponding partition that for a positive constant $\mu$,

$$\Delta t \geq ch^{2\mu}.$$  

(3.8)

It is well known that this inequality implies

$$\|\chi\|_{L^\infty} \leq c h^{-\nu} \|\chi\| \quad \forall \chi \in S_h^k,$$

(cf. Nitsche [10]). Let now $V^0 = U^0$ and $V^0, \ldots, V^N \in S_h^k$ satisfy

$$\langle \vec{\partial}V^n, \chi \rangle + (V^{n+1/2}V_x^{n+1/2}, \chi) - (V_x^{n+1/2}, \chi') + \nu (V_x^{n+1/2}, \chi'') = 0$$

\quad \forall \chi \in S_h^k

for $n = 0, \ldots, N - 1$. Letting $E^n := U^n - V^n, n = 0, \ldots, N$, from (1.16), (3.10) we obtain

$$\langle \vec{\partial}E^n, \chi \rangle + \nu (E_x^{n+1/2}, \chi'') - (E_x^{n+1/2}, \chi') = (E_x^{n+1/2}, \chi) + (U^{n+1/2}E_x^{n+1/2}, \chi') \quad \forall \chi \in S_h^k.$$

Taking $\chi := E_x^{n+1/2}$ we obtain by periodicity

$$\frac{1}{2k} \left( \|E_x^{n+1}\|^2 - \|E^{n}\|^2 \right) + \nu \|E_x^{n+1/2}\|^2 - \|E_x^{n+1/2}\|^2$$

$$= \left( U^{n+1/2}E_x^{n+1/2}, E_x^{n+1/2} \right)$$

$$\leq \frac{1}{2} (\|W^{n+1/2}\|^2_{L^\infty} + \|\xi^{n+1/2}\|^2_{L^\infty}) \|E_x^{n+1/2}\|^2 + \|E_x^{n+1/2}\|^2$$

$$\leq (c + ch^{-2\nu} (k^4 + h^{2\mu})) \|E_x^{n+1/2}\|^2 + \|E_x^{n+1/2}\|^2$$

where (2.9), (3.9) and (3.7) have been used. Then (1.5) yields

$$\|E^{n+1}\|^2 - \|E^n\|^2 \leq C k (1 + k^4 h^{-2\nu} + h^{2(\nu - \mu)} (\|E_x^{n+1}\|^2 + \|E^n\|^2).$$

(3.11)

For $k^5 h^{-2\nu}$ and $kh^{2(\nu - \mu)}$ sufficiently small, assuming $E^n = 0$, (3.11) implies $E^{n+1} = 0$. Summarizing, for sufficiently smooth $u$ and $k^5 h^{-2\nu}$, $kh^{2(\nu - \mu)}$ sufficiently small, assuming (3.9) we deduce uniqueness of the Crank–Nicolson approximations.

Odd initial value. We assume here that the initial value $u^0$ is an odd function. Then $v(x, t) := -u(-x, t)$ is a solution of (1.1)–(1.2). Thus $v = u$, i.e., $u(\cdot, t)$ is odd for $0 \leq t \leq T$.

Assume now that if $x_i$ is a knot of our spline space then $-x_i$ is a knot as well, and moreover that the same differentiability conditions are posed at $x_i$ and $-x_i$, $i \in \mathbb{Z}$. As a consequence, $\chi \in S_h^k$ implies $\chi(-\cdot) \in S_h^k$. Let $u^0_h$ be an odd function as is natural for odd $u^0$. Then the semidiscrete approximation $u_h(\cdot, t)$ is odd for $0 \leq t \leq T$, and moreover under our assumptions implying uniqueness of the Crank–Nicolson approximations $U^n$, they are odd, since $V^n := -U^n(-\cdot)$ are also Crank–Nicolson approximations. This fact is of significant practical importance, since in (1.16) we only have to take the odd $\chi$’s thus reducing the number of equations to about 50%.
References

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