Limitation to the asymptotic formula in Waring’s problem

by

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1. Introduction. In 1920’s, Hardy and Littlewood introduced an analytic method for solving Waring’s problem: That is, they showed that every sufficiently large natural number can be expressed as a sum of at most $s$ $k$th powers, where $s$ depends only on $k$. Let $R_s(n)$ denote the number of representations of $n$ as the sum of $s$ $k$th powers. The idea of the Hardy–Littlewood method is to show that there is an asymptotic formula for $R_s(n)$ when $n$ is sufficiently large, i.e.

$$R_s(n) = (\mathfrak{G}_s(n) + o(1)) \Gamma \left(1 + \frac{1}{k}\right)^s \Gamma \left(\frac{s}{k}\right)^{-1} n^{s/k - 1},$$

where $\mathfrak{G}_s(n)$ is called the singular series and defined by

$$\mathfrak{G}_s(n) = \sum_{q=1}^{\infty} \sum_{a=1 \atop (a,q)=1}^q (S(q,a)/q)^s e(-an/q),$$

with

$$S(q,a) = \sum_{m=1}^q e(am^k/q).$$

Let $\tilde{G}(k)$ denote the least integer $t$ such that (1) holds for all $s \geq t$. Hardy and Littlewood [3] also obtained $\tilde{G}(k) \leq (k - 2)2^{k-1} + 5$ for $k \in \mathbb{N}$. Hua [5] obtained $\tilde{G}(k) \leq 2^k + 1$ for small $k$, and Vaughan [10, 11] improved this to $\tilde{G}(k) \leq 2^k$ for $k \geq 3$. In 1988, Heath-Brown [4] showed that $\tilde{G}(k) \leq 7 \cdot 2^{k-3} + 1$ for $k \geq 6$ and Boklan [1] recently obtained $\tilde{G}(k) \leq 7 \cdot 2^{k-3}$. For large $k$ Vinogradov [12] proved that $\tilde{G}(k) \leq 183k^9(\log k + 1)^2$ and then Hua [6] showed that $\tilde{G}(k) \leq (4 + o(1)) k^2 \log k$ as $k \to \infty$. Recently, Wooley [13] obtained $\tilde{G}(k) \leq (2 + o(1)) k^2 \log k$ as $k \to \infty$ by using an improved form of

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Vinogradov’s Mean Value Theorem. It seems likely that $\tilde{G}(k) = O(k)$, and Vaughan has conjectured that (1) holds whenever $s \geq \max(k+1, \Gamma_0(k))$ where $\Gamma_0(k)$ is the least $s$ such that for every $n$ and $q$ the congruence $x_1^k + \ldots + x_s^k \equiv n \pmod{q}$ has a solution with $(x_1, q) = 1$.

In this paper, we wish to show that the usual approximation to $\mathcal{R}_s(n)$ cannot always be very precise. We will obtain some analogues of the theorems in [7].

First of all, we restrict ourselves to $k > 2$.

**Theorem 1.** Suppose that $1/2 \leq r < 1$ and $k+1 \leq s < 2k$. Then

$$\sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) s \Gamma \left( \frac{s}{k} \right) \mathcal{G}_s(n)n^{s/k-1} \right)^2 r^n \gg R^{s/k},$$

where $R = \left( 1 - r \right)^{-1}$.

**Corollary 1.** Suppose that $k+1 \leq s < 2k$. As $x \to \infty$, we have

$$\sum_{n \leq x} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) s \Gamma \left( \frac{s}{k} \right) \mathcal{G}_s(n)n^{s/k-1} \right)^2 = \Omega(x^{s/k}).$$

**Theorem 2.** Suppose that $s \geq k+2$ is fixed and $1/2 \leq r < 1$. Then

$$\sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) s \Gamma \left( \frac{s}{k} \right) \mathcal{G}_s(n)n^{s/k-1} \right)^2 r^n = -\frac{s}{2} \Gamma \left( 1 + \frac{1}{k} \right) s^{s-1} R^{(s-1)/k} + O(R^{(s-2)/k}),$$

where $R = \left( 1 - r \right)^{-1}$.

**Corollary 2.** Suppose that $s \geq k+2$ is fixed and $1/2 \leq r < 1$. Then

$$\sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) s \Gamma \left( \frac{s}{k} \right) \mathcal{G}_s(n)n^{s/k-1} \right)^2 r^n \geq \frac{s^2}{4} \Gamma \left( 1 + \frac{1}{k} \right) 2^{s-2} R^{(2s-2)/k-1} + O(R^{(2s-3)/k-1}).$$

**Corollary 3.** Suppose that $s$ is fixed and $s \geq k+2$. As $x \to \infty$, we have

$$\sum_{n \leq x} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) s \Gamma \left( \frac{s}{k} \right) \mathcal{G}_s(n)n^{s/k-1} \right)^2 = \Omega(x^{(2s-2)/k-1}).$$

**Remark.** Note that when $k = 2$, Theorem 2 and Corollaries 2 and 3 hold for $s \geq 5$. The proofs of these results are exactly the same as in the case $k > 2$, except that the condition $s \geq k+2$ is replaced by $s \geq 5$. 

The following corollary shows that the approximation of $R_s(n)$ by the asymptotic formula cannot be very precise.

**Corollary 4.** For $k \geq 3$, $R_{k+1}(n) - \Gamma \left(1 + \frac{1}{k}\right)^k \mathcal{S}_{k+1}(n)n^{1/k} = \Omega(n^{1/(2k)})$,

and for $s \geq k + 2$ and $k \geq 3$, $R_s(n) - \Gamma \left(1 + \frac{1}{k}\right)^s \Gamma \left(\frac{s}{k}\right)^{-1} \mathcal{S}_s(n)n^{s/k-1} = \Omega(n^{(s-1)/(k-1)})$.

When $k = 2$, the analogue of Theorem 2 cannot apply for $s = 4$. However, we can use some elementary arguments to obtain a similar result.

**Theorem 3.** For $k = 2$, $R_4(n) - \frac{\pi^2}{16} \mathcal{S}_4(n)n = \Omega(n^{1/2})$,

and for $k = 2$ and $s \geq 5$, $R_s(n) - \frac{\pi^{s/2}}{2^s} \Gamma \left(\frac{s}{2}\right)^{-1} \mathcal{S}_s(n)n^{s/2-1} = \Omega(n^{s/2-3/2})$.

Note that $r_4(n) = \text{card}\{(x_1, \ldots, x_4) \in \mathbb{Z}^4 : x_1^2 + \ldots + x_4^2 = n\}$ satisfies $r_4(n) = \pi^2 \mathcal{S}_4(n)n$.

2. Preliminary lemmas

**Lemma 1.** Suppose that $1/2 \leq r < 1$ and $R = (1 - r)^{-1}$. Then, as $r \to 1-$,

\begin{equation}
 f(r) \sim L(r),
\end{equation}

where $f(r) = \sum_{n=1}^{\infty} r^n$ and

\begin{equation}
 L(r) = \Gamma \left(1 + \frac{1}{k}\right) \left(1 - r\right)^{-1/k},
\end{equation}

In addition, $f(r) - L(r) = -1/2 + O((1 - r)^{1/k})$, where $k \geq 2$.

**Proof.** Suppose that $\Phi$ has a continuous second derivative on $[0, \infty)$. Then, by the Euler–Maclaurin summation formula, we have

\begin{equation}
 \sum_{1 \leq n \leq x} \Phi(n) = \int_1^x \Phi(y) dy + \frac{1}{2} \Phi(1) - B_1(x) \Phi(x) + \int_1^x B_1(y) \Phi'(y) dy
\end{equation}
\[ \begin{align*}
&= \int_1^x \Phi(y) \, dy + \frac{1}{2} \Phi(1) - B_1(x) \Phi(x) + [B_2(y) \Phi'(y)]_1^x \\
&\quad - \int_1^x B_2(y) \Phi''(y) \, dy,
\end{align*} \]

where \( B_j(x) = b_j(\{x\}) \), \( b_1(y) = y - \frac{1}{2} \), \( b_2(y) = \frac{1}{2} y^2 - \frac{1}{2} + \frac{1}{12} \). Put \( \Phi(y) = r^{y^k} \). Then

\( \Phi'(y) = -k y^{k-1} r^{y^k} \left( \log \frac{1}{r} \right) \),

\( \Phi''(y) = -k(k-1) y^{k-2} r^{y^k} \left( \log \frac{1}{r} \right) + (ky)^{k-1} r^k \left( \log \frac{1}{r} \right)^2 \),

and \( \Phi(1) = r \).

Let \( y_0 = \left( \frac{k-1}{k \log(1/r)} \right)^{1/k} \). Then, by (11), \( \Phi''(y) \leq 0 \) for \( y \leq y_0 \), and \( \Phi''(y) \geq 0 \) for \( y \geq y_0 \). Hence, assuming \( r \geq 1/\sqrt{e} \),

\( \int_1^x B_2(y) \Phi''(y) \, dy \leq \frac{1}{12} \int_1^{y_0} -\Phi''(y) \, dy + \frac{1}{12} \int_{y_0}^\infty \Phi''(y) \, dy \)

\( = \frac{1}{12} \Phi'(1) - \frac{1}{6} \Phi'(y_0) \)

\( = -\frac{kr}{12} \log \frac{1}{r} + \frac{1}{6} ky_0^{k-1} r^{y_0} \left( \log \frac{1}{r} \right) \) (by (10))

\( = -\frac{kr}{12} \log \frac{1}{r} + \frac{1}{6} y_0^{k-1} \frac{k-1}{k \log(1/r)} r^{y_0} \left( \log \frac{1}{r} \right) \)

\( = -\frac{kr}{12} \log \frac{1}{r} + \frac{k-1}{6} r^{y_0} \left( \frac{k \log(1/r)}{k-1} \right)^{1/k} \).

Put \( \Phi(y) = r^{y^k} \) in (9). By (12), we have

\( \sum_{n=1}^\infty r^{n^k} = \int_1^\infty r^{y^k} \, dy + \frac{r}{2} + O \left( \left( \log \frac{1}{r} \right)^{1/k} \right) \).

By changing variable \( u = y^k \log(1/r) \), this is

\( \int_{\log(1/r)}^\infty \left( \log \frac{1}{r} \right)^{-1/k} \frac{1}{k} u^{1/k-1} e^{-u} \, du + \frac{r}{2} + O \left( \left( \log \frac{1}{r} \right)^{1/k} \right). \)

We will extend the range of the integral, so we need to estimate the value of the integral from 0 to \( \log(1/r) \), and note that then \( e^{-y} = 1 + O(y) \). Thus
Asymptotic formula in Waring’s problem

\[ \int_0^{\log(1/r)} \left( \log \frac{1}{r} \right)^{-1/k} \frac{1}{k} y^{1/k-1} e^{-y} dy \]

\[ = \left( \log \frac{1}{r} \right)^{-1/k} \int_0^{\log(1/r)} \frac{1}{k} y^{1/k-1} e^{-y} dy \]

\[ = \left( \log \frac{1}{r} \right)^{-1/k} \int_0^{\log(1/r)} \frac{1}{k} y^{1/k-1} (1 + O(y)) dy \]

\[ = \left( \log \frac{1}{r} \right)^{-1/k} \left( \log \frac{1}{r} \right)^{1/k} + O\left( \left( \log \frac{1}{r} \right)^{1/k} \right) \]

\[ = 1 + O\left( \log \frac{1}{r} \right). \]

Combine this with (14). Then we have

\[ \sum_{n=1}^{\infty} r^{nk} \]

\[ = \int_0^{\infty} \left( \log \frac{1}{r} \right)^{-1/k} \frac{1}{k} y^{1/k-1} e^{-y} dy - 1 + r/2 + O\left( \left( \log \frac{1}{r} \right)^{1/k} \right). \]

Obviously,

\[ \log \frac{1}{r} = \log \frac{1}{1 - (1 - r)}. \]

By Taylor’s expansion, this is \((1 - r) + O((1 - r)^2)\). Hence

\[ \left( \log \frac{1}{r} \right)^{-1/k} = (1 - r)^{-1/k} (1 + O(1 - r)) = (1 - r)^{-1/k} + O((1 - r)^{1/k}), \]

provided that \(k \geq 2\). Combine this with (15) to get

\[ \sum_{n=1}^{\infty} r^{nk} = (1 - r)^{-1/k} \Gamma\left( 1 + \frac{1}{k} \right) - \frac{1}{2} + O((1 - r)^{1/k}) \]

as \(r \to 1^-\).

**Lemma 2.** Suppose that \(s \geq k + 1\). Then

\[ \sum_{q \leq Q} q^{1/k} |S_n(q)| \ll (nQ)^s, \]

where

\[ S_n(q) = \sum_{\substack{a=1 \\atop (a,q)=1}}^q (S(q,a)/q)^s e(-an/q). \]
Proof. See Lemma 4.8 of [9].

Lemma 3. Suppose \( y \geq 1, \varepsilon > 0 \) and \( s \geq k + 1 \). Let

\[
\mathcal{G}_s(n, y) = \sum_{q \leq y} \sum_{a=1}^{q} (S(q, a)/q)^{\varepsilon} e(-\alpha n/q),
\]

and

\[
E_s(n, y) = \mathcal{G}_s(n) - \mathcal{G}_s(n, y).
\]
Then \( E_s(n, y) \ll n^\varepsilon y^{\varepsilon - 1/k} \).

Proof. By Lemma 2, we have

\[
\sum_{R < q \leq 2R} q^{1/k} |S_n(q)| \ll n^\varepsilon R^\varepsilon.
\]

Also

\[
\sum_{R < q \leq 2R} |S_n(q)| \leq \left( \frac{1}{R} \right)^{1/k} \sum_{R < q \leq 2R} q^{1/k} |S_n(q)| \ll n^\varepsilon R^{\varepsilon - 1/k}.
\]

Sum over \( R = y, 2y, 4y, 8y, \ldots \) to get

\[
\sum_{q > y} |S_n(q)| \ll n^\varepsilon y^{\varepsilon - 1/k}.
\]

Lemma 4. Suppose that \( 1/2 \leq r < 1, R = (1 - r)^{-1} \) and \( \alpha > -1 \). Then

\[
\sum_{n=2}^{\infty} n^\alpha (\log n)^\beta r^n \ll R^{\alpha + 1} (\log R)^\beta.
\]
The implicit constant may depend on \( \alpha \) and \( \beta \).

Proof. See Lemma 2 of [7].

Lemma 5. Let \( \alpha > 0 \). Then for every \( t \), we have

\[
(-1)^n \binom{-\alpha}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left\{ 1 + \sum_{j=1}^{t} b_j(\alpha) n^{-j} \right\} + O(n^{\alpha-t-2}),
\]
as \( n \to \infty \), where the coefficients \( b_k(\alpha) \) are real numbers which depend at most on \( k \) and \( \alpha \).

Proof. See Lemma 4.1 of [8].

Lemma 6. Let \( \mathcal{G}_s(n) \) be given by (2) and \( s \geq k + 2 \). Then

\[
\sum_{n \leq x} \mathcal{G}_s(n) = x + O(1).
\]
Proof. The term with \( q = 1 \) in the definition of \( S_s(n) \) contributes \( \lfloor x \rfloor \) when summed. Thus, we need to show that the terms with \( q \geq 2 \) contribute \( O(1) \) when summed. By Lemma 4.4 of [9], if \( p \nmid a \) and \( l > \gamma \), then

\[
S(p^l, a) = \begin{cases} 
p^{k-1}S(p^{l-k}, a) & \text{when } l > k, \\
p^{l-1} & \text{when } l \leq k,
\end{cases}
\]

where \( \gamma \) is defined by

\[
\gamma = \begin{cases} 
\tau + 2 & \text{when } p = 2 \text{ and } \tau = 0, \\
\tau + 1 & \text{when } p > 2 \text{ or } p = 2 \text{ and } \tau > 0,
\end{cases}
\]

and \( \tau \) is the largest \( t \) such that \( p^t \) divides \( k \). Note that \( \gamma \leq k \) unless \( k = p = 2 \) in which case \( \gamma = 3 \). Suppose that \( 2 \leq l \leq \gamma \). Then

\[
|S(p^l, a)| \leq p^{l-1},
\]

since \( l \leq k \) and \( p \mid k \). For \( l = 1 \), by (3.54) of Hardy and Littlewood [3], we have

\[
|S(p, a)| \leq (k-1)p^{1/2}.
\]

Let \( q = \prod_p p^{r_p} \). Rewrite \( q \) as \( q_1 q_2^2 q_3^3 \ldots q_k^k \), where \( q_1, q_2, \ldots, q_{k-1} \) are square-free and pairwise coprime. By Lemma 2.10 of [9],

\[
S(q, a) = \prod_{p \mid q} S(p^{r_p}, a_{p^{r_p}}),
\]

where \( a_{p^{r_p}} \equiv a \pmod{p} \). By (18), we have

\[
S(q, a) = \prod_{u=1}^{k-1} \prod_{\substack{p \mid q_u \\langle p \rangle > 2}} S(p^{u_p}, a_{p^{u_p}}) \prod_{\substack{p \mid q_k \\langle p \rangle > 2}} p^{u_p(k-1)} S(2^{u_p(k-1)}, a_{2^{u_p(k-1)}}).
\]

Therefore,

\[
|S(q, a)| \leq \prod_{u=2}^{k-1} \prod_{\substack{p \mid q_u \\langle p \rangle > 2}} p^{u_p-1} \prod_{\substack{p \mid q_k \\langle p \rangle > 2}} kp^{u_p-1} \times \prod_{\substack{p \mid q_1 \\langle p \rangle > 2}} kp^{1/2} \prod_{\substack{p \mid q_k \\langle p \rangle > 2}} p^{u_p(k-1)} (4 \cdot 2^{u_p(k-1)/2})
\]

\[
\ll \left( \prod_{u=2}^{k-1} q_u^{u_p-1} \right) \left( \prod_{p \leq k} q_1^{1/2} \left( \prod_{p \mid q} q_k^{k-1} \right) \right)
\]

\[
\ll q^k q_1^{1/2} q_2 q_3^2 \ldots q_k^{k-1}.
\]
If $q > 1$ and $(a, q) = 1$, then
\[
\sum_{n \leq x} e(-an/q) \ll |\sin(\pi a/q)|^{-1} \ll \|a/q\|,
\]
where $\|y\|$ is the distance of $y$ from the nearest integer. So the terms with $q \geq 2$ in (17) contribute
\[
\ll \sum_{q=2}^{\infty} q^{-s} \sum_{a=1}^{q-1} (q^{-s/q} q_{1/2} q_{2}^{-s/2} \cdots q_{k}^{-s/k})^{s} q^{-s} \|a/q\|^{-1}
\]
where $\eta = \varepsilon(s+1)$. The last sum is
\[
\leq \sum_{q_{1}=1}^{\infty} \sum_{q_{2}=1}^{\infty} \cdots \sum_{q_{k}=1}^{\infty} \frac{1+\eta-s/2}{q_{1}^{2}} + \frac{2+2\eta-s-2s}{q_{2}^{2}} + \frac{3+3\eta+2s-3s}{q_{3}^{2}} + \cdots \frac{k+k\eta+(k-1)s-ks}{q_{k}^{2}}
\]
\[
= \sum_{q_{1}=1}^{\infty} \sum_{q_{2}=1}^{\infty} \cdots \sum_{q_{k}=1}^{\infty} \frac{1+\eta-s/2}{q_{1}^{2}} + \frac{2+2\eta-s-3s}{q_{2}^{2}} + \frac{3+3\eta-s}{q_{3}^{2}} + \cdots \frac{k+k\eta-s}{q_{k}^{2}}.
\]
When $s \geq k+2$, it is convergent. Hence, the lemma follows.

**Lemma 7.** Let $1/2 \leq r < 1$ and $L(r)$ be as in Lemma 1 and suppose that $s \geq \max(5, k+2)$. Then
\[
\sum_{n=1}^{\infty} \Theta_{s}(n) \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right) n^{-s/k-1} r^{n} = L^{s}(r) + O(R^{s/k-1}).
\]

**Proof.** Clearly,
\[
L^{s}(r) = \Gamma \left( 1 + \frac{1}{k} \right)^{s} (1-r)^{-s/k}.
\]
By the binomial expansion, we have
\[
L^{s}(r) = \Gamma \left( 1 + \frac{1}{k} \right)^{s} \sum_{n=0}^{\infty} (-1)^{n} \binom{-s/k}{n} r^{n}.
\]
Hence, by Lemma 5, we have
\[
L^{s}(r) = \Gamma \left( 1 + \frac{1}{k} \right)^{s} \sum_{n=1}^{\infty} \Gamma \left( \frac{s}{k} \right) n^{s/k-1} r^{n} + O \left( 1 + \sum_{n=1}^{\infty} n^{s/k-2} r^{n} \right).
\]
By Lemma 4, this is
\[
\Gamma \left( 1 + \frac{1}{k} \right)^{s} \Gamma \left( \frac{s}{k} \right) \sum_{n=1}^{\infty} n^{s/k-1} r^{n} + O(R^{s/k-1}).
\]
The difference between the main terms in (23) is
\[ \sum_{n=1}^{\infty} (\mathcal{S}_s(n) - 1) \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right) n^{s/k-1} r^n, \]
which by partial summation is
\[ \sum_{n=1}^{\infty} \left( \sum_{m \leq n} \mathcal{S}_s(m) - n \right) \frac{\Gamma \left( 1 + \frac{1}{k} \right) s}{\Gamma \left( s/k \right)} n^{s/k-1} r^n = (n + 1)^{s/k-1} r^n - (n + 1)^{s/k-1} r^{n+1}. \]
From Lemma 6, we see that the first factor \( \ll 1 \). By the binomial expansion, the last factor is
\[ (n^{s/k-1} - (n + 1)^{s/k-1}) r^n + (1 - r)(n + 1)^{s/k-1} r^n \]
\[ = - \left( \frac{s}{k} - 1 \right) n^{s/k-2} r^n + (1 - r)(n + 1)^{s/k-1} r^n + O(n^{s/k-3} r^n), \]
Thus, by Lemma 4, (25) becomes \( \ll R^{s/k-1} \). Combining this with (24) gives the lemma.

3. Proof of theorems

Proof of Theorem 2. We have to show that
\[ \sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right) \mathcal{S}_s(n) n^{s/k-1} \right) r^n \]
\[ = - \frac{s}{2} \Gamma \left( 1 + \frac{1}{k} \right) \left( 1 - r \right)^{(s-1)/k} + O((1 - r)^{(s-2)/k}). \]
From Lemma 7 we see that this is simply a matter of establishing that
\[ f^s(r) - L^s(r) = - \frac{s}{2} \Gamma \left( 1 + \frac{1}{k} \right) R^{(s-1)/k} + O(R^{(s-2)/k}), \]
where \( R = (1 - r)^{-1} \). By Lemma 1, it follows that
\[ f^s(r) - L^s(r) = (s + O(r^{-1/k}))(f(r) - L(r)) L^{s-1}(r) \]
\[ = - \frac{s}{2} \Gamma \left( 1 + \frac{1}{k} \right) R^{(s-1)/k} + O(R^{(s-2)/k}), \]
as required.

Proof of Theorem 1. Choose \( y = R^k \). First of all, we show that it suffices to prove
\[ \sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right) \mathcal{S}_s(n, y) n^{s/k-1} \right)^2 r^{2n} \gg R^{s/k}, \]
where \( \mathcal{S}_s(n, y) \) is as in Lemma 3.
By definition of \( \mathcal{S}_s(n, y) \), the left hand side is
\[
\ll \sum_{n=1}^\infty \left( R_s(n) - \Gamma\left(1 + \frac{1}{k}\right)^s \frac{s}{k}^{-1} \mathcal{S}_s(n)n^{s/k-1} \right)^2 r^{2n}
\]
\[
+ \sum_{n=1}^\infty (E_s(n, y))^2 n^{2(s/k-1)}r^{2n}.
\]

By Lemma 3, the second sum is
\[
\ll \sum_{n=1}^\infty n^{2\varepsilon - 2/k} n^{2(s/k-1)} r^{2n}.
\]

By Lemma 4, this is \( \ll y^{2\varepsilon - 2/k} R^{2s/k-1+2\varepsilon} \). Since \( y = R^k \), this is \( \ll R^{2s/k-3+\varepsilon'} \). For \( k + 1 \leq s < 2k \), this is \( o(R^{s/k}) \).

Now, we prove (26). By Parseval’s identity, we may write the left hand side of (26) as
\[
\int_0^1 \left| \sum_{n=1}^\infty \left( R_s(n) - \Gamma\left(1 + \frac{1}{k}\right)^s \frac{s}{k}^{-1} \mathcal{S}_s(n)y^{s/k-1} \right)r^n e(n\alpha) \right|^2 d\alpha.
\]

By the Cauchy–Schwarz inequality, this is at least \( T^2 \), where
\[
T = \int_0^1 \left| \sum_{n=1}^\infty R_s(n)r^n e(n\alpha) - \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \sum_{n=1}^\infty \mathcal{S}_s(n,y)n^{s/k-1}r^n e(n\alpha) \right| d\alpha.
\]

Clearly,
\[
T \geq \int_1^2 - \int_2^3,
\]
where
\[
\int_1 = \int_0^1 \left| \sum_{n=1}^\infty R_s(n)r^n e(n\alpha) \right| d\alpha,
\]
\[
\int_2 = \int_0^1 \left| \sum_{n=1}^\infty \Gamma\left(1 + \frac{1}{k}\right)^s \frac{s}{k}^{-1} \mathcal{S}_s(n,y)n^{s/k-1}r^n e(n\alpha) \right| d\alpha.
\]

By Parseval’s identity, we have
\[
\sum_{n=1}^\infty r^{2n^k} = \int_0^1 \sum_{n=1}^\infty r^k e(n^k \alpha)^2 d\alpha.
\]
By Hölder’s inequality, this is
\[
\leq \left( \int_0^1 \left| \sum_{n=1}^\infty r_n e(n^k \alpha) \right|^s \, d\alpha \right)^{2/s} \left( \int_0^1 \, d\alpha \right)^{1-2/s} = \left( \int_0^1 \left| \sum_{n=1}^\infty r_n e(n^k \alpha) \right|^s \, d\alpha \right)^{2/s}.
\]
By Lemma 1 with \( r \) replaced by \( r^2 \), we have
\[
\left( \int_0^1 \left| \sum_{n=1}^\infty r_n^2 e(n^k \alpha) \right|^s \, d\alpha \right)^{2/s} \gg \frac{1}{(1-r)^{1/k}}
\]
as \( r \to 1^- \). Since \( R = (1-r)^{-1} \), therefore,
\[
(30) \int_1 \gg R^{s/(2k)}.
\]
Finally, we estimate the integral \( \int_2 \). By definition of \( S_s(n, y) \) and (29), we have
\[
(31) \int_2 = \int_0^1 \left| \sum_{n=1}^\infty \Gamma(1+1/k)^s \sum_{q \leq y} \sum_{(a,q)=1} \left( \frac{S(q,a)}{q} \right)^s \right| \, d\alpha
\]
\[
\leq \Gamma \left( 1 + \frac{1}{k} \right)^s \sum_{q \leq y} \sum_{(a,q)=1} \left| \frac{S(q,a)}{q} \right|^s
\]
\[
\times \int_0^1 \left| \sum_{n=1}^\infty n^{s/k-1} r_n e\left( n \left( \alpha - \frac{a}{q} \right) \right) \right| \, d\alpha.
\]
Now, our task is to estimate the integral in (31). Suppose that \( |\beta| \leq 1/2 \) and \( |\beta| > 1 - r \). By Lemma 5, we may write
\[
\frac{N^\gamma}{\Gamma(\gamma+1)} = \sum_{j=1}^t f_j (-1)^N \left( -\gamma - 2 + j \right) N + O(N^{\gamma-t}),
\]
where the \( f_j \) depend at most on \( \gamma \) and \( t \). This enables us to write
\[
(32) \sum_{n=1}^\infty \frac{n^{s/k-1}}{(s/k)} r_n e(n\beta) = \sum_{n=1}^\infty \sum_{j=1}^t f_j (-1)^N \left( -\frac{s/k - 1 + j}{n} \right) r_n e(n\beta)
\]
\[
+ \sum_{n=1}^\infty (O(n^{s/k-1-t})) r_n e(n\beta).
\]
Put $t = 2$. Since $s < 2k$, the last sum is
\[(33) \quad \sum_{n=1}^{\infty} n^{s/k-3} \ll 1.\]

Therefore,
\[
\sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e(n\beta) = \sum_{n=0}^{2} \sum_{j=1}^{s/k-1} f_j \left( -\frac{s}{k} - 1 - j \right) r^n e(n\beta) + O(1).
\]

Hence, we have
\[(34) \quad \sum_{n=1}^{\infty} \frac{n^{s/k-1}}{\Gamma(s/k)} r^n e(n\beta) = f_1(1 - re(n\beta))^{-s/k} + f_2(1 - re(n\beta))^{-s/k+1} + O(1).\]

Since $|1 - re(\beta)|^2 = (1 - r)^2 + 4r(\sin \pi \beta)^2$, we have
\[(35) \quad \left| \frac{1}{1 - re(\beta)} \right|^{s/k} = \left( \frac{1}{\sqrt{(1-r)^2 + 4r(\sin \pi \beta)^2}} \right)^{s/k} \ll \min((1 - r)^{-s/k}, |\beta|^{-s/k}).\]

Replace $\alpha - a/q$ by $\beta$ in the integral of right hand side of (31) and by periodicity replace the interval $[-a/q, 1-a/q]$ by $[-1/2, 1/2]$. Then the integral becomes
\[
\int_{-1/2}^{1/2} \sum_{n=1}^{\infty} n^{s/k-1} r^n e(n\beta) d\beta.
\]

Hence, by (34) and (35), this is
\[
\ll \int_{-1/2}^{1/2} \min((1 - r)^{-s/k}, |\beta|^{-s/k}) d\beta
\]
\[
= \int_{|\beta| \leq 1-r} (1 - r)^{-s/k} d\beta + \int_{1-r}^{1/2} \beta^{-s/k} d\beta + \int_{-1/2}^{-(1-r)} (-\beta)^{-s/k} d\beta
\]
\[
\ll (1 - r)^{1-s/k}.
\]

By (31), we have
\[
\int_{2} \ll \sum_{q \leq y} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| S(a,q) \right|^s (1 - r)^{1-s/k}.
\]

By Lemma 4.9 of [9] with $s \geq k + 1$ and since $R = (1 - r)^{-1}$, we have
\[ f_2 \ll y^{s/k} R^{s/k-1} \]. Since \( y = R^k \) and \( s < 2k \), we have
\[ (36) \quad \int f_2 = o(R^{s/(2k)}). \]

By (27)-(29) and noting that \( s < 2k \), we obtain \( T \gg R^{s/(2k)} \). Hence, the theorem follows.

**Proof of Theorem 3.** We divide the solutions counted by \( r_4(n) \) according to how many of the \( x_i \) are non-zero. Let
\[ \varrho_j(n) = \text{card}\{x_i \in \mathbb{Z}/\{0\} : x_i^2 + \ldots + x_j^2 = n\}. \]

Then
\[ r_4(n) = \varrho_4(n) + 4\varrho_3(n) + 6\varrho_2(n) + 4\varrho_1(n) + \varrho_0(n). \]

Now we have
\[ \varrho_4(n) = 2^{-4} R_4(n) \quad \text{and} \quad r_4(n) = \pi^2 \mathcal{S}_4(n)n \]
(see Hardy [2], Section 3.11) and \( 4\varrho_3(n) + 6\varrho_2(n) + 4\varrho_1(n) + \varrho_0(n) \) is readily seen to be \( \Omega(n^{1/2}) \), which gives the first part of the theorem. The second part of the theorem follows at once from Theorem 2.

4. **Proof of corollaries**

**Proof of Corollary 1.** Multiply both sides of (3) by
\[ R = (1 - r)^{-1} = \sum_{l=0}^{\infty} r^l. \]

Then the left hand side of (3) becomes
\[ \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right)^s \left( \frac{1}{k} \right) \mathcal{S}_s(n) n^{s/k-1} \right)^2 r^{n+l}. \]

Obviously, this is
\[ \sum_{n=1}^{\infty} \sum_{m \leq n} \left( R_s(m) - \Gamma \left( 1 + \frac{1}{k} \right)^s \left( \frac{1}{k} \right) \mathcal{S}_s(m) m^{s/k-1} \right)^2 r^n. \]

The right hand side of (3) becomes \( R^{s/k+1} \). Hence, we have
\[ (37) \quad \sum_{n=1}^{\infty} \sum_{m \leq n} \left( R_s(m) - \Gamma \left( 1 + \frac{1}{k} \right)^s \left( \frac{1}{k} \right) \mathcal{S}_s(m) m^{s/k-1} \right)^2 r^n \gg R^{s/k+1}. \]

If (4) were false, then we would have
\[ \sum_{m \leq n} \left( R_s(m) - \Gamma \left( 1 + \frac{1}{k} \right)^s \left( \frac{1}{k} \right) \mathcal{S}_s(m) m^{s/k-1} \right)^2 = o(n^{s/k}). \]
Multiply both sides by $r^n$ and sum over $n$. Then
\[
\sum_{n=1}^{\infty} \sum_{m \leq n} \left( R_s(m) - \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right)^{-1} \mathcal{S}_s(m) m^{s/k-1} \right)^2 r^n = o(R^{s/k+1}).
\]
This contradicts (37), and hence (4) is true.

**Proof of Corollary 2.** By Cauchy’s inequality,
\[
\left( \sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right)^{-1} \mathcal{S}_s(n) n^{s/k-1} \right)^2 r^n \right) \left( \sum_{n=1}^{\infty} r^n \right) \geq \left( \sum_{n=1}^{\infty} \left( R_s(n) - \Gamma \left( 1 + \frac{1}{k} \right) \Gamma \left( \frac{s}{k} \right)^{-1} \mathcal{S}_s(n) n^{s/k-1} \right) r^n \right)^2.
\]
By Theorem 2, the right hand side is
\[
\frac{s^2}{4} \Gamma \left( 1 + \frac{1}{k} \right)^{2s-2} R^{(2s-2)/k} + O(R^{(2s-3)/k})
\]
and the second sum on the left hand side is $rR = R + O(1)$. Hence, the result follows.

**Proof of Corollary 3.** This is similar to the proof of Corollary 1.

**Proof of Corollary 4.** The first part of the corollary is immediate from Corollary 1 and the second part from Theorem 2.

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**References**


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