On elementary abelian 2-Sylow $K_2$ of rings of integers of certain quadratic number fields

by

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I. Introduction. A large number of papers have contributed to determining the structure of the tame kernel $K_2\mathcal{O}_F$ of algebraic number fields $F$. Recently, for quadratic number fields $F$ whose discriminants have at most three odd prime divisors, 4-rank formulas for $K_2\mathcal{O}_F$ have been made very explicit by Qin Hourong in terms of the indefinite quadratic form $x^2 - 2y^2$ (see [7], [8]).

We have made a successful effort, for quadratic number fields $F = \mathbb{Q}(\sqrt{\pm p_1 p_2})$, to characterize in terms of positive definite binary quadratic forms, when the 2-Sylow subgroup of the tame kernel of $F$ is elementary abelian.

This makes determining exactly when the 4-rank of $K_2\mathcal{O}_F$ is zero, computationally even more accessible. For arbitrary algebraic number fields $F$ with 4-rank of $K_2\mathcal{O}_F$ equal to zero, it has been pointed out that the Leopoldt conjecture for the prime 2 is valid for $F$, compare [6].

We consider this paper to be an addendum to the Acta Arithmetica publications [7], [8]. It grew out of our circulated 1989 notes [3].

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II. Statement of results. We consider quadratic fields $\mathbb{Q}(\sqrt{\pm p_1 p_2})$ with two odd (positive) prime numbers $p_1, p_2$.

For real quadratic fields, concerning the question of when the 2-Sylow subgroup of the tame kernel is elementary abelian, we concentrate on the most involved case $p_1 \equiv p_2 \equiv 1 \bmod 8$ and prove:

Theorem 1. Let $E = \mathbb{Q}(\sqrt{p_1 p_2})$ with rational primes $p_1 \equiv p_2 \equiv 1 \bmod 8$. Then 2-Sylow $K_2\mathcal{O}_E$ is elementary abelian if and only if
(i) \( (p_1/p_2) = -1 \) and
(ii) exactly one of the two primes \( p_1, p_2 \) fails to be represented over \( \mathbb{Z} \) by the quadratic form \( x^2 + 32y^2 \).

For imaginary quadratic fields, we concentrate on the most involved case (up to the order of \( p_1, p_2 \))
\[
p_1 \equiv 7 \mod 8, \quad p_2 \equiv 1 \mod 8 \quad \text{and} \quad (p_1/p_2) = 1
\]
and prove:

**Theorem 2.** Let \( L = \mathbb{Q}(\sqrt{-p_1p_2}) \) with rational primes \( p_1 \equiv 7 \mod 8, \ p_2 \equiv 1 \mod 8, \ (p_1/p_2) = 1 \). Let \( h(K) \) denote the class number of \( K = \mathbb{Q}(\sqrt{-2p_1}) \). Then \( 2\)-Sylow \( K_2\mathcal{O}_L \) is elementary abelian if and only if
\[
p_2 = x^2 + 32y^2 \quad \text{and} \quad p_2^{h(K)/4} = 2a^2 + p_1b^2 \quad \text{with} \ b \not\equiv 0 \mod p_2
\]
either both have integral solutions, or neither one has an integral solution.

**III. Proof of Theorem 1.** We consider \( E = \mathbb{Q}(\sqrt{p_1p_2}) \) with primes \( p_1 \equiv p_2 \equiv 1 \mod 8 \). By definition, \( 2\)-Sylow \( K_2\mathcal{O}_E \) is elementary abelian if and only if the \( 4\)-rank of \( K_2\mathcal{O}_E \) is zero. By [4, 2.3] we have
\[
4\text{-rk} K_2\mathcal{O}_E = 0 \quad \text{if and only if} \quad 2\text{-rk} \ker \chi = 1
\]
where \( \chi : H_E \to C_S(E)/C_S(E)^2 \) is the homomorphism given in [4, 2.1]. Here \( C_S(E) \) denotes the \( S \)-ideal class group of \( E \) with \( S \) being the set of infinite and dyadic places of \( E \). Since the square class of 2 lies in the kernel of \( \chi \) we can restate (1) as
\[
4\text{-rk} K_2\mathcal{O}_E = 0 \quad \text{if and only if} \quad \ker \chi \text{ is generated by the class of } 2 \text{ in } E^*/E^{*2}.
\]

Let \( C(E) \) denote the (ordinary) ideal class group of \( E \). We have \( 2\text{-rk} C(E) = 1 \), compare [2, 18.3] and \( 2\text{-rk} C_S(E) = 1 \) also since \( C_S(E)/C_S(E)^2 \cong C(E)/C(E)^2 \). Let \( \mathfrak{P}_1 \) denote the prime ideal of \( \mathcal{O}_E \) lying over the ramified prime \( p_1 \), say.

Assume now that \( 2\)-Sylow \( K_2\mathcal{O}_E \) is elementary abelian. If the class of \( \mathfrak{P}_1 \) were a square in \( C(E) \), then the class of \( p_1 \) would be in the kernel of both the homomorphisms \( \chi_1 \) and \( \chi_2 \) defined in [4, 2.5 and 3.1] and hence in the kernel of \( \chi = \chi_1\chi_2 \) (see [4, 3.2]). However by (2), the class of \( p_1 \) in \( E^*/E^{*2} \) does not lie in \( \ker \chi \). Thus, the class of \( \mathfrak{P}_1 \), whose square is 1, is a nonsquare in \( C(E) \). So, \( 2\)-Sylow \( C(E) \) is generated by the class of \( \mathfrak{P}_1 \) and \( 4\)-rk \( C(E) = 0 \).

We have shown that \( 2\)-Sylow \( C(E) \) is elementary abelian. This implies that \( (p_1/p_2) = -1 \) (compare [2, 19.6]), and in that case the norm of the fundamental unit of \( E \) is \( -1 \) (see [2, 19.9]). In other words, we concluded
that the 2-Sylow subgroup of the narrow ideal class group of $E$ is elementary abelian. In terms of the graph $\Gamma(E)$ of $E$ (see [5]) this means that $\Gamma(E)$ is given by $p_1 \bullet - \bullet p_2$, which is equivalent to the Legendre symbol $(p_1/p_2)$ being $-1$.

Thus we have:

(3) $$(p_1/p_2) = -1$$ if and only if 2-Sylow $C(E)$ is elementary abelian and the norm of the fundamental unit of $E$ is $-1$.

In order to finish the proof of Theorem 1 it now suffices to prove that under the assumption of 2-Sylow $C(E)$ being elementary abelian and $N\varepsilon = -1$ for the fundamental unit of $\varepsilon$ of $E$, we have:

2-Sylow $K_2\mathcal{O}_E$ is elementary abelian if and only if exactly one of the primes $p_1, p_2$ fails to be represented over $\mathbb{Z}$ by the quadratic form $x^2 + 32y^2$.

Consider the subgroup $U^+_S$ of $E^*/E^{*2}$ consisting of square classes of totally positive $S$-units of $E$. The 2-rank of $U^+_S$ is 2; the kernel of $\chi$ is generated by the class of 2 in $E^*/E^{*2}$ if and only if $U^+_S \cap H_E$ is generated by the class of 2. Since the elements of $H_E$ are square classes of elements in $E^*$ which are norms from $E(\sqrt{-1})$ over $E$, we have obtained so far:

(4) 2-Sylow $K_2\mathcal{O}_E$ is elementary abelian if and only if $(p_1/p_2) = -1$ and there exists a totally positive $S$-unit $\pi$ of $E$ that fails to be a norm from $E(\sqrt{-1})$ over $E$.

We will now use reciprocity of Hilbert symbols to relate the last condition to the positive definite form $x^2 + 32y^2$. Let $D_1$ be one of the two dyadic primes of $E$. For a totally positive $S$-unit $\pi$ of $E$, all we have to characterize is $$(\pi, -1)_{D_1} = -1.$$ Now, $(\pi, -1)_{D_1} = (2, \varepsilon)_{D_1}$, where $\varepsilon$ is the fundamental unit of $E$. We are going to characterize

$$(2, \varepsilon)_{D_1} = -1.$$ Let $\mathfrak{D}$ be the dyadic prime of $E(\sqrt{-1})$ over $D_1$. We have $(2, \varepsilon)_{D_1} = (1 + i, \varepsilon)_{\mathfrak{D}}$, where $i^2 = -1$. So, exactly when is

$$(1 + i, \varepsilon)_{\mathfrak{D}} = -1 ?$$

We want to characterize this in terms of the quadratic field $\mathbb{Q}(\sqrt{-1})$. Since $\varepsilon$ is of norm $-1$, there exists a $\delta$ in $\mathbb{Q}(\sqrt{-1})$ such that $\delta$ and $\varepsilon \in E$ have the same square class in $E(\sqrt{-1})$ and $N_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}(\delta) = p_1p_2$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$. We ask: when is

$$(1 + i, \delta)_{\mathfrak{D}} = -1 ?$$
With \( D = (1+i) \), the dyadic prime in \( \mathbb{Q}(\sqrt{-1}) \), this amounts to: when is 
\[
(1+i, \delta)_D = -1?
\]

Let \( P_j \) and \( \overline{P}_j \) be the primes of \( \mathbb{Q}(\sqrt{-1}) \) lying over \( p_j, j = 1, 2 \). Since \( \text{ord}_{P_j}(\delta) + \text{ord}_{\overline{P}_j}(\delta) \equiv 1 \mod 2 \), we may assume that \( \text{ord}_{P_j}(\delta) \equiv 1 \mod 2, j = 1, 2 \). Now we can make the essential step: we have 
\[
(1+i, \delta)_D = (1+i, \delta)_{P_1} (1+i, \delta)_{P_2}
\]
with the Hilbert symbols on the right hand side given by the 4-th power symbols \( \left[ \frac{2i}{P_j} \right]_4, j = 1, 2 \). So
\[
(1+i, \delta)_D = \left[ \frac{2i}{P_1} \right]_4 \left[ \frac{2i}{P_2} \right]_4
\]
and, by [1], the symbol \( \left[ \frac{2i}{P_j} \right]_4 \) is -1 if and only if the rational prime \( p_j \) is not of the form \( x^2 + 32y^2 \) over \( \mathbb{Z} \).

We have obtained
\[
(\pi, -1)_{D_1} = \left[ \frac{2i}{p_1} \right]_4 \left[ \frac{2i}{p_2} \right]_4 = -1
\]
if and only if exactly one of the primes \( p_1, p_2 \) fails to be represented over \( \mathbb{Z} \) by the quadratic form \( x^2 + 32y^2 \).

In view of (4), this completes the proof of Theorem 1.

We have given the proof of Theorem 1 via (3) and (5) in order to suggest the following generalizations.

**IV. Conjectures**

**Conjecture 1.** Let \( E = \mathbb{Q}(\sqrt{p_1 \cdots p_k}) \) with distinct rational primes \( p_i \equiv 1 \mod 8, i = 1, \ldots, k \). Then 2-Sylow \( K_2 \mathcal{O}_E \) is elementary abelian if and only if

(i) 2-Sylow \( C(E) \) is elementary abelian and the norm of the fundamental unit of \( E \) is -1 and

(ii) an odd number of the primes \( p_1, \ldots, p_k \) fail to be represented over \( \mathbb{Z} \) by the quadratic form \( x^2 + 32y^2 \).

Since the analogy with Theorem 1 is so beautiful we are going to state without proof:

**Theorem 1’.** Let \( F = \mathbb{Q}(\sqrt{2p_1p_2}) \) with rational primes \( p_1 \equiv p_2 \equiv 1 \mod 8 \). Then 2-Sylow \( K_2 \mathcal{O}_F \) is elementary abelian if and only if

(i) \( (p_1/p_2) = -1 \) and

(ii) exactly one of the two primes \( p_1, p_2 \) fails to be represented over \( \mathbb{Z} \) by the quadratic form \( x^2 + 64y^2 \).
Regarding Theorem 1′ we suggest the generalization:

**Conjecture 1′.** Let \( F = \mathbb{Q}(\sqrt{2p_1 \ldots p_k}) \) with distinct rational primes \( p_i \equiv 1 \mod 8, i = 1, \ldots, k \). Put \( E = \mathbb{Q}(\sqrt{p_1 \ldots p_k}) \). Then 2-Sylow \( K_2O_F \) is elementary abelian if and only if

(i) 2-Sylow \( C(E) \) is elementary abelian and the norm of the fundamental unit of \( E \) is \(-1\) and

(ii) an odd number of the primes \( p_1, \ldots, p_k \) fail to be represented over \( \mathbb{Z} \) by the quadratic form \( x^2 + 64y^2 \).

By the above and [3], the conjectures are valid for \( k = 1 \) and \( k = 2 \).

In Theorem 1′ and Conjecture 1′ the quadratic form \( x^2 + 64y^2 \) replaces naturally the quadratic form \( x^2 + 32y^2 \) from Theorem 1 and Conjecture 1 in view of Gauss’s famous result: For a prime \( p \equiv 1 \mod 8 \), the fourth power symbol \( \left[ \frac{2}{p} \right]_4 \) is \(-1\) if and only if \( p \) is not of the form \( x^2 + 64y^2 \) over \( \mathbb{Z} \); see e.g. [9, p. 84].

**V. Numerical illustration in the real case.** Among the three primes 17, 41, and 73, the prime 41 = \( 3^2 + 32 \cdot 1^2 \) is the only one that is represented over \( \mathbb{Z} \) by the form \( x^2 + 32y^2 \). We have \((17/41) = (17/73) = -1\) and \((41/73) = +1\). Hence, by Theorem 1:

For \( E = \mathbb{Q}(\sqrt{17 \cdot 41}) \), 2-Sylow \( K_2O_E \) is elementary abelian.

For \( E = \mathbb{Q}(\sqrt{17 \cdot 73}) \), 2-Sylow \( K_2O_E \) is not elementary abelian.

For \( E = \mathbb{Q}(\sqrt{41 \cdot 73}) \), 2-Sylow \( K_2O_E \) is not elementary abelian.

Among the three primes 17, 41, and 73, the prime 73 = \( 3^2 + 64 \cdot 1^2 \) is the only one that is represented over \( \mathbb{Z} \) by the form \( x^2 + 64y^2 \). Hence, by Theorem 1′:

For \( F = \mathbb{Q}(\sqrt{2 \cdot 17 \cdot 41}) \), 2-Sylow \( K_2O_F \) is not elementary abelian.

For \( F = \mathbb{Q}(\sqrt{2 \cdot 17 \cdot 73}) \), 2-Sylow \( K_2O_F \) is elementary abelian.

For \( F = \mathbb{Q}(\sqrt{2 \cdot 41 \cdot 73}) \), 2-Sylow \( K_2O_F \) is not elementary abelian.

**VI. Proof of Theorem 2.** We consider \( L = \mathbb{Q}(\sqrt{-p_1p_2}) \) with primes \( p_1 \equiv 7 \mod 8, p_2 \equiv 1 \mod 8 \) and \((p_1/p_2) = 1\). Let \( S \) be the set of infinite and dyadic places of \( L \). The 2-rank of the \( S \)-ideal class group of \( L \) is 1, compare [4, 7.1]; let \( h_S(L) \) denote the \( S \)-class number of \( L \). This time, we have by [4, 2.3]:

\[
(6) \quad 4\text{-rk}K_2O_L = 0 \text{ if and only if } 2\text{-rk} \ker \chi = 2.
\]

In terms of the homomorphism \( \chi_2 \) one concludes:

\[
(7) \quad 2\text{-Sylow } K_2O_L \text{ is elementary abelian if and only if either } h_S(L) \equiv 2 \mod 4 \text{ and } \chi_2 \text{ is trivial, or } h_S(L) \equiv 0 \mod 4 \text{ and } \chi_2 \text{ is nontrivial.}
\]
We can express the 2-rank of the kernel of $\chi_2$ in terms of the field $L(\sqrt{-1})$ (see [4, 3.9]):

$$2\text{-rk ker } \chi_2 = 1 + 2\text{-rk}C_S(L(\sqrt{-1})).$$

Thus, by [4, 7.3] we find that $\chi_2$ is trivial if and only if $2\text{-rk}C_S(L(\sqrt{-1})) = 2$ if and only if $p_2$ is represented by $x^2 + 32y^2$ over $\mathbb{Z}$. So, we conclude:

(8) $2$-Sylow $K_2O_L$ is elementary abelian if and only if $h_S(L) \equiv 2 \mod 4$ and $p_2$ is represented by $x^2 + 32y^2$ over $\mathbb{Z}$, or $h_S(L) \equiv 0 \mod 4$ and $p_2$ is not represented by $x^2 + 32y^2$ over $\mathbb{Z}$.

The issue left is to identify such pairs of primes $p_1, p_2$ for which $h_S(L) \equiv 2 \mod 4$. The 2-Sylow subgroup of the ideal class group of the quadratic field $K = \mathbb{Q}(\sqrt{-2p_1})$ is cyclic of order divisible by four (see [2, 18.6 and 19.6]). Hence $K$ admits a unique unramified cyclic extension $N$ of degree 4 over $K$. The field $N$ has the following properties:

- $N$ is a quadratic extension of $\mathbb{Q}(\sqrt{-p_1, \sqrt{2}})$,
- $N$ is normal over $\mathbb{Q}$, and the Galois group of $N$ over $\mathbb{Q}$ is the dihedral group of order 8.

The rational prime $p_2$ splits in $\mathbb{Q}(\sqrt{-p_1, \sqrt{2}})$. Thus the Artin symbol $\mathfrak{A}(p_2, N/\mathbb{Q})$ is a well-defined central element of $\text{Gal}(N/\mathbb{Q})$. In terms of the Artin symbol we have the following characterization:

(9) $h_S(L) \equiv 2 \mod 4$ if and only if $\mathfrak{A}(p_2, N/\mathbb{Q}) \neq 1$ if and only if $p_2$ is not completely split in $N$ over $\mathbb{Q}$.

The characterization (9) does make it possible to restate result (8) in definite terms. The prime $p_2$ splits in $K$ and $p_2$ is a norm from $K$ over $\mathbb{Q}$. We write $p_2O_K = P_2 \mathfrak{P}_2$; the class of $P_2$ is a square in the ideal class group $C(K)$. The prime $P_2$ of $K$ splits completely in $N$ over $K$ if and only if its class is a fourth power in $C(K)$. Since the 2-Sylow subgroup of $C(K)$ is cyclic we conclude that either $\text{cl}(P_2)^{h(K)/4}$ is trivial in $C(K)$, or $\text{cl}(P_2)^{h(K)/4}$ is the element of order 2 in $C(K)$.

Thus either $P_2^{h(K)/4}$ is principal which occurs if and only if $p_2$ splits completely in $N$ over $\mathbb{Q}$, or $D \cdot P_2^{h(K)/4}$ is principal, where $D$ is the dyadic prime of $K$. In view of (9) this yields

(10) $h_S(L) \equiv 2 \mod 4$ if and only if $p_2^{h(K)/4} = 2a^2 + p_1b^2$ for some $a, b \in \mathbb{Z}$ with $b \not\equiv 0 \mod p_2$.

Thus, (8) and (10) combined yield the characterization stated in Theorem 2. □

We note that Theorem 2 has been given in definite terms, since there is an effective algorithm to determine the class number of $K$. If the class number of $K$ is equal to 4, so $h(K)/4 = 1$, then we can drop the restriction $b \not\equiv 0 \mod p_2$ in the statement of Theorem 2. For example, for $p_1 = 7$ we obtain:
Abelian 2-Sylow $K_2$ of rings of integers

Corollary. Let $L = \mathbb{Q}(\sqrt{-7p})$ with a rational prime $p \equiv 1 \pmod{8}$, $(7/p) = 1$. Then 2-Sylow $K_2(\mathcal{O}_L)$ is elementary abelian if and only if $p = x^2 + 32y^2$ and $p = 2a^2 + 7b^2$ either both have integral solutions or neither one has an integral solution.

VII. Numerical illustration in the imaginary case. For $p_1 = 7$ or 23, and $p_2 = 193$ we have $(p_1/p_2) = 1$ and $K = \mathbb{Q}(\sqrt{-2p_1})$ has class number $h(K) = 4$. We have $p_2 = 193 = 2 \cdot 3^2 + 7 \cdot 5^2$ is neither represented by $x^2 + 32y^2$ nor by $2a^2 + 23b^2$ over $\mathbb{Z}$. Hence by Theorem 2:

For $L = \mathbb{Q}(\sqrt{-7 \cdot 193})$, 2-Sylow $K_2\mathcal{O}_L$ is not elementary abelian.

For $L = \mathbb{Q}(\sqrt{-23 \cdot 193})$, 2-Sylow $K_2\mathcal{O}_L$ is elementary abelian.

For $p_1 = 31$ and $p_2 = 193$ we have $(p_1/p_2) = 1$ and $K = \mathbb{Q}(\sqrt{-2 \cdot 31})$ has class number $h(K) = 8$. Since neither $p_2 = 193$ is represented by $x^2 + 32y^2$ nor $p_2^2 = 193^2$ is represented by $2a^2 + 31b^2$, we have by Theorem 2:

For $L = \mathbb{Q}(\sqrt{-31 \cdot 193})$, 2-Sylow $K_2\mathcal{O}_L$ is elementary abelian.

References


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